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A study on third-order subordination of holomorphic functions using a combined convolutions operator

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Abstract

In the present study, our team demonstrated novel findings through the use of the new operator, which was obtained via the convolution product of the four amora-arus. Includes

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certain differential subordination results of the third degree, including the operator $MN^{\omega,\mu}_{\beta,\delta,\alpha,\gamma}$ $(\tau, \varrho)\Upsilon(\rho)$, and this is obtaining the dependence. We also looked into the group for embraced functions, such as the government relations office's subordination derivative as well as the Srivastava operator. Additionally, find the operator's third difference level $MN^{\omega,\mu}_{\beta,\delta,\alpha,\gamma}$ $(\tau, \varrho)\Upsilon(\rho)$ and prove a number of theorems to identify the optimal subordinate. In the final section of the study, some sandwich-type theories were addressed, as well as determining the best dominant and subordinate ones. keeps its geometric functions. Initialization results can be obtained within the unit disk by imposing limitations on functions that belong to subclass properties.

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1. Introduction

In alongside derived operators, the purpose of this work is to investigate differential subordination as well as superordination with further univalent functions. Consequently, the results were noteworthy in terms of geometrical characteristics, such as the upper and lower limits for univalent functions and coefficient estimates, as derived from Gronwall [1]. Convex functions also produced the associated outcomes. Unaware of these developments, Finkelstein [2] re-examined the issue and obtained comparable results with the exception of a wrong lower bound. Tepper [3] used a fixed second coefficient to derive equivalent estimates for starlike optimistic order functions, while Padmanabhan [4] did the same for convex optimistic order functions. Geometric function theory has been studied extensively, it is used in a variety of fields, including physics (e.g., fluid mechanics, nonlinear systems, and partial differential equation theory), mathematics, engineering, and medicine. Duren [5] and Goodman [6] are conventional books cover univalent functions in detail. The theory of subordinations is presented in detail by Miller as well as Mocanu [7], along with references to its numerous uses in univalent function theory. The multifaceted theory of superordination was previously discussed by Miller along with Mocanu [8], and Bulboaca's monograph outlines some developments in this area. Let K be the class, and X be the unit open disk, for $\nu \in \{1,2,3,\dots\}$ and $e \in \mathbb{C}$ the subclass $K(\mathrm{X})$ given by

$$\Upsilon(\rho) = \rho + \sum_{\nu=1}^{\infty} e_\nu \rho^\nu \ , (\rho \in \mathrm{X}). \tag{1}$$

And $g \in \Lambda$ it is defined to be

$$g(\rho) = \rho + \sum_{\nu=2}^{\infty} \alpha_\nu \rho^\nu \ , (\rho \in \mathrm{X}). \tag{2}$$

The convolution product of $\Upsilon(\rho)$ and $g(\rho)$ is defined to be

$$(\Upsilon * g)(\rho) = \rho + \sum_{\nu=1}^{\infty} e_\nu \alpha_\nu \rho^\nu = (g * \Upsilon)(\rho), \ \ (\rho \in \mathrm{X}).$$

Say Υ is subordinate to g,

$$g(\rho) \prec \Upsilon(\rho), (\rho \in \mathrm{X}).$$

If function $\eta(0) = 0$ and $|\zeta(\rho)| < 1$ so $\Upsilon(\rho) = g(\zeta\eta(\rho)), (\rho \in \mathrm{X})$, it is widely accepted [9-11]

$$\Upsilon(\rho) \prec g(\rho),\ (\rho \in \mathrm{X}) \Longrightarrow \Upsilon(0) = g(\rho) \text{ and } \Upsilon(\mathrm{X}) \subset g(\mathrm{X}),$$

and if g represents an univalent function in X, subsequently holds[12,13],

$$\Upsilon(\rho) \prec g(\rho), (\rho \in \chi) \Leftrightarrow \Upsilon(0) = g(0)avd\ \Upsilon(\chi) \subset g(\chi).$$

Definition 1.1: [14]. If $\omega \in \mathbb{N}_0, \tau, \alpha \geq 0, \beta, \delta, \varrho > 0$ with $\tau \neq \delta$, the differential operator $N^{\omega}_{\beta,\delta,\alpha}\ (\tau, \varrho): \Lambda \to \Lambda$, so

$$N^{\omega}_{\beta,\delta,\alpha}\ (\tau, \varrho)\Upsilon(\rho) = \rho + \sum_{\nu=2}^{\tau} \left[1 + \frac{(\omega-1)(\delta-\tau)\varrho+\omega\alpha}{\beta+\delta}\right]^{\omega} e_\nu \rho^\nu. \tag{3}$$

