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A new sequential fractional problem of Lane-Emden type : An analytical and a numerical study

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Abstract

We study a class of three parameters-sequential integro-differential problem of Lane-Emden type with anti periodic conditions. We begin by presenting some results of existence and uniqueness, by application of BCP of Banach theorem. Then, by using Schaefer fixed point theorem, we prove an existence result. Some illustrative examples are also discussed. In another part of this paper, we apply the Adomian decomposition method for our problem under some initial conditions to obtain new solutions. Some graphs of the ADM method obtained solutions are plotted to show the behaviour of the solutions.

Subject Classification (2020): *30C45, 39A72, 39B85.*

Keywords: *Lane-emden integro differential equation, Caputo derivative, Existence and uniqueness, fixed point.*

1. Introduction

There are many applications of fractional differential equations on physics and engineering science, see for instance the papers [1, 2, 3, 4, 5]. In order understand the complex nonlinear physical phenomena, most research has focused on study of nonlinear FDE problems, see the articles to nonlinear FDEs, see papers [6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. One of the applications of FDEs is on the Lane-Emden differential problems. The classical problem was first studied by Lane in 1870 [16] and then more detailed by Emden [17]. The Lane-Emden problem is a problem that describes phenomena in mathematical physics, see [18, 19, 20].

The Lane-Emden equation is given by [21]:

$$z''(t) + \frac{a}{t}z'(t) + H\big(t, z(t)\big) = G(t), t \in [0,1], \tag{1.1}$$

with

$$z(0) = A_1, z'(0) = A_2, \tag{1.2}$$

and such that H, G are some given functions.

The examples of the application of this theory.To cite afew, [22, 23, 24, 25, 26].

The idea of this article in the first part, is to study the class of problems of three sequential FDEs of Lane Emden type:

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$$D^{\alpha}D^{\beta}D^{\gamma}z(t)+\frac{k}{t^{\lambda}}D^{\alpha}z(t)+a_1F(t,z(t),D^{\gamma}z(t),J^{p}z(t))+$$
$$a_2G(t,z(t),D^{\gamma}z(t))+a_3H(t,z(t))=L(t). \quad (1.3)$$

with the conditions:

$$z(0)+z(1)=0,\quad D^{\beta}z(0)+D^{\beta}z(1)=0,\quad D^{\gamma}z(0)+D^{\gamma}z(1)=0,$$
$$k>0,p>0\quad 0<\lambda<1,0\leq\alpha,\beta,\gamma\leq 1,\quad ,t\in]0,1].$$

In Eq. (1.8), we shall take into account that the derivatives of Caputo and the Riemann-Liouville integral, $F: I\times\mathbb{R}^3\to\mathbb{R}$, $G: I\times\mathbb{R}^2\to\mathbb{R}$, $H: I\times\mathbb{R}\to\mathbb{R}$ and $L: I\to\mathbb{R}$ are four given functions and $I=[0,1]$.

In the second part, we introduce the ADM method [27, 28, 29, 30, 31, 32] for solving the three sequential FDEs of Lane Emden type under some initial condition:

$$\begin{cases} D^{\alpha}D^{\beta}D^{\gamma}z(t)+\frac{k}{t^{\lambda}}D^{\alpha}z(t)+a_1F(t,z(t),D^{\gamma}z(t),J^{p}z(t))+ \\ a_2G(t,z(t),D^{\gamma}z(t))+a_3H(t,z(t))=f(t).\ a_1,a_2,a_3\in\mathbb{R}. \\ k>0,p>0\quad 0<\lambda<1,0\leq\alpha,\beta,\gamma\leq 1,1<\alpha+\beta+\gamma\leq 2, \\ t\in]0,1]. \end{cases} \quad (1.4)$$

2. Preliminaries

The following lemmas are to be recalled.

Lemma 1: *For* $\alpha>0$ *and* $n=[\alpha]+1$, *we have*

$$J^{\alpha}[D^{\alpha}z(t)]=z(t)+c_0+c_1t+c_2t^2+\ldots+c_{n-1}t^{n-1},t\in I,$$

for some $c_i\in\mathbb{R},i=0,1,\cdots,n-1$.

Lemma 2: *Let* $q>p>0$, $f\in L^1(I)$. *Then, it yields that*

$$D^{p}J^{q}[f(t)]=J^{q-p}[f(t)].$$

By the following lemma, we give the integral representation for (1.3):

Lemma 3; *Let* $L_1\in C(I)$, $t\in I$, $0\leq\alpha,\beta,\gamma\leq 1$. *Then the solution of*

$$D^{\alpha}\left(D^{\beta}[D^{\gamma}z(t)]\right)=L_1(t), \quad (2.1)$$

such as

$$L_1(t)=L(t)-a_1F(t,z(t),D^{\gamma}z(t),J^{p}y(t))-a_2G(t,z(t),D^{\gamma}z(t))-$$
$$a_3H(t,z(t))-\frac{k}{t^{\lambda}}D^{\alpha}z(t)$$

with the conditions:

$$z(0)+z(1)=0,\ D^{\beta}z(0)+D^{\beta}z(1)=0,\ D^{\gamma}z(0)+D^{\gamma}z(1)=0 \quad (2.2)$$

is:

$$z(t)=\int_0^t\frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}\int_0^s\frac{(s-u)^{\beta-1}}{\Gamma(\beta)}\int_0^u\frac{(u-v)^{\alpha-1}}{\Gamma(\alpha)}L_1(v)dvduds$$

$$-\frac{1}{2}\int_0^1 \frac{(1-s)^{\gamma-1}}{\Gamma(\gamma)}\int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)}\int_0^u \frac{(u-v)^{\alpha-1}}{\Gamma(\alpha)}L_1(v)dvduds$$

$$-\left[\frac{B_1 t^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)}+\frac{B_3 t^{\gamma}}{\Gamma(\gamma+1)}+B_5\right]\int_0^1 \frac{(1-s)^{\gamma-1}}{\Gamma(\gamma)}\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}L_1(u)duds$$

$$-\left[\frac{B_2 t^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)}+\frac{B_4 t^{\gamma}}{\Gamma(\gamma+1)}+B_6\right]\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}L_1(u)duds,$$

such that

$$B_1=\frac{2\Gamma(\gamma+1)\Gamma(\beta+1)\Gamma(\gamma-\beta+1)}{2\Gamma(\gamma-\beta+1)\Gamma(\beta+1)-\Gamma(\gamma+1)}, \qquad B_2=\frac{\Gamma(\beta+1)}{\Gamma(\gamma+1)-2\Gamma(\gamma-\beta+1)\Gamma(\beta+1)}$$

$$B_3=\frac{\Gamma(\gamma-\beta+1)\Gamma(\gamma+1)}{\Gamma(\gamma+1)-2\Gamma(\gamma-\beta+1)\Gamma(\beta+1)}, \qquad B_4=\frac{2\Gamma(\gamma-\beta+1)\Gamma(\beta+1)-\Gamma(\gamma+1)+1}{2[2\Gamma(\gamma-\beta+1)\Gamma(\beta+1)-\Gamma(\gamma+1)]}$$

$$B_5=\frac{\Gamma(\gamma+\beta+1)\Gamma(\gamma-\beta+1)-2\Gamma(\gamma+1)\Gamma(\gamma-\beta+1)\Gamma(\beta+1)}{\Gamma(\gamma+\beta+1)[2\Gamma(\gamma-\beta+1)\Gamma(\beta+1)-\Gamma(\gamma+1)]}$$

$$B_6=\frac{\Gamma(\beta+1)[\Gamma(\gamma+1)-2\Gamma(\beta+\gamma+1)\Gamma(\gamma-\beta+1)]-\Gamma(\gamma+1)\Gamma(\beta+\gamma+1)-\Gamma(\beta+\gamma+1)}{2\Gamma(\gamma+\beta+1)\Gamma(\gamma+1)[2\Gamma(\gamma-\beta+1)\Gamma(\beta+1)-\Gamma(\gamma+1)]}$$

Proof: In view of Lemma 1, relation (2.1) is equivalent to:

$$z(t)=\int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}\int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)}\int_0^u \frac{(u-v)^{\alpha-1}}{\Gamma(\alpha)}L_1(v)dvduds$$
$$.-c_0\frac{t^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)}-c_1\frac{t^{\gamma}}{\Gamma(\gamma+1)}-c_2.$$

By (2.2) and taking

$$\Delta_{\alpha,\beta}[f(t)]=\int_0^1 (1-s)^{\beta-1}\int_0^s (s-u)^{\alpha-1}f(u)duds$$

and

$$\Delta_{\alpha,\beta,\gamma}[f(t)]=\int_0^1 (1-s)^{\gamma-1}\int_0^s (s-u)^{\beta-1}\int_0^u (u-v)^{\alpha-1}f'(u)dvduds,$$

we get:

$$c_0=\frac{B_1}{\Gamma(\alpha)\Gamma(\gamma)}\Delta_{\alpha,\gamma}[L_1(t)]+\frac{B_2}{\Gamma(\alpha)\Gamma(\beta)}\Delta_{\alpha,\beta}[L_1(t)],$$

$$c_1=\frac{B_3}{\Gamma(\alpha)\Gamma(\gamma)}\Delta_{\alpha,\gamma}[L_1(t)]+\frac{B_4}{\Gamma(\alpha)\Gamma(\beta)}\Delta_{\alpha,\beta}[L_1(t)],$$

$$c_2=\frac{1}{2\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\Delta_{\alpha,\beta,\gamma}[L_1(t)]+\frac{B_5}{\Gamma(\alpha)\Gamma(\gamma)}\Delta_{\alpha,\gamma}[L_1(t)]+\frac{B_6}{\Gamma(\alpha)\Gamma(\beta)}\Delta_{\alpha,\beta}[L_1(t)]$$

Therefore,

$$z(t)=\int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}\int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)}\int_0^u \frac{(u-v)^{\alpha-1}}{\Gamma(\alpha)}L_1(v)dvduds$$

$$-\frac{1}{2}\Delta_{\alpha,\beta,\gamma}[L_1(t)]$$
$$-\left[\frac{B_1 t^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)}+\frac{B_3 t^{\gamma}}{\Gamma(\gamma+1)}+B_5\right]\Delta_{\alpha,\gamma}[L_1(t)]$$
$$-\left[\frac{B_2 t^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)}+\frac{B_4 t^{\gamma}}{\Gamma(\gamma+1)}+B_6\right]\Delta_{\alpha,\beta}[L_1(t)]$$

3. Results

We begin by introducing the space

$$Z:=\{z, D^{\alpha}z, D^{\gamma}z \in C(I,\mathbb{R})\},$$

with:

$$\|y\|_Z = \max\{\|y\|_\infty, \|D^{\alpha}y\|_\infty, \|D^{\gamma}y\|_\infty\},$$

$$\text{where,} \qquad \|x\|_\infty = \sup_{t\in I}|x(t)|.$$

And we define the nonlinear operator $\Phi: Z \to Z$:

$$\Phi y(t) = \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}\int_0^s \frac{(s-u)^{\beta-1}}{\Gamma(\beta)}\int_0^u \frac{(u-v)^{\alpha-1}}{\Gamma(\alpha)}[L_1(v)]dvduds$$
$$-\frac{1}{2\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\Delta_{\alpha,\beta,\gamma}[L_1(t)]-\left[\frac{B_1 t^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)}+\frac{B_3 t^{\gamma}}{\Gamma(\gamma+1)}+B_5\right]\Delta_{\alpha,\gamma}[L_1(t)]$$
$$-\left[\frac{B_2 t^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)}+\frac{B_4 t^{\gamma}}{\Gamma(\gamma+1)}+B_6\right]\Delta_{\alpha,\beta} \tag{3.1}$$

We impose the conditions:

(H1): For $i = 1,2,3,4,5,6$ There are positive constants Wi, such that for all $(a_1, a_2, a_3), (b_1, b_2, b_3) \in Z$, such that:

- $|F(t, a_1, a_2, a_3) - F(t, b_1, b_2, b_3))| \le W_1|a_1 - b_1| + W_2|a_2 - b_2| + W_3|a_3 - b_3|$
- $|G(t, a_1, a_2) - G(t, b_1, b_2))| \le W_4|a_1 - b_1| + W_5|a_2 - b_2|$.
- $|H(t, a_1) - H(t, b_1)| \le W_6|a_1 - b_1|$.

(H2): The functions F, G, H and L are continuous.

(H3): For $i = 1,2,3$ there exist constants $E_i > 0$,such that for each $t \in I$ and all $a_1, a_2, a_3 \in \mathbb{R}$, we have:

$$|F(t, a_1, a_2, a_3)| \le E_1, \quad |G(t, a_1, a_2)| \le E_2, \quad |H(t, a_1)| \le E_3.$$

(H4): We put $\|L\|_\infty := E_L$.

Then,we set the following quantities:

$$T_1 = \frac{1}{\Gamma(\alpha+1)}, \quad T_2 = \frac{1}{\Gamma(\beta+1)} \quad T_3 = \frac{1}{\Gamma(\gamma+1)}, \quad T_4 = \frac{1}{\Gamma(\alpha+\beta+1)},$$
$$T_5 = \frac{1}{\Gamma(\alpha+\gamma+1)}, \quad T_6 = \frac{1}{\Gamma(\beta+\gamma+1)}, \quad T_7 = \frac{1}{\Gamma(\alpha+\beta+\gamma+1)}, \quad T_8 = \frac{1}{\Gamma(\beta+\gamma-\alpha+1)},$$
$$T_9 = \frac{1}{\Gamma(\beta-\gamma+1)}, T_9 = \frac{1}{\Gamma(\gamma-\alpha+1)}, \quad T_{10} = \frac{1}{\Gamma(\gamma-\alpha+1)}, T_{11} = \frac{1}{\Gamma(\alpha+\beta-\gamma+1)},$$

$$T_{12}=\frac{1}{\Gamma(\beta+\gamma+p)},\quad T_{13}=\frac{1}{\Gamma(\gamma+p+1)},\quad T_{14}=\frac{1}{\Gamma(\beta+p+1)},\quad T_{15}=\frac{1}{\Gamma(\alpha+\beta+\gamma-\lambda+1)},$$
$$T_{16}=\frac{1}{\Gamma(\alpha+\beta-\lambda+1)},\quad T_{17}=\frac{1}{\Gamma(\alpha+\gamma-\lambda+1)},\quad T_{18}=\frac{1}{\Gamma(\beta+\gamma-\lambda+1)},\quad T_{19}=\frac{1}{\Gamma(\gamma-\lambda+1)},$$
$$T_{20}=\frac{1}{\Gamma(\beta-\lambda+1)},\quad T_{21}=\frac{1}{\Gamma(\alpha-\gamma+1)},\quad T_{22}=\frac{1}{\Gamma(\alpha+\beta-2\gamma+1)},T_{23}=\frac{1}{\Gamma(\alpha+\beta+p+1)},$$
$$T_{24}=\frac{1}{\Gamma(\alpha+p+1)},\quad T_{25}=\frac{1}{\Gamma(\alpha+\beta-\gamma+p+1)},\quad T_{26}=\frac{1}{\Gamma(\beta-2\gamma+1)},\quad T_{27}=\frac{1}{\Gamma(\alpha-\lambda+1)},$$
$$T_{28}=\frac{1}{\Gamma(\alpha+\beta+\gamma+p)},\quad T_{29}=\frac{1}{\Gamma(\alpha+\gamma+p+1)},\quad T_{30}=\frac{1}{\Gamma(\alpha+\beta-\gamma)},\quad T_{31}=\frac{1}{\Gamma\alpha+\beta-\gamma-\lambda+1)}$$

and

$$S_1=|B_1|T_6+|B_3|T_3+|B_5|,\quad S_2=|B_2|T_6+|B_4|T_3+|B_6|,$$
$$S_3=|B_1|T_8+|B_3|T_9,\quad S_4=|B_1|T_2+|B_3|,$$
$$S_5=|B_2|T_2+|B_4|,\quad S_6=|B_2|T_8+|B_4|T_9.$$