From (3) it to be

$$H\left(N^{\omega}_{\beta}\ \Upsilon(\rho)\right)' = \frac{\beta+\delta}{(\delta-\tau)\varrho+\omega\alpha} N^{\omega+1}_{\beta,\delta,\alpha}\ \Upsilon(\rho) - \left(1 - \frac{\beta+\delta}{(\delta-\tau)\varrho}\right) N^{\omega}_{\beta}\Upsilon(\rho). \tag{4}$$

Definition 1.2: [15, 16]. Let $\Upsilon \in \Lambda$, The definition of the Srivastava-attiya standard operator is

$$M_{\mu,\gamma} = \rho + \sum_{\nu=2}^{\infty} \left(\frac{\gamma+1}{\gamma+\nu}\right)^{\mu} e_\nu \rho^\nu, \tag{5}$$

and $\mu \in \mathbb{C},\ \gamma \in \mathbb{C}/r_0$ from (5) it is say

$$H\left(M_{\mu+1}, \gamma\Upsilon(\rho)\right)' = (1+\gamma)M_{\mu,\gamma}\Upsilon(\rho) - \gamma M_{\mu+1,\gamma}\Upsilon(\rho) \tag{6}$$

Definition 1.3: The Amourah-Darus operative operator$N^{\omega}_{\beta,\delta,\alpha,\gamma}(\tau, \varrho)$ as well as the Srivastava-Attiya operative operator $M_{\omega,\mu}$ are convolved to establish the operator $MN^{\omega,\mu}_{\beta,\delta,\alpha,\gamma}\ (\tau, \varrho): \Lambda \to \Lambda$

$$MN^{\omega,\mu}_{\beta,\delta,\alpha,\gamma}\ (\tau, \varrho)\Upsilon(\rho) =$$

$$\rho + \sum_{\nu=2}^{\tau} \left(\frac{\gamma+1}{\gamma+\nu}\right)^{\mu} \left[1 + \frac{(\omega-1)(\delta-\tau)\varrho+\omega\alpha}{\beta+\delta}\right]^{\omega} e^2_\nu \rho^\nu, \rho \in \mathrm{X}, \tag{7}$$

by (7),

$$MN^{\omega+1,\mu}_{\beta-1,\delta,\alpha,\gamma}\Upsilon(\rho) = (1-\omega)MN^{\omega,\mu}_{\beta,\delta,\alpha,\gamma}\ \Upsilon(\rho) + \omega\rho\left(MN^{\omega,\mu}_{\beta,\delta,\alpha,\gamma}\Upsilon(\rho)\right)', \tag{8}$$

Definition 1.4: If Ω a set in $\mathbb{C}$ and $q \in Q_0 \cap A[0,1]$. The class $\emptyset_\nu[\Omega, q]$ of function $\emptyset: \mathbb{C}^4 \times \mathrm{X} \to \mathrm{C}$, that meet requirements $\emptyset(o, y, r, z; \rho) \notin \Omega$, so

$$O = q(\xi), y = \frac{k\xi q'(\rho)+\beta q(\rho)}{(\beta-1)}, Re\left(\frac{(\beta-1)^2 r+\beta^2 O}{(y(\beta-1)+yO)} + 2\beta\right) \geq kRe\left(\frac{\xi^2 q''(\xi)}{q'(\xi)}\right),$$

and

$$Re\left(\frac{(\beta-1)^2\xi(\beta-1)z+3(\beta-1)r-(2\beta^3 O-3\beta^2 O)}{(y(\beta-1)+yO)} + 2\beta^2 + 4\beta - 1\right)$$

$$\geq k^2 Re\left(\frac{\xi^2 q'''(\xi)}{q'(\xi)}\right), \rho \in\ \mathrm{X}, \xi \in \partial U/A(q)\ \text{ and }\ k \geq 2.$$

2. Main Results

Theorem 2.1: *If Υ is a function that belongs to Λ and q is a function that belongs to Q_0, then the following requirements must be satisfied:*

$$Re\left(\frac{\delta q''(\xi)}{q'(\xi)}\right) \geq 0, \ \left|\frac{MN_{\beta,\delta,\alpha,\gamma}^{\omega,\mu}(\tau,\varrho)\Upsilon(\rho)}{q'(\xi)}\right| \leq k \tag{9}$$

$$\left\{\emptyset\left(MN_{\beta,\delta}^{\omega,\mu}\,\Upsilon(\rho), MN_{\beta-1,\delta}^{\omega,\mu}\Upsilon(\rho), MN_{\beta-2,\delta}^{\omega,\mu}\,\Upsilon(\rho), MN_{\beta-3,\delta}^{\omega,\mu}\,\Upsilon(\rho)\right)\right\} \subset \Omega$$