also

$$M_1:=(a_1W_1+a_2W_4+a_3W_6)\left[\tfrac{3}{2}T_7+T_5S_1+T_4S_2\right]+$$
$$(a_1W_2+a_2W_5)\left[\tfrac{3}{2}T_4+T_1S_1+T_{11}S_2\right]$$
$$+(a_1W_3)\left[\tfrac{3}{2}T_{12}+T_{13}S_3+T_{14}S_2\right]+$$
$$|k|\Gamma(1-\lambda)\left[\tfrac{3}{2}T_{15}+T_{17}S_1+T_{16}S_2\right],$$

$$M_2:=(a_1W_1+a_2W_4+a_3W_6)\left[\tfrac{3}{2}T_6+T_3S_3+T_2S_2\right]+$$
$$(a_1W_2+a_2W_5)\left[\tfrac{3}{2}T_2+S_3+T_{17}S_2\right]$$
$$+(a_1W_3)\left[\tfrac{3}{2}T_{12}+T_{13}S_3+T_{14}S_2\right]+$$
$$|k|\Gamma(1-\lambda)\left[\tfrac{3}{2}T_{18}+T_{19}S_3+T_{20}S_2\right],$$

$$M_3:=(a_1W_1+a_3W_6)\left[\tfrac{3}{2}T_4+T_1S_4+T_{11}S_5\right]+$$
$$(a_1W_2)\left[\tfrac{3}{2}T_{11}+T_{21}S_4+T_{22}S_5\right]$$
$$+(a_1W_3)\left[\tfrac{3}{2}T_{23}+T_{24}S_4+T_{25}S_5\right]$$
$$+(a_2W_4)\left[\tfrac{3}{2}T_4+T_1S_4+T_9S_5\right]$$
$$+(a_2W_5)\left[\tfrac{3}{2}T_{11}+T_{21}S_4+T_{26}S_5\right]$$
$$+|k|\Gamma(1-\lambda)\left[\tfrac{3}{2}T_{16}+T_{27}S_4+T_{16}S_5.\right]$$

$$M:=max(M_1,M_2,M_3)$$

3.1 *Application of BCP of Banach*

We apply BCP, the contraction principle of Banach to obtain the existence of a unique solution of our problem. So,we prove the following theorem:

Theorem 1: *Assume that (H1) holds and* $0 < M < 1$.

Then, there will be a unique solution of (1.3) on I.

Proof A: Let $x, y \in Z$, Thanks to (H1), we have:

$$\|\Phi y - \Phi x\|_\infty \leq (a_1W_1 + a_2W_4 + a_3W_6)\left(\frac{3}{2}T_7 + T_5S_1 + T_4S_2\right)\|y - x\|_Z$$
$$+(a_1W_2 + a_2W_5)\left(\frac{3}{2}T_4 + T_1S_1 + T_{11}S_2\right)\|y - x\|_Z$$
$$+(a_1W_3)\left(\frac{3}{2}T_{28} + T_{29}S_1 + T_{23}S_2\right)\|y - x\|_Z$$
$$+|k|\Gamma(1-\lambda)\left(\frac{3}{2}T_{15} + T_{17}S_1 + T_{16}S_2\right)\|y - x\|_Z.$$

Consequently:

$$\|\Phi y - \Phi x\|_\infty \leq M_1\|y - x\|_Z.$$

B: Let $x, y \in Z$, then we have:

$$\|D^\alpha\Phi y - D^\alpha\Phi x\|_\infty \leq (a_1W_1 + a_2W_4 + a_3W_6)\left(\frac{3}{2}T_6 + T_3S_3 + T_2S_2\right)\|y - x\|_Z$$
$$+(a_1W_2 + a_2W_5)\left(\frac{3}{2}T_2 + S_3 + T_9S_2\right)\|y - x\|_Z$$
$$+(a_1W_3)\left(\frac{3}{2}T_{12} + T_{13}S_3 + T_{14}S_2\right)\|y - x\|_Z$$
$$+|k|\Gamma(1-\lambda)\left(\frac{3}{2}T_{18} + T_{19}S_3 + T_{20}S_2\right)\|y - x\|_Z.$$

Thus,

$$\|D^\alpha\Phi y - D^\alpha\Phi x\|_\infty \leq M_2\|y - x\|_Z.$$

C: Also, if $x, y \in Z$, then we have:

$$\|D^\gamma\Phi y - D^\gamma\Phi x\|_\infty \leq (a_1W_1 + a_3W_6)\left(\frac{3}{2}T_4 + T_1S_4 + T_{11}S_5\right)\|y - x\|_Z$$
$$+(a_1W_2)\left(\frac{3}{2}T_{11} + T_{21}S_4 + T_{22}S_5\right)\|y - x\|_Z$$
$$+(a_1W_3)\left(\frac{3}{2}T_{23} + T_{24}S_4 + T_{25}S_5\right)\|y - x\|_Z$$
$$+(a_2W_4)\left(\frac{3}{2}T_4 + T_1S_4 + T_9S_5\right)\|y - x\|_Z$$
$$+(a_2W_5)\left(\frac{3}{2}T_{16} + T_{27}S_4 + T_{26}S_5\right)\|y - x\|_Z$$
$$+|k|\Gamma(1-\lambda)\left(\frac{3}{2}T_{16} + T_{27}S_4 + T_{16}S_5\right)\|y - x\|_Z.$$

Therefore,

$$\|D^\gamma\Phi y - D^\gamma\Phi x\|_\infty \leq M_3\|y - x\|_Z.$$

So, we can conclude that

$$\|\Phi x - \Phi y\|_Z \leq M\|x - y\|_Z.$$

According of BCP of Banach, Φ has a unique fixed point. Then (1.3). has a unique solution

3.2 *Application of Schaefer Fixed Point Theorem*

We apply the Schaefer fixed point theorem to show that the solution set of (1.3) is not empty. So,we prove the following theorem:

Theorem 2: *If* $(H2), (H3)$ *and* $(H4)$ *are verified. Then, the solution set of (1.3) is not empty.*

Proof: To apply Schaefer's fixed point theorem, we proceed as follows:

Step 1: It is clear that Φ is a continuous operator on Z:

Step 2: Suppose that $r > 0$ and consider

$$Br := \{x \in Z; \|x\|_Z \leq r\}$$

Let $y \in Br$, we can observe that:

$$\|\Phi y\|_\infty \leq (E_L + a_1E_1 + a_2E_2 + a_3E_3)\left(\frac{3}{2}T_7 + T_5S_1 + T_4S_2\right)$$
$$+|k|r\Gamma(1-\lambda)\left(\tfrac{3}{2}T_{15} + T_{17}S_1 + T_{16}S_2\right) < +\infty \quad (3.2)$$

and we have

$$\|D^\alpha \Phi y\|_\infty \leq (E_L + a_1E_1 + a_2E_2 + a_3E_3)\left(\tfrac{3}{2}T_6 + T_3S_3 + T_2S_6\right) +$$
$$|k|r\Gamma(1-\lambda)\left(\tfrac{3}{2}T_{18} + T_{19}S_3 + T_9S_6\right) < +\infty. \quad (3.3)$$

On the other hand, one can prove that:

$$\|D^\gamma \Phi y\|_\infty \leq (E_L + a_1E_1 + a_2E_2 + a_3E_3)\left(\tfrac{3}{2}T_4 + T_1S_4 + T_{30}S_5\right) +$$
$$|k|r\Gamma(1-\lambda)\left(\tfrac{3}{2}T_{16} + T_{27}S_4 + T_{31}S_5\right) < +\infty. \quad (3.4)$$

From (3.2), (3.3) and (3.4), we have $\|\Phi y\|_Z < +\infty$.

Hence, the operator Φ is uniformly bounded on B_r.

Step 3: Let B_r be any bounded set of Z. So by considering $y \in Br$, we can state that for each $t_1, t_2 \in [0,1], t_1 \neq t_2$,

$$|\Phi y(t_2) - \Phi y(t_1)| \leq \frac{(E_L + a_1E_1 + a_2E_2 + a_3E_3)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}$$
$$\left(\begin{array}{l}\int_0^{t_1}(t_1-s)^{\gamma-1} - (t_2-s)^{\gamma-1}\int_0^s (s-u)^{\beta-1}\int_0^u (u-v)^{\alpha-1}dvduds \\ \\ +\int_{t_1}^{t_2}(t_2-s)^{\gamma-1}\int_0^s (s-u)^{\beta-1}\int_0^u (u-v)^{\alpha-1}dvduds\end{array}\right)$$

$$+\frac{|k|}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}\begin{pmatrix}\int_0^{t_1}(t_1-s)^{\gamma-1}-(t_2-s)^{\gamma-1}\int_0^s(s-u)^{\beta-1}\int_0^u(u-v)^{\alpha-1}dvduds\\ \int_{t_1}^{t_2}(t_2-s)^{\gamma-1}\int_0^s(s-u)^{\beta-1}\int_0^u(u-v)^{\alpha-1}dvduds\end{pmatrix}$$

$$+\begin{pmatrix}\frac{|B_1|\left|t_1^{\beta+\gamma}-t_2^{\beta+\gamma}\right|}{\Gamma(\beta+\gamma+1)}\\ +\frac{|B_3||t_1^{\gamma}-t_2^{\gamma}|}{\Gamma(\gamma+1)}\end{pmatrix}\begin{pmatrix}\frac{(E_L+a_1E_1+a_2E_2+a_3E_3)}{\Gamma(\alpha)\Gamma(\gamma)}\int_0^1(1-s)^{\gamma-1}\int_0^s(s-u)^{\alpha-1}duds\\ \frac{|k|}{\Gamma(\alpha)\Gamma(\gamma)}\int_0^1(1-s)^{\gamma-1}\int_0^s(s-u)^{\alpha-1}\frac{|D^{\alpha}y(u)|}{u^{\lambda}}duds.\end{pmatrix} \tag{3.5}$$

So we notice that:

$$|D^{\alpha}\Phi y(t_2)-D^{\alpha}\Phi y(t_1)|\leq\frac{(E_L+a_1E_1+a_2E_2+a_3E_3)}{\Gamma(\beta)\Gamma(\gamma)}\begin{pmatrix}\int_0^{t_1}(t_1-s)^{\gamma-1}-(t_2-s)^{\gamma-1}\int_0^s(s-u)^{\beta-1}duds\\ +\int_{t_1}^{t_2}(t_2-s)^{\gamma-1}\int_0^s(s-u)^{\beta-1}duds\end{pmatrix}$$

$$+\frac{|k|}{\Gamma(\beta)\Gamma(\gamma)}\begin{pmatrix}\int_0^{t_1}(t_1-s)^{\gamma-1}-(t_2-s)^{\gamma-1}\int_0^s(s-u)^{\beta-1}\frac{|D^{\alpha}y(u)|}{u^{\lambda}}duds\\ +\int_{t_1}^{t_2}(t_2-s)^{\gamma-1}\int_0^s(s-u)^{\beta-1}\frac{|D^{\alpha}y(u)|}{u^{\lambda}}duds\end{pmatrix}$$

$$+\left(\frac{|B_1|}{\Gamma(\beta+\gamma-\alpha+1)}\left|t_1^{\beta+\gamma-\alpha}-t_2^{\beta+\gamma-\alpha}\right|+\frac{|B_3|}{\Gamma(\gamma-\alpha+1)}\left|t_1^{\gamma-\alpha}-t_2^{\gamma-\alpha}\right|\right)$$
$$\times\frac{(E_L+a_1E_1+a_2E_2+a_3E_3)}{\Gamma(\gamma)}\int_0^1(1-s)^{\gamma-1}ds+\frac{|k|}{\Gamma(\gamma)}\int_0^1(1-s)^{\gamma-1}\frac{|D^{\alpha}y(s)|}{s^{\lambda}}ds$$

$$+\left(\frac{|B_2|}{\Gamma(\beta+\gamma-\alpha+1)}\left|t_1^{\beta+\gamma-\alpha}-t_2^{\beta+\gamma-\alpha}\right|+\frac{|B_4|}{\Gamma(\gamma-\alpha+1)}\left|t_1^{\gamma-\alpha}-t_2^{\gamma-\alpha}\right|\right)$$
$$\times\frac{(E_L+a_1E_1+a_2E_2+a_3E_3)}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}ds+\frac{|k|}{\Gamma(\beta)}\int_0^1(1-s)^{\gamma-1}\frac{|D^{\alpha}y(s)|}{s^{\lambda}}ds. \tag{3.6}$$

On the other hand,we have:

$$|D^{\gamma}\Phi y(t_2)-D^{\gamma}\Phi y(t_1)|\leq\frac{(E_L+a_1E_1+a_2E_2+a_3E_3)}{\Gamma(\alpha)\Gamma(\beta)}\begin{pmatrix}\int_0^{t_1}(t_1-s)^{\beta-1}-(t_2-s)^{\beta-1}\int_0^s(s-u)^{\alpha-1}duds\\ +\int_{t_1}^{t_2}(t_2-s)^{\beta-1}\int_0^s(s-u)^{\alpha-1}duds\end{pmatrix}$$

$$+\frac{|k|}{\Gamma(\alpha)\Gamma(\beta)}\begin{pmatrix}\int_0^{t_1}(t_1-s)^{\beta-1}-(t_2-s)^{\beta-1}\int_0^s(s-u)^{\alpha-1}\frac{|D^{\alpha}y(u)|}{u^{\lambda}}duds\\ +\int_{t_1}^{t_2}(t_2-s)^{\beta-1}\int_0^s(s-u)^{\alpha-1}\frac{|D^{\alpha}y(u)|}{u^{\lambda}}duds\end{pmatrix}$$

$$+\left(\frac{|B_1|}{\Gamma(\beta+1)}\left|t_1^{\beta}-t_2^{\beta}\right|\right)\begin{pmatrix}(E_L+a_1E_1+a_2E_2+a_3E_3)\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds\\ \\ +|k|\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{|D^{\alpha}y(s)|}{s^{\lambda}}ds\end{pmatrix}$$

$$+\left(\frac{|B_2|}{\Gamma(\beta+1)}\left|t_1^{\beta}-t_2^{\beta}\right|\right)\begin{pmatrix}\frac{(E_L+a_1E_1+a_2E_2+a_3E_3)}{\Gamma(\alpha)\Gamma(\beta-\gamma)}\int_0^1(1-s)^{\beta-\gamma-1}\int_0^s(s-u)^{\alpha-1}duds\\ \\ +\frac{|k|}{\Gamma(\alpha)\Gamma(\beta-\gamma)}\int_0^1(1-s)^{\beta-\gamma-1}\int_0^s(s-u)^{\alpha-1}\frac{|D^{\alpha}y(u)|}{u^{\lambda}}duds.\end{pmatrix}$$

(3.7)

We observe that in (3.5), (3.6) and (3.7) the right hand sides tend to zero, if $t_1 \to t_2$.

Therefore, taking into account step1, step2 and to step3 and thanks to Arzela-Ascoli theorem, we notice that Φ is a completely continuous operator.