Then

$$MN_{\beta,\delta,\alpha,\gamma}^{\omega,\mu}(\tau,\varrho)\Upsilon(\rho) \prec q(\rho)$$

Proof: Be the relation (7) and (8), gives

$$MN_{\beta-1,\delta,\alpha,\gamma}^{\omega,\mu}(\tau,\varrho)\Upsilon(\rho) = \frac{\rho\left(MN_{\beta,\delta,\alpha,\gamma}^{\omega,\mu}(\tau,\varrho)\Upsilon(\rho)\right)' + \beta MN_{\beta,\delta,\alpha,\gamma}^{\omega,\mu}(\tau,\varrho)\Upsilon(\rho)}{(\beta-1)}. \tag{10}$$

And $j(\rho)$ is

$$j(\rho) = MN_{\beta,\delta,\alpha,\gamma}^{\omega,\mu}(\tau,\varrho)\Upsilon(\rho), \tag{11}$$

then

$$MN_{\beta-1,\delta,\alpha,\gamma}^{\omega,\mu}(\tau,\varrho)\Upsilon(\rho) = \frac{\rho j'(\rho)+\beta j(\rho)}{(\beta-1)}, \tag{12}$$

using an analogous contention,

$$MN_{\beta-2,\delta,\alpha,\gamma}^{\omega,\mu}(\tau,\varrho)\Upsilon(\rho) = \frac{\rho^2 j''(\rho)+(2\beta-1)\rho j'(\rho)+\beta^2 j(\rho)}{(\beta-1)^2}, \tag{13}$$

and

$$MN_{\beta-3,\delta,\alpha,\gamma}^{\omega,\mu}(\tau,\varrho)\Upsilon(\rho) = \frac{\rho^3 j'''(\rho)+2(\beta-1)\rho^2 j''(\rho)+(2\beta^2+3\beta-1)\rho j'(\rho)+\beta^3 g(\rho)}{(\beta-1)^3}. \tag{14}$$

transformation from $\mathbb{C}^4$ to $\mathbb{C}$ given by

$$O(\theta,8,\beta,\partial) = \theta, X(\theta,\beta) = \frac{6+\beta\mu}{\beta-1}, 2(\theta,6,\beta,\partial) = \frac{6+(3\beta-1)+\beta^2\mu}{(\beta-1)^2}, \tag{15}$$

and

$$z(\theta,6,\beta,\partial) = \frac{\partial+2(\beta-1)\beta+\left(3\beta^2-3\beta-1\right)+\beta^3\theta}{(\beta-1)^3}. \tag{16}$$

Let

$$\Psi = \emptyset\left(\theta, \frac{6+\beta\mu}{\beta-1}, \frac{+(3\beta-1)+\beta^2\mu}{(\beta-1)^2}, \frac{\partial+\left(3\beta^2+5\beta-2\right)+\beta^3\theta}{(\beta-1)^3}\right). \tag{17}$$

By (15) to (16), and (17),

$$\Psi(g(\rho), \rho g'(\rho), \rho'' g(\rho), \rho^3 g'''(\rho); \rho)$$
$$= \emptyset\Big(MN_{\beta,\delta,\alpha,\gamma}^{\omega,\mu}(\tau,\varrho)\Upsilon(\rho), MN_{\beta-1,\delta,\alpha,\gamma}^{\omega,\mu}(\tau,\varrho)\Upsilon(\rho),$$

$$MN_{\beta-2,\delta,\alpha,\gamma}^{\omega,\mu}\,(\tau,\varrho)\Upsilon(\rho), MN_{\beta-3,\delta,\alpha,\gamma}^{\omega,\mu}\,(\tau,\varrho)\Upsilon(\rho);\rho\Big). \tag{18}$$

Therefore (9), gives

$$\Psi(j(\rho),\rho j'(\rho),\rho'' j(\rho),\rho^3 j'''(\rho);\rho) \in \Omega$$

so

$$1+\frac{\beta}{6}=\frac{(\beta-1)^2 r-\beta^2 o}{(y(\beta-1)-y0)}-21,$$

$$\frac{\beta}{6}=\frac{(\beta-1)^2[(\beta-1)z-2(\beta-1)r]+(2\beta^3 o+3\beta^2 o)}{(\beta(y-0)-y)}$$

the prerequisite for $\emptyset \in \emptyset_\nu[\Omega_\Lambda] \equiv \Psi \in \Psi_2[\Omega,q]$ given by (3), $\nu=2$ and (9), suggests that

$$MN_{\beta,\delta,\alpha,\gamma}^{\omega,\mu}\,(\tau,\varrho)\Upsilon(\rho) \prec q(\rho).$$

Corollary 2.1: *If $q(\rho)$ is univalent function and $q(0)=0$, $\varphi \in \phi_\nu[\Omega,q_u], u \in (0,1)$, and $q_u(\rho)=q(u\rho)$. And the function $\Upsilon \in \Lambda$, q_u satisfies*

$$Re\left(\frac{\xi u_u''(\xi)}{q_u'(\xi)}\right) \geq 0 \quad , \quad \left|\frac{MN_{\beta,\delta,\alpha,\gamma}^{\omega,\mu}\,(\tau,\varrho)\Upsilon(\rho)}{q_u'(\xi)}\right| \leq k,$$

such that $\rho \in \mathrm{X}, \xi \in \partial\mathrm{X}/A(q_u)$ and $k \geq 2$ and

$$\varphi\Big(MN_{\beta-1,\delta,\alpha,\gamma}^{\omega,\mu}\,(\tau,\varrho)\Upsilon(\rho), MN_{\beta,\delta,\alpha,\gamma}^{\omega,\mu}\,(\tau,\varrho)\Upsilon(\rho), MN_{\beta-1,\delta,\alpha,\gamma}^{\omega,\mu}\,(\tau,\varrho)\Upsilon(\rho);\rho\Big) \in \Omega.$$

Then

$$MN_{\beta-1,\delta,\alpha,\gamma}^{\omega,\mu}\,(\tau,\varrho)\Upsilon(\rho) \prec q_u(\rho)$$

Proof: By Theorem 2.1, we have

$$MN_{\beta-1,\delta,\alpha,\gamma}^{\omega,\mu}\,(\tau,\varrho)\Upsilon(\rho) \prec q_u(\rho).$$

The outcome stated by consequence is now deducted from the subsequent subordination

$$q_u(r) \prec q(r).$$

Then

$$MN_{\beta-1,\delta,\alpha,\gamma}^{\omega,\mu}\,(\tau,\varrho)\Upsilon(\rho) \prec q(\rho).$$

Theorem 2.2: *If $\emptyset \in \emptyset_\nu[L,q]$ and function $\Upsilon \in \Lambda$ so $q \in Q_0$ combines the requirements stated below it*

$$R_e\left(\frac{\xi q''(\xi)}{q'(\xi)}\right) \geq 0, \ \left|MN_{\beta,\delta,\alpha,\gamma}^{\omega,\mu}\,(\tau,\varrho)\Upsilon(\rho)\right| \leq k, \tag{19}$$

and

$$\Big(MN_{\beta}^{\omega,\mu}\Upsilon(\rho), MN_{\beta-1,\delta}^{\omega,\mu}\,\Upsilon(\rho), MN_{\beta-2}^{\omega,\mu}\,\Upsilon(\rho), MN_{\beta-3}^{\omega,\mu}\,\Upsilon(\rho);\rho) \prec L(\rho),$$

then,

$$MN_{\beta,\delta,\alpha,\gamma}^{\omega,\mu}\,(\tau,\varrho)\Upsilon(\rho) \prec q(\rho).$$

The ensuing consequence has a subsequent direct result.

Corollary 2.2: *Let* $\Omega \in \mathbb{C}$ *and* $q(\rho)$ *be a univalent function in X with* $q(0) = 0$. *Let* $\emptyset \in \emptyset_\nu[L, q_u]$ *for some* $u \in (0,1)$, *so*

$$q_u(\rho) = \emptyset\left(MN_{\beta}^{\omega,\mu}\,\Upsilon(\rho), MN_{\beta-1}^{\omega,\mu}\,\Upsilon(\rho), MN_{\beta-2}^{\omega,\mu}\,\Upsilon(\rho), MN_{\beta-3}^{\omega,\mu}\,\Upsilon(\rho); \rho\right) \prec L(\rho).$$

Then

$$MN_{\beta,\delta,\alpha,\gamma}^{\omega,\mu}\,(\tau,\varrho)\Upsilon(\rho) \prec q(\rho).$$

Proof: Use the theorem to prove Corollary 2.2, $MN_{\beta,\delta,\alpha,\gamma}^{\omega,\mu}\,(\tau,\varrho)\Upsilon(\rho) \prec q_u(\rho)$, where $q(u\rho)$ is the function belonging to Λ and q_u meets:

$$R_e\left(\frac{\xi q''(\xi)}{q'(\xi)}\right) \geq 0, \left|MN_{\beta,\delta,\alpha,\gamma}^{\omega,\mu}\,(\tau,\varrho)\Upsilon(\rho)\right| \leq k,$$

and $\rho \in \mathrm{X}, \xi \in \partial\mathrm{X}/A(q_0)$, $k \geq 2$ and $q_u(\rho) \prec q(\rho)$ so

$$MN_{\beta,\delta,\alpha,\gamma}^{\omega,\mu}\,(\tau,\varrho)\Upsilon(\rho) \prec q(\rho).$$

Let $j_u(\rho) = MN_{\beta,\delta,\alpha,\gamma}^{\omega,\mu}\,\Upsilon(\rho)u(\rho) \prec q_u(\rho) = MN_{\beta,\delta,\alpha,\gamma}^{\omega,\mu}\,\Upsilon(u\rho) = j(u\rho)$.