Step 4: Let the set $V := \{x \in Z : x = \eta\Phi x, 0 < \eta < 1\}$ and let $y \in V$ and for some $0 < \eta < 1$, then $y = \eta\Phi(y)$. So, we have:

$$\|y\|_{\infty} \leq \eta\left(\leq (E_L + a_1E_1 + a_2E_2 + a_3E_3)\left(\frac{3}{2}T_7 + T_5S_1 + T_4S_2\right) + |k|r\Gamma(1-\lambda)\left(\frac{3}{2}T_{15} + T_{17}S_1 + T_{16}S_2\right).\right)$$

In the same manner, we can prove that:

$$\|D^{\alpha}\Phi y\|_{\infty} \leq \eta\left(\|D^{\alpha}\Phi y\|_{\infty} \leq (E_L + a_1E_1 + a_2E_2 + a_3E_3)\left(\frac{3}{2}T_6 + T_3S_3 + T_2S_6\right) + |k|r\Gamma(1-\lambda)\left(\frac{3}{2}T_{18} + T_{19}S_3 + T_9S_6\right).\right)$$

Also, we have:

$$\|D^{\gamma}\Phi y\|_{\infty} \leq$$
$$\eta\begin{pmatrix}\|D^{\gamma}\Phi y\|_{\infty} \leq (E_L + a_1E_1 + a_2E_2 + a_3E_3)\left(\frac{3/2}{\Gamma(\alpha+\beta+1)} + \frac{S_4}{\Gamma(\alpha+1)} + \frac{S_5}{\Gamma(\alpha+\beta-\gamma)}\right) + \\ |k|r\Gamma(1-\lambda)\left[\frac{3/2}{\Gamma(\alpha+\beta-\lambda+1)} + \frac{S_4}{\Gamma(\alpha-\lambda+1)} + \frac{S_5}{\Gamma\alpha+\beta-\gamma-\lambda+1)}.\right]\end{pmatrix}$$

From (3.2), (3.3) and (3.4), we can deduce that $\|y\|_Z < \infty$. This shows that V is bounded.

It follows that Φ has at least one fixed point.

Example 1: In this example we have the following parameters and initial conditions:

$$\begin{aligned}\alpha &= 0.97, \quad \beta = 0.98, \quad \gamma = 0.007,\\ k &= 0.06, \quad p = 2.1, \quad \lambda = 0.5,\\ a_1 &= 0.003, \quad a_2 = 0.002, \quad a_3 = 0.001.\end{aligned}$$

Bounded Nonlinear Functions:

$$\begin{aligned} F(t,x_1,x_2,x_3) &= \frac{\sin(1+t)}{\sqrt{12+t^2}}+\frac{1}{8}x_1+\frac{5}{1000}x_2+\frac{91}{100}x_3, \\ G(t,x_1,x_2) &= \frac{1}{100+t^5}+\frac{\sqrt{\pi-1}}{7}x_1+\frac{2}{3}x_2, \\ H(t,x_1) &= \frac{\cos(1+t^2)}{\sqrt{1+t^3}}+\frac{3}{8}x_1, L(t)=\frac{19}{8}\exp(-t)\cot(t). \end{aligned}$$

Example 2: Let us now, the second parameters and initial conditions:

$$\alpha = 0.96,\quad \beta = 0.97,\quad \gamma = 0.011,$$
$$k = 0.07,\quad p = 1.9,\quad \lambda = 0.6,$$
$$a_1 = 0.001,\quad a_2 = 0.0003,\quad a_3 = 0.0002.$$

Bounded Nonlinear Functions:

$$F(t,x_1,x_2,x_3) = \frac{\sin(x_1)-\cos(2+x_2x_3)}{2+e^{-t^2}},$$
$$G(t,x_1,x_2) = \frac{\cos(x_1+x_2)}{1+t},$$
$$H(t,x_1) = \frac{e^{-t}}{4+tx_1},\quad L(t) = \frac{3t+2}{5}.$$

These two examples satisfy the conditions of the above existence theorem.

4. ADM Methodology

To use the ADM method to the problem (1.4):

$$D^{\alpha}D^{\beta}D^{\gamma}z(t)+\frac{k}{t^{\lambda}}D^{\alpha}z(t)+a_1F(t,z(t),D^{\gamma}z(t),J^{p}z(t))+ a_2G(t,z(t),D^{\gamma}z(t))+a_3H(t,z(t))=f(t),$$

under some initial conditions that will be specified later.

We consider the following problem:

$$L\big(y(t)\big)+R\big(y(t)\big)+N\big(y(t)\big)=f(t), \tag{4.1}$$

where:

- L: the linear fractional differential operator of higher order which is easily invertible,
- N: the nonlinear operator,
- R: the linear operator of order less than L,
- f: a given function,
- y: the function to be determined.

Applying L^{-1} to the members of Eq.(4.1) we find:

$$y(t)-\psi_0 = L^{-1}(f)-L^{-1}\big(R(y)\big)-L^{-1}\big(N(y)\big), \tag{4.2}$$

where ψ_0 is presented from the initial conditions or from the boundary conditions and satisfies $L\psi_0 = 0$.

The ADM method admits the decomposition:

$$y(t) = \sum_{n=0}^{\infty} y_n(t), \tag{4.3}$$

and the nonlinear term N will be decomposed by:

$$N(y) = \sum_{n=0}^{\infty} A_n, \tag{4.4}$$

where A_n are the Adomian polynomials [27] given by

$$A_n = \frac{1}{n!}\frac{d^n}{ds^n}\left[N\left(\sum_{i=0}^{\infty} s^i y_i\right)\right]_{s=0}, \quad n = 0,1,2,\ldots \tag{4.5}$$

The first three term of A_n are

$$\begin{aligned} A_0 &= N(y_0) \\ A_1 &= y_1 N'(y_0) \\ A_2 &= \frac{1}{2} y_1^2 N''(y_0) + y_2 N'(y_0). \end{aligned} \tag{4.6}$$

Substituting the decomposition series equations (4.3) and (4.4) into equation (4.2), gives:

$$\sum_{n=0}^{\infty} y_n = \psi_0 + L^{-1}(f) - L^{-1}\sum_{n=0}^{\infty} R(y_n) - L^{-1}\sum_{n=0}^{\infty} A_n, \tag{4.7}$$

we can obtain the solution algorithm as follows:

$$\begin{aligned} y_0(t) &= \psi_0 + L^{-1}(f) \\ y_{n+1}(t) &= -L^{-1}\big(R(y_n)\big) - L^{-1}(A_n), \quad n = 0,1,2,\ldots \end{aligned} \tag{4.8}$$

4.1 *The Studied Problem*

Consider the following equation:

$$\begin{aligned} D^{\alpha}D^{\beta}D^{\gamma}y(t) + \frac{k}{t^{\lambda}}D^{\alpha}y(t) + a_1F(t, y(t), D^{\gamma}y(t), J^{p}y(t)) + \\ a_2G(t, y(t), D^{\gamma}y(t)) + a_3H(t, y(t)) = f(t) \\ 0 < \lambda < 1, k > 0, p > 0 \quad 0 < \alpha \le 1, 0 \le \beta \le 1, 0 < \gamma \le 1, \\ 1 < \alpha + \beta + \gamma \le 2, t \in I, \end{aligned} \tag{4.9}$$

with the conditions

$$y(0) = a, y'(0) = b, y''(0) = c, \ a, b, c \in \mathbb{R}. \tag{4.10}$$

Applying the inverse of $L = D^{\alpha}D^{\beta}D^{\gamma}$ in Eq. (4.9) and using the condition (4.10), yields:

$$\begin{aligned} y(t) = y(0) + ty'(0) + \frac{t^2}{2}y''(0) + J^{\gamma}(D^{\gamma}y(0)) + J^{\gamma+\beta}\big(D^{\beta}D^{\gamma}y(0)\big) + \\ J^{\gamma+\beta+\alpha}f(t) - J^{\gamma+\beta+\alpha}\left(\frac{k}{t^{\lambda}}D^{\alpha}y(t)\right) - J^{\gamma+\beta+\alpha}\big(R(y(t))\big) - J^{\gamma+\beta+\alpha}\big(N(y(t))\big), \end{aligned}$$

where N is the nonlinear operator of $a_1F + a_2G + a_3H$, R is a linear fractional differential operator of $a_1F + a_2G + a_3H$.

So, we get the following recursive formula:

$$y_0(x) = a + bt + \frac{c}{2}t^2 + J^{\gamma}(D^{\gamma}a) + J^{\gamma+\beta}\big(D^{\beta}D^{\gamma}a\big) + J^{\gamma+\beta+\alpha}f(t) = a +$$

$$bt + \frac{c}{2}t^2 + J^{\gamma+\beta+\alpha}f(t)$$

$$y_{n+1}(t) = -J^{\gamma+\beta+\alpha}\left(\frac{k}{t^{\lambda}}D^{\alpha}y(t)\right) - J^{\gamma+\beta+\alpha}\big(R(y(t))\big) - J^{\gamma+\beta+\alpha}\big(N(y(t))\big) \tag{4.11}$$

4.2 *Illustrative examples*

We present two examples

4.2.1 Example 1

We consider the following first problem:

$$D^{\alpha}D^{\beta}D^{\gamma}y(t) + \frac{2}{t^{\lambda}}D^{\alpha}y(t) + 2y(t)D^{\gamma}y(t) - y^2(t) = 0, t \in [0,1] \tag{4.12}$$

with

$$y(0) = 1, y'(0) = 0, y'(0) = 0. \tag{4.13}$$

Then

$F(t, y(t), D^{\gamma}y(t), J^{p}y(t)) = y(t)D^{\gamma}y(t)$ $G(t, y(t), D^{\gamma}y(t)) = 0$

$H(t, y(t)) = y^2(t)$ $f(t) = 0.$

Let

$$y = \sum_{n=0}^{\infty} y_n, \quad yD^{\gamma}y = \sum_{n=0}^{\infty} A_n \quad , \quad y^2 = \sum_{n=0}^{\infty} B_n.$$

The A_n and B_n are given as:

$A_0 = y_0D^{\gamma}y_0,$ $B_0 = y_0^2,$

$A_1 = y_0D^{\gamma}y_1 + y_1D^{\gamma}y_0,$ $B_1 = 2y_0y_1,$

$A_2 = y_2D^{\gamma}y_0 + y_1D^{\gamma}y_1 + y_0D^{\gamma}y_2,$ $B_2 = 2y_0y_2 + y_1^2,$

$A_3 = y_3D^{\gamma}y_0 + y_2D^{\gamma}y_1 + y_1D^{\gamma}y_2 + y_0D^{\gamma}y_3,$ $B_3 = 2y_0y_3 + 2y_1y_2,$

. .

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. .

To estimate the numerical solution of Eq.(4.12), we get the Adomian solution.

$$y_0(t) = 1 + J^{\gamma}(D^{\gamma}1) + J^{\gamma+\beta}\left(D^{\beta}D^{\gamma}1\right)$$

$$y_{n+1}(t) = -2J^{\gamma+\beta+\alpha}\left(\frac{1}{t^{\lambda}}D^{\alpha}y_n(t)\right) - 2J^{\gamma+\beta+\alpha}(A_n) + J^{\gamma+\beta+\alpha}(B_n)$$

The first terms of the series are given by:

$$y_0(t) = 1 + J^{\gamma}(D^{\gamma}1) + J^{\gamma+\beta}\left(D^{\beta}D^{\gamma}1\right) = 1,$$

$$y_1(t) = -2J^{\gamma+\beta+\alpha}\left(\frac{1}{t^{\lambda}}D^{\alpha}y_0(t)\right) - 2J^{\gamma+\beta+\alpha}(A_0) + J^{\gamma+\beta+\alpha}(B_0)$$

$$= -2J^{\gamma+\beta+\alpha}\left(\frac{1}{t^{\lambda}}D^{\alpha}y_0(t)\right) - 2J^{\gamma+\beta+\alpha}(y_0D^{\gamma}y_0) + J^{\gamma+\beta+\alpha}((y_0)^2)$$

$$= J^{\gamma+\beta+\alpha}(1) = \frac{1}{\Gamma(\gamma+\beta+\alpha+1)}t^{\gamma+\beta+\alpha},$$

$$y_2(t) = -2J^{\gamma+\beta+\alpha}\left(\frac{1}{t^{\lambda}}D^{\alpha}y_1(t)\right) - 2J^{\gamma+\beta+\alpha}(A_1) + J^{\gamma+\beta+\alpha}(B_1)$$
$$= -2J^{\gamma+\beta+\alpha}\left(\frac{1}{t^{\lambda}}D^{\alpha}y_1(t)\right) - 2J^{\gamma+\beta+\alpha}(y_0D^{\gamma}y_1 + y_1D^{\gamma}y_0) + J^{\gamma+\beta+\alpha}(2y_0y_1)$$
$$= -\frac{2\Gamma(\beta+\gamma-\lambda+1)}{\Gamma(\beta+\gamma+1)\Gamma(\alpha+2\beta+2\gamma-\lambda+1)}t^{2\gamma+2\beta+\alpha-\lambda} - \frac{2}{\Gamma(\gamma+2\beta+2\alpha+1)}t^{\gamma+2\beta+2\alpha+1} + \frac{2}{\Gamma(2\alpha+2\beta+2\gamma+1)}t^{2\gamma+2\beta+2\alpha},$$

$$y_3(t) = -2J^{\gamma+\beta+\alpha}\left(\frac{1}{t^{\lambda}}D^{\alpha}y_2(t)\right) - 2J^{\gamma+\beta+\alpha}(A_2) + J^{\gamma+\beta+\alpha}(B_2),$$
$$= -2J^{\gamma+\beta+\alpha}\left(\frac{1}{t^{\lambda}}D^{\alpha}y_2(t)\right) - 2J^{\gamma+\beta+\alpha}(y_2D^{\gamma}y_0 + y_1D^{\gamma}y_1 + y_0D^{\gamma}y_2) + J^{\gamma+\beta+\alpha}(2y_0y_2 + (y_1)^2)$$
$$= \frac{4\Gamma(\beta+\gamma-\lambda+1)\Gamma(2\gamma+2\beta-2\lambda+1)}{\Gamma(\beta+\gamma+1)\Gamma(2\gamma+2\beta-\lambda+1)\Gamma(3\gamma+3\beta+\alpha-2\lambda+1)}t^{3\gamma+3\beta+\alpha-2\lambda} +$$
$$\frac{4\Gamma(\gamma+2\beta+2\alpha+2)\Gamma(\gamma+2\beta+\alpha-\lambda+2)}{\Gamma(\gamma+2\beta+2\alpha+1)\Gamma(\gamma+2\beta+\alpha+2)\Gamma(2\gamma+3\beta+2\alpha-\lambda+2)}t^{2\gamma+3\beta+2\alpha-\lambda+1} -$$
$$\frac{4\Gamma(2\gamma+2\beta+2\alpha-\lambda)}{\Gamma(2\gamma+2\beta+\alpha+1)\Gamma(3\gamma+3\beta+3\alpha-\lambda)}t^{3\gamma+3\beta+3\alpha-\lambda} -$$
$$\frac{2}{\Gamma(2\gamma+3\beta+3\alpha+1)}\left(\frac{\Gamma(\gamma+2\beta+2\alpha+1)}{\Gamma(\beta+\alpha+1)\Gamma(\gamma+\beta+\alpha+1)} + 2\right)t^{2\gamma+3\beta+3\alpha} +$$
$$\frac{4\Gamma(\beta+\gamma-\lambda+1)}{\Gamma(\beta+\gamma+1)\Gamma(2\gamma+3\beta+2\alpha-\lambda+1)}t^{2\gamma+3\beta+2\alpha-\lambda} +$$
$$\frac{4\Gamma(\gamma+2\beta+2\alpha+2)}{\Gamma(\gamma+2\beta+2\alpha+1)\Gamma(\gamma+3\beta+3\alpha+2)}t^{\gamma+3\beta+3\alpha+1} -$$
$$\frac{4\Gamma(\beta+\gamma-\lambda+1)\Gamma(2\gamma+2\beta+\alpha-\lambda+1)}{\Gamma(\beta+\gamma+1)\Gamma(\alpha+2\beta+2\gamma-\lambda+1)\Gamma(3\gamma+3\beta+2\alpha-\lambda+1)}t^{3\gamma+3\beta+2\alpha-\lambda} -$$
$$\frac{4\Gamma(\gamma+2\beta+2\alpha+1)}{\Gamma(\gamma+2\beta+2\alpha+1)\Gamma(2\gamma+3\beta+3\alpha+1)}t^{2\gamma+3\beta+3\alpha+1}$$
$$+\left(\frac{4}{\Gamma(2\alpha+2\beta+2\gamma+1)} + \frac{1}{(\Gamma(\gamma+\beta+\alpha+1))^2}\right)\frac{\Gamma(2\gamma+2\beta+2\alpha+1)}{\Gamma(3\gamma+3\beta+3\alpha+1)}t^{3\gamma+3\beta+3\alpha}$$

Then a numerical solution of equation (4.12) is given in series form:

$$y(t) = 1 + \frac{1}{\Gamma(\gamma+\beta+\alpha+1)}t^{\gamma+\beta+\alpha} - \frac{2\Gamma(\beta+\gamma-\lambda+1)}{\Gamma(\beta+\gamma+1)\Gamma(\alpha+2\beta+2\gamma-\lambda+1)}t^{2\gamma+2\beta+\alpha-\lambda}$$
$$-\frac{2}{\Gamma(\gamma+2\beta+2\alpha+1)}t^{\gamma+2\beta+2\alpha+1} + \frac{2}{\Gamma(2\alpha+2\beta+2\gamma+1)}t^{2\gamma+2\beta+2\alpha} + \ldots \tag{4.14}$$

The figures Fig 1 (a) and Fig 1 (b) show the solution curve (4.14) for some particular values.