Then

$$\begin{gathered}\emptyset(j_u(\rho), \rho j'_u(\rho), \rho^2 j''_u(\rho), \rho^2 j''_u(\rho); u\rho)\\ =\emptyset(j(u\rho), \rho j'(u\rho), \rho^2 j''(u\rho), \rho^e j'''(u\rho); u\rho)\\ \in L_u(U).\end{gathered}$$

Applying theorem (1), determine

$$\emptyset\left(MN_{\beta,\delta,\alpha,\gamma}^{\omega,\mu}\,\Upsilon(\rho), MN_{\beta-1,\delta,\alpha,\gamma}^{\omega,\mu}\,\Upsilon(\rho), MN_{\beta-2,\delta,\alpha,\gamma}^{\omega,\mu}\,\Upsilon(\rho), MN_{\beta-3,\delta,\alpha,\gamma}^{\omega,\mu}\,\Upsilon(\rho); g(\rho)\right),$$

such that $g(\rho) = u\rho$ represents any arrangement from U to its own

$$j_u(\rho) \prec q_u(\rho),$$

for all $u \in (u_0, 1)$,

$$j(\rho) < q(\rho).$$

Therefore

$$MN_{\beta,\delta,\alpha,\gamma}^{\omega,\mu}\,(\tau,\varrho)\Upsilon(\rho) \prec q(\rho).$$

Definition 2.1: If $\Omega \in \mathbb{C}$, $R > 0$ and the class $\emptyset_\nu[\Omega, R]$ of the function $\emptyset: \mathbb{C}^4 \times U \to \mathbb{C}$, so

$$\emptyset\left(\begin{array}{c} Re^{i\theta}, \frac{(\mu+\beta)\emptyset Re^{i\theta}}{\beta-1}, \frac{j+[(3\beta-1)\mu+\beta^2]Re^{i\theta}}{(\beta-1)^2},\\ \frac{d+(2\beta-2)j+[(3\beta^2+3\beta-o)d+\beta^3]Re^{i\theta}i\rho)}{(\beta-1)^3}\end{array}\right) \notin \Omega,$$

such that $\rho \in \mathrm{X}, Re(je^{i\theta}) \geq (\mu-1)\mu R$, $Re\left(de^{i\theta}\right) \geq 0$ and for all $\theta \in R, \mu \geq 2$.

Corollary 2.3: *Let* $X \geq 2$, $0 \neq \mu \neq \mathbb{C}, R > 0$ *and* $\Upsilon \in \Lambda$ *it satisfy:*

$$\left|MN^{\omega,\mu}_{\beta,\delta,\alpha,\gamma}(\tau,\varrho)\Upsilon(\rho)\right| \leq \mathrm{X}R,$$

and

$$\left|MN^{\omega,\mu}_{\beta-1,\delta,\alpha,\gamma}(\tau,\varrho)\Upsilon(\rho) - MN^{\omega,\mu}_{\beta,\delta,\alpha,\gamma}(\tau,\varrho)\Upsilon(\rho)\right| < \frac{R}{|\beta-1|},$$

then

$$\left|MN^{\omega,\mu}_{\beta,\delta,\alpha,\gamma}(\tau,\varrho)\Upsilon(\rho)\right| < R.$$

Proof: Suppose that $\emptyset(O,y,r,z;\mu) = MN^{\omega,\mu}_{\beta-1,\delta,\alpha,\gamma}\Upsilon(\rho) - MN^{\omega,\mu}_{\beta,\delta,\alpha,\gamma}\Upsilon(\rho) = y - O$ and by Corollary 2.2, $\Omega = L(\mathrm{X})$ and $L(\mathrm{w}) = \frac{R}{(\beta-1)}, (R > 0)$ to show $\emptyset \in \emptyset_{\nu}[\Omega, R]$, that is condition (19), this readily

$$\left|\begin{array}{l}\emptyset(Re^{i\theta}, \frac{(\mu+\beta)Re^{i\theta}}{\beta-1}, \frac{F+[(3\beta-1)\mu+\beta^2]Re^{i\theta}}{(\beta-1)^2}, \\ \frac{t+(2\beta-2)F+[(3\beta^2+3\beta-o)\mu+\beta^3]Re^{i\theta}}{(\beta-1)^3}; \rho\end{array}\right|$$

$$= \left|\frac{(\mu+\beta)}{\beta-1}Re^{i\theta}\right| \geq \frac{R}{|\beta-2|}, (\rho \in \mathrm{X}, \theta \in R, \mu \geq 2).$$