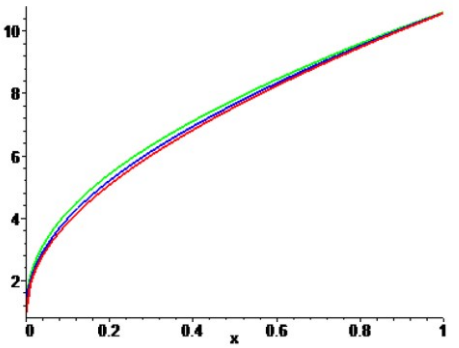


Fig 1(a)
Graph of solution (4.14) with

$\alpha = 1, \beta = \frac{1}{2}, \gamma = \frac{1}{4}, \lambda = 2$(red),

$\alpha = 0.98, \beta = \frac{1}{2}, \gamma = \frac{1}{4}, \lambda = 2$(blue),

$\alpha = 0.95, \beta = \frac{1}{2}, \gamma = \frac{1}{4}, \lambda = 2$(green).

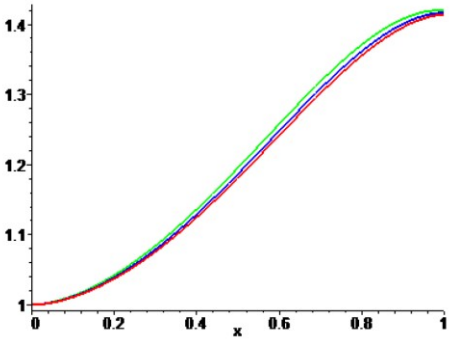


Fig 1(b)
Graph of solution (4.14) with

$\alpha = 1, \beta = \frac{1}{2}, \gamma = \frac{1}{4}, \lambda = -2$(red),

$\alpha = 0.98, \beta = \frac{1}{2}, \gamma = \frac{1}{4}, \lambda = -2$(blue),

$\alpha = 0.95, \beta = \frac{1}{2}, \gamma = \frac{1}{4}, \lambda = -2$(green).

4.2.2 Example 2

The second fractional differential problem:

$$D^{\alpha}D^{\beta}D^{\gamma}y(t) + \frac{1}{t^{\lambda}}D^{\alpha}y(t) - y(t) + 2J^{p}y(t) - y^{3}(t) = t, t \in [0,1] \quad (4.15)$$

with the conditions

$$y(0) = 0, y'(0) = 0, y'(0) = 0. \quad (4.16)$$

Let

$$y = \sum_{n=0}^{\infty} y_n,\ y^3 = \sum_{n=0}^{\infty} C_n.$$

The C_n are given as:

$$C_0 = y_0^3, C_1 = 3y_0^2 y_1, C_2 = 3y_0^2 y_2 + 3y_0 y_1^2, C_3 = 3y_0^2 y_3 + 6y_0 y_1 y_2 + y_1^3, \ldots\ldots.$$

To estimate the numerical solution of equation.(4.15), we get the Adomian solution.

$$y_0(t) = J^{\gamma+\beta+\alpha}(t)$$

$$y_{n+1}(t) = -J^{\gamma+\beta+\alpha}\left(\frac{1}{t^{\lambda}} D^{\alpha} y_n(t)\right) + J^{\gamma+\beta+\alpha}(y_n(t)) - 2J^{\gamma+\beta+\alpha+p}(y_n(t)) + J^{\gamma+\beta+\alpha}(C_n)$$

We introduce the first few terms of series:

$$y_0(t) = J^{\gamma+\beta+\alpha}(t) = \frac{1}{\Gamma(\gamma+\beta+\alpha+2)} t^{\gamma+\beta+\alpha+1},$$

$$y_1(t) = -J^{\gamma+\beta+\alpha}\left(\frac{1}{t^{\lambda}} D^{\alpha} y_0(t)\right) + J^{\gamma+\beta+\alpha}\left(y_0(t)\right) - 2J^{\gamma+\beta+\alpha+p}\left(y_0(t)\right) + J^{\gamma+\beta+\alpha}(C_0)$$

$$= -J^{\gamma+\beta+\alpha}\left(\frac{1}{t^{\lambda}} D^{\alpha} y_0(t)\right) + J^{\gamma+\beta+\alpha}(y_0(t)) - 2J^{\gamma+\beta+\alpha+p}(y_0(t)) + J^{\gamma+\beta+\alpha}(y_0^3)$$

$$= -\frac{\Gamma(\gamma+\beta+2-\lambda)}{\Gamma(\gamma+\beta+2)\Gamma(2\gamma+2\beta+\alpha+2-\lambda)} t^{2\gamma+2\beta+\alpha+1-\lambda} + \frac{1}{\Gamma(2\gamma+2\beta+2\alpha+2)} t^{2\gamma+2\beta+2\alpha+1} - \frac{2}{\Gamma(2\gamma+2\beta+2\alpha+p+2)} t^{2\gamma+2\beta+2\alpha+p+1} + \frac{\Gamma(3\gamma+3\beta+3\alpha+4)}{\left(\Gamma(\gamma+\beta+\alpha+2)\right)^3 \Gamma(4\gamma+4\beta+4\alpha+4)} t^{4\gamma+4\beta+4\alpha+3}$$

$$y_2(t) = -J^{\gamma+\beta+\alpha}\left(\frac{1}{t^{\lambda}} D^{\alpha} y_1(t)\right) + J^{\gamma+\beta+\alpha}(y_1(t)) - 2J^{\gamma+\beta+\alpha+p}(y_1(t)) + J^{\gamma+\beta+\alpha}(C_1)$$

$$= -J^{\gamma+\beta+\alpha}\left(\frac{1}{t^{\lambda}} D^{\alpha} y_1(t)\right) + J^{\gamma+\beta+\alpha}(y_1(t)) - 2J^{\gamma+\beta+\alpha+p}(y_1(t)) + 3J^{\gamma+\beta+\alpha}(y_0^2 y_1)$$