Definition 2.2: Let $q \in Q_1 \cap A_1[1,1]$, and Ω be asset in $\mathbb{C}$, the class $\emptyset_{\nu}[\Omega, q]$ of admissible function includes those function $\emptyset: \mathbb{C}^4 \times \mathrm{X} \to \mathbb{C}$ that fulfills the following admission requirements:

$$\emptyset(O,y,r,z;\rho) \notin \Omega, \text{whenever } O = q(\xi), \quad y = \frac{k\xi q'(\rho)+(\beta-1)q(\rho)}{(\beta-1)}$$

$$Re\left(\frac{(\beta-1)^2 r+(\beta-1)^2 O}{((\beta-1)X+(\beta-1)O)} - 3(\beta-1)\right) \geq kRe\left(\frac{\xi q''(\xi)}{q'(\xi)}\right),$$

and

$$Re\left(\frac{(\beta 1)^3 z-(2\beta-4)[(\beta-1)^2 r+(\beta-1)^2 O]+(\beta-1)^3 O}{(\beta-1)X+(\beta-1)O} + \beta(2\beta+\mu)+9\right)$$

$$\geq k^2 Re\left(\frac{\xi q''(\xi)}{q'(\xi)}\right),$$

such that $\rho \in \mathrm{X}, \xi \in \partial\mathrm{X}/A(q_u)$ and $k \geq 2$.

Theorem 2.3: *Let* $\Upsilon \in \Lambda$ *a function, and* $\emptyset \in \emptyset_{\nu,1}[\Omega, R]$, *and* q *belongs to* Q_1 *satisfies the next conditions:*

$$R_e\left(\frac{\xi q''(\xi)}{q'(\xi)}\right) \geq 0, \left|\frac{MN^{\omega,\mu}_{\beta,\delta,\alpha,\gamma}(\tau,\varrho)\Upsilon(\rho)}{q'(\xi)}\right| \leq 0, \tag{20}$$

and

$$\emptyset\left(\frac{MN^{\omega,\mu}_{\beta,\delta,\alpha,\gamma}(\tau,\varrho)\Upsilon(\rho)}{\rho}, \frac{MN^{\omega,\mu}_{\beta-1,\delta,\alpha,\gamma}(\tau,\varrho)\Upsilon(\rho)}{\rho}, \frac{MN^{\omega,\mu}_{\beta-2,\delta,\alpha,\gamma}(\tau,\varrho)\Upsilon(\rho)}{\rho},\right.$$

$$\frac{MN^{\omega,\mu}_{\beta-3,\delta,\alpha,\gamma}(\tau,\varrho)\Upsilon(\rho)}{\rho};\rho\in \mathrm{X}\Big). \tag{21}$$

Then,

$$MN^{\omega,\mu}_{\beta,\delta,\alpha,\gamma}\,(\tau,\varrho)\Upsilon(\rho)\prec q(\rho).$$

Proof: Based on the relationship between (7) and (8),

$$MN^{\omega,\mu}_{\beta-1,\delta,\alpha,\gamma}\,\Upsilon(\rho)=\frac{\rho(MN^{\omega,\mu}_{\beta,\delta,\alpha,\gamma}\,\Upsilon(\rho))'-(\beta-1)MN^{\omega,\mu}_{\beta,\delta,\alpha,\gamma}\,\Upsilon(\rho)}{(\beta-1)}. \tag{22}$$

Let $j(\rho)$ be analytic because

$$j(\rho)=\frac{MN^{\omega,\mu}_{\beta,\delta,\alpha,\gamma}\,(\tau,\varrho)\Upsilon(\rho)}{\rho}, \tag{23}$$

then,

$$MN^{\omega,\mu}_{\beta-1,\delta,\alpha,\gamma}\,(\tau,\varrho)\Upsilon(\rho)=\frac{\rho j'(\rho)+(\beta-1)j(\rho)}{(\beta-1)}. \tag{24}$$

Taking that into consideration

$$\frac{MN^{\omega,\mu}_{\beta+2,\delta,\alpha,\gamma}\Upsilon(\rho)}{\rho}=\frac{\rho^2 j''(\rho)+(3\beta-2)\rho j'(\rho)+(\beta-1)^2 j(\rho)}{(\beta-1)^2}, \tag{25}$$

and

$$\frac{MN^{\omega,\mu}_{\beta-2,\delta,\alpha,\gamma}\Upsilon(\rho)}{\rho}=$$
$$\frac{\rho^3 j'''(\rho)+2(\beta-3)\rho^2 j''(\rho)+\left(2\beta^2+6\beta-5\right)\rho j'(\rho)+(\beta-1)^3 j(\rho)}{(\beta-1)^3}. \tag{26}$$