$$= \frac{\Gamma(\gamma+\beta+2-\lambda)\Gamma(2\gamma+2\beta+2-2\lambda)}{\Gamma(\gamma+\beta+2)\Gamma(2\gamma+2\beta+2-\lambda)\Gamma(3\gamma+3\beta+\alpha+2-2\lambda)} t^{3\gamma+3\beta+\alpha+1-2\lambda} - \frac{\Gamma(2\gamma+2\beta+\alpha+2-\lambda)}{\Gamma(2\gamma+2\beta+\alpha+2)\Gamma(3\gamma+3\beta+2\alpha+2-\lambda)} t^{3\gamma+3\beta+2\alpha+1-\lambda} + \frac{2\Gamma(2\gamma+2\beta+\alpha+p+2-\lambda)}{\Gamma(2\gamma+2\beta+\alpha+p+2)\Gamma(3\gamma+3\beta+2\alpha+p+2-\lambda)} t^{3\gamma+3\beta+2\alpha+p+1-\lambda} - \frac{\Gamma(3\gamma+3\beta+3\alpha+4)\Gamma(4\gamma+4\beta+3\alpha+4-\lambda)}{\Gamma(4\gamma+4\beta+3\alpha+4)\Gamma(5\gamma+5\beta+3\alpha+4-\lambda)\left(\Gamma(\gamma+\beta+\alpha+2)\right)^3} t^{5\gamma+5\beta+4\alpha+3-\lambda} - \frac{\Gamma(\gamma+\beta+2-\lambda)}{\Gamma(\gamma+\beta+2)\Gamma(3\gamma+3\beta+2\alpha+1-\lambda)} t^{3\gamma+3\beta+2\alpha+1-\lambda} + \frac{1}{\Gamma(3\gamma+3\beta+3\alpha+2)} t^{3\gamma+3\beta+3\alpha+1} - \frac{2}{\Gamma(3\gamma+3\beta+3\alpha+p+2)} t^{3\gamma+3\beta+3\alpha+p+1} + \frac{\Gamma(3\gamma+3\beta+3\alpha+4)}{\Gamma(5\gamma+5\beta+5\alpha+4)\left(\Gamma(\gamma+\beta+\alpha+2)\right)^3} t^{5\gamma+5\beta+5\alpha+3}$$

$$+\frac{2\Gamma(\gamma+\beta+2-\lambda)\Gamma(2\gamma+2\beta+\alpha+2-\lambda+p)}{\Gamma(\gamma+\beta+2)\Gamma(2\gamma+2\beta+\alpha+2-\lambda)\Gamma(3\gamma+3\beta+2\alpha+1-\lambda+p)}t^{3\gamma+3\beta+2\alpha+1-\lambda+p}-$$
$$\frac{2\Gamma(2\gamma+2\beta+2\alpha+2+p)}{\Gamma(2\gamma+2\beta+2\alpha+2)\Gamma(3\gamma+3\beta+3\alpha+2+p)}t^{3\gamma+3\beta+3\alpha+p+1}$$
$$+\frac{4\Gamma(2\gamma+2\beta+2\alpha+2p+2)}{\Gamma(2\gamma+2\beta+2\alpha+p+2)\Gamma(3\gamma+3\beta+3\alpha+2p+2)}t^{3\gamma+3\beta+3\alpha+2p+1}-$$
$$\frac{2\Gamma(3\gamma+3\beta+3\alpha+4)\Gamma(4\gamma+4\beta+4\alpha+p+4)}{(\Gamma(\gamma+\beta+\alpha+2))^3\Gamma(4\gamma+4\beta+4\alpha+4)\Gamma(5\gamma+5\beta+5\alpha+p+4)}t^{5\gamma+5\beta+5\alpha+p+3}$$
$$-\frac{3\Gamma(\gamma+\beta+2-\lambda)\Gamma(3\gamma+3\beta+2\alpha+3-\lambda)}{(\Gamma(\gamma+\beta+\alpha+2))^2\Gamma(\gamma+\beta+2)\Gamma(2\gamma+2\beta+\alpha+2-\lambda)\Gamma(4\gamma+4\beta+3\alpha+3-\lambda)}t^{4\gamma+4\beta+3\alpha+2-\lambda}$$
$$+\frac{3\Gamma(3\gamma+3\beta+3\alpha+3)}{(\Gamma(\gamma+\beta+\alpha+2))^2\Gamma(2\gamma+2\beta+2\alpha+2)\Gamma(4\gamma+4\beta+4\alpha+3)}t^{4\gamma+4\beta+4\alpha+2}$$
$$-\frac{6\Gamma(3\gamma+3\beta+3\alpha+p+3)}{(\Gamma(\gamma+\beta+\alpha+2))^2\Gamma(2\gamma+2\beta+2\alpha+p+2)\Gamma(4\gamma+4\beta+4\alpha+p+3)}t^{4\gamma+4\beta+4\alpha+p+2}$$
$$+\frac{3\Gamma(3\gamma+3\beta+3\alpha+4)\Gamma(5\gamma+5\beta+5\alpha+5)}{(\Gamma(\gamma+\beta+\alpha+2))^2(\Gamma(\gamma+\beta+\alpha+2))^3\Gamma(4\gamma+4\beta+4\alpha+4)\Gamma(6\gamma+6\beta+6\alpha+5)}t^{6\gamma+6\beta+6\alpha+4}$$

Then, we give a numerical solution of equation (4.15) in series form:

$$y(t)=\frac{1}{\Gamma(\gamma+\beta+\alpha+2)}t^{\gamma+\beta+\alpha+1}-\frac{\Gamma(\gamma+\beta+2-\lambda)}{\Gamma(\gamma+\beta+2)\Gamma(2\gamma+2\beta+\alpha+2-\lambda)}t^{2\gamma+2\beta+\alpha+1-\lambda}+$$
$$\frac{1}{\Gamma(2\gamma+2\beta+2\alpha+2)}t^{2\gamma+2\beta+2\alpha+1}-\frac{2}{\Gamma(2\gamma+2\beta+2\alpha+p+2)}t^{2\gamma+2\beta+2\alpha+p+1}+$$
$$\frac{\Gamma(3\gamma+3\beta+3\alpha+4)}{(\Gamma(\gamma+\beta+\alpha+2))^3\Gamma(4\gamma+4\beta+4\alpha+4)}t^{4\gamma+4\beta+4\alpha+3}+\ldots \quad (4.17)$$

The figures Fig 2 (a) and Fig 2 (b) show the solution curve (4.17) for some particular values.

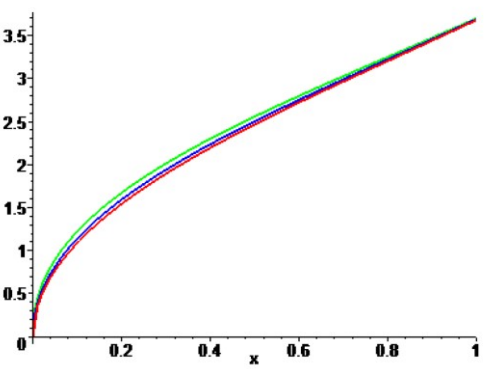


Fig 2(a)
Graph of solution (4.17) with

$\alpha=1,\beta=\frac{1}{2},\gamma=\frac{1}{4},\lambda=3,p=1$(red),

$\alpha=0.98,\beta=\frac{1}{2},\gamma=\frac{1}{4},\lambda=3,p=1$(blue),

$\alpha=0.95,\beta=\frac{1}{2},\gamma=\frac{1}{4},\lambda=3,p=1$(green).

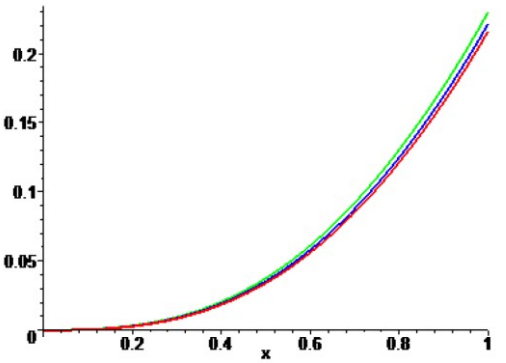


Fig 2 (b)
Graph of solution (4.17) with

$\alpha = 1, \beta = \frac{1}{2}, \gamma = \frac{1}{4}, \lambda = -3, p = 1$(red),

$\alpha = 0.98, \beta = \frac{1}{2}, \gamma = \frac{1}{4}, \lambda = -3, p = 1$(blue),

$\alpha = 0.95, \beta = \frac{1}{2}, \gamma = \frac{1}{4}, \lambda = -3, p = 1$(green).

5. Conclusion

In the first part of this work, we have studied a class of three parameter-sequential integro-differential problems of Lane-Emden type with anti periodic conditions. We have proved a first result, using the BCP theorem. Then, by the application of the Schaefer fixed point theorem we have proved the second result. Two examples have also been presented. In another part of this paper, we have applied the ADM method to obtain new solutions for our problem under some initial conditions. Some graphs of the ADM method obtained solutions have been plotted.

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