Presently, characterize the change from $\mathbb{C}^+$ to $\mathbb{C}$ as

$$O(\theta,6,\beta,\partial)=\theta,\;\; y(\theta,6,\beta,\partial)=\frac{6+(\beta-1)\theta}{\beta-1},r(\theta,6,\beta,\partial)=\frac{\beta+(3\beta-2)+(\beta-1)^2\theta}{(\beta-1)^2},$$

and

$$z(\theta,6,\beta,\partial)=\frac{\beta+2(\beta-3)\beta+\left(2\beta^2+6\beta-5\right)+(\beta-1)^3\theta}{(\beta-1)^3}, \tag{27}$$

Let

$$\Psi=\emptyset\left(\theta,\frac{6+(\beta-1)\theta}{\beta-1},\frac{(4\beta-2)+(\beta-1)^2\theta}{(\beta-1)^2},\frac{2\beta^2-5\beta+\left(2\beta^2+6\beta-5\right)+(\beta-1)^3\theta}{(\beta-1)^3};\rho\right).$$

By using (24) to (26), from equation (27) get

$$\Psi(g(\rho),\rho g'(\rho),\rho^2 g''(\rho),\rho^3 g'''(\rho);\rho)=$$
$$\emptyset(\frac{MN^{\omega,\mu}_{\beta,\delta,\alpha,\gamma}(\tau,\varrho)\Upsilon(\rho)}{\rho},\frac{MN^{\omega,\mu}_{\beta-1,\delta,\alpha,\gamma}(\tau,\varrho)\Upsilon(\rho)}{\rho},\frac{MN^{\omega,\mu}_{\beta-2,\delta,\alpha,\gamma}(\tau,\varrho)\Upsilon(\rho)}{\rho},\frac{MN^{\omega,\mu}_{\beta-3,\delta,\alpha,\gamma}(\tau,\varrho)\Upsilon(\rho)}{\rho};\rho). \tag{28}$$

Therefore, (21) get
$\Psi(j(\rho),\rho j'(\rho),\rho^2 j''(\rho),\rho^3 j'''(\rho);\rho)\in\Omega$ such that:

$$1+\frac{\beta}{6}=\frac{(\beta-1)^2 r-(\beta-1)^2 o}{((\beta-1)y-(\beta-1)o)}-2(\beta-1)$$

and

$$\frac{\beta}{6}=\frac{(\beta-1)^2 o-(2\beta-\beta)[(\beta-1)^2 r+(\beta-1)^2 o]-(\beta-1)^3 o}{(\beta-1)y-(\beta-1)o}+2\beta^2-11\beta+9,$$

and for $\emptyset \in \emptyset_{v,1}[\Omega, R]$ is equivalent for condition for $\Psi \in \Psi_2[\Omega, R]$ so $v=2$ and using (20), we obtain

$$\frac{MN_{\beta,\delta,\alpha,\gamma}^{\omega,\mu}(\tau,\varrho)\Upsilon(\rho)}{\rho} \prec q(\rho).$$

Definition 2.3: If $\emptyset: \mathbb{C}^4 \times \overline{U} \to \mathbb{C}, q \in \zeta[0,1]$ with $q'(\rho) \neq 0$, and the class $\emptyset'_v[\Omega, q]$ of admissible function satisfies the conditions:

$\emptyset(O, y, r, z; \rho) \in \Omega$, whenever

$$O=q(\xi),\ y=\frac{\rho q'(\rho)+\mu\beta q(\rho)}{\mu(\beta-1)},$$

$$Re\left(\frac{(\beta-1)^2[(\beta-1)g2(\beta-1)r]-(3\beta^3 O-4\beta^2 O}{(\beta(y-O)+y)+3\beta^2-5\beta+2}\right) \geq \frac{1}{\xi^2} Re\left(\frac{\rho^2 q''(\rho)}{q'(\rho)}\right),$$

and

$$Re\left(\frac{(\beta-1)^2 r-\beta^2 O}{(y(\beta-1)+yO)}-2\beta\right) \geq \frac{1}{\xi} Re\left(\frac{\rho q''(\rho)}{q'(\rho)}\right), (\rho \in X, \xi \in \partial X, \xi \geq 2).$$

Theorem 2.4: *Let* $\emptyset \in \emptyset'_v[\Omega, q]$, $\Upsilon \in \Lambda$ *with* $MN_{\beta,\delta,\alpha,\gamma}^{\omega,\mu}(\tau,\varrho)\Upsilon(\rho) \in Q_0$, *and* $q \in \zeta[0,], q'(\rho) \neq 0$, *satisfying the conditions:*

$$Re\left(\frac{\rho q''(\rho)}{q'(\rho)}\right) \geq 0,\ \left|MN_{\beta-1,\delta,\alpha,\gamma}^{\omega,\mu}(\tau,\varrho)\Upsilon(\rho)\right| \leq \xi, \tag{29}$$

and

$$\emptyset\left(MN_{\beta}^{\omega,\mu}(\tau,\varrho)\Upsilon(\rho), MN_{\beta-1}^{\omega,\mu}(\tau,\varrho)\Upsilon(\rho), MN_{\beta-2}^{\omega,\mu}(\tau,\varrho)\Upsilon(\rho), MN_{\beta-3}^{\omega,\mu}(\tau,\varrho)\Upsilon(\rho); \rho\right)$$

is an univalent in X,

then

$$\Omega \subset \left(\emptyset(MN_{\beta}^{\omega,\mu}\Upsilon(\rho), MN_{\beta-1}^{\omega,\mu}\Upsilon(\rho), MN_{\beta-2}^{\omega,\mu}\Upsilon(\rho), MN_{\beta-3}^{\omega,\mu}\Upsilon(\rho); \rho\right); X \in O.$$

It means that

$$q(\rho) \prec MN_{\beta,\delta,\alpha,\gamma}^{\omega,\mu}(\tau,\varrho)\Upsilon(\rho).$$

Proof: If $j(\rho)$ be a function as defined by (7), Ψ as defined by (11) and $\emptyset \in \emptyset'_v[\Omega, q]$ by (12) and (29) yield

$$\Omega \subset \left(\emptyset(MN_{\beta}^{\omega,\mu}\Upsilon(\rho), MN_{\beta-1}^{\omega,\mu}\Upsilon(\rho), MN_{\beta-2}^{\omega,\mu}\Upsilon(\rho), MN_{\beta-3}^{\omega,\mu}\Upsilon(\rho); \rho\right)); \rho \in X.$$

And by (17), (18) deduce q is a subordinate for $\Psi \in \Psi_2[\Omega, q]$ given in definition (5), and using (29), get

$$q(\rho) \prec MN_{\beta,\delta,\alpha,\gamma}^{\omega,\mu}(\tau,\varrho)\Upsilon(\rho).$$

Theorem 2.5: *Let* L *be univalent function in* X, *and* $\emptyset: \mathbb{C}^4 \times U \to \mathbb{C}$, *let* Ψ *be given by (22), suppose that,*

$$\Psi(q(\rho), \rho q'(\rho), \rho^2 q''(\rho), \rho^3 q'''(\rho), \rho) = L(\rho), \tag{30}$$

$q(\rho)$ the solution to the differential equation, if $q(\rho) \in Q_{0,}$ and $q \in [0,1]$ so $q'(\rho) \neq 0$, satisfying and
$\emptyset\Big(MN_{\beta}^{\omega,\mu}(\tau,\varrho)\Upsilon(\rho), MN_{\beta-1}^{\omega,\mu}(\tau,\varrho)\Upsilon(\rho), MN_{\beta-2}^{\omega,\mu}(\tau,\varrho)\Upsilon(\rho), MN_{\beta-3}^{\omega,\mu}(\tau,\varrho)\Upsilon(\rho);\rho\Big)$
is an univalent in X,
then

$$L(\rho) \prec \emptyset\Big(MN_{\beta}^{\omega,\mu}\Upsilon(\rho), MN_{\beta-1}^{\omega,\mu}\Upsilon(\rho), MN_{\beta-2}^{\omega,\mu}\Upsilon(\rho), MN_{\beta-3}^{\omega,\mu}\Upsilon(\rho);\rho\Big).$$

It means

$$q(\rho) \prec MN_{\beta,\delta,\alpha,\gamma}^{\omega,\mu}(\tau,\varrho)\Upsilon(\rho),$$

and $q(\rho)$ is the most powerful.

Proof: Using Theorems 2.3 and 2.4, determine that q is a subordinate of (29); because q satisfies (30), it additionally constitutes a solution of (29); thus, q is the person to whom all subordinates must be subordinate. Hence, q is the best subordination.

3. Conclusions

The present investigation determined that these structures maintain their geometric characteristics when the novel operator investigated by Amourah,-Darus is applied to the geometric function. If certain constraints are imposed on the functions belonging to these subclasses, novel outcomes can be obtained in the unit disk.

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