Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 29 (2026), No. 6, pp. 1835–1847
DOI : 10.47974/JIM-2319

Weighted Hardy-type inequalities with Diamond-alpha derivative on time scales

S. Sahu [†]
S. K. Sunanda *
Department of Mathematics
International Institute of Information Technology, Bhubaneswar
Gothapatna
Bhubaneswar 751003
Odisha
India

Abstract

We establish new weighted Hardy-type inequalities involving the Diamond-alpha derivative within time scales calculus, unifying discrete and continuous settings. By introducing weight functions, we generalize classical Hardy-type inequalities and derive conditions ensuring the sharpness of the constants. The results extend known inequalities by providing optimal bounds and covering various time scales, such as real numbers, integers, and q-calculus. Applications to dynamic equations and illustrative examples are presented, demonstrating the interaction between weighted functions and dynamic derivatives. These findings deepen the understanding of weighted inequalities on time scales, offering new avenues for generalization and application.

Subject Classification: *26E70, 26D15, 26D10, 33A40.*

Keywords: *Weighted Hardy-type inequalities, Diamond-alpha derivatives, Time scales, Dynamic inequalities.*

Introduction

Hardy-type inequalities are key tools in analysis, with applications in differential equations and dynamic systems. They offer estimates linking integrals or sums of a function to weighted averages, highlighting properties of the function. Time scale theory, introduced by Hilger, unifies

[†] *E-mail:* **c519002@iiit-bh.ac.in**
* *E-mail:* **shanta@iiit-bh.ac.in (Corresponding Author)**

discrete and continuous analysis, while the Diamond-alpha derivative blends delta and nabla derivatives, enhancing this unified framework.

Hardy inequality in discrete form is given in 1920, [1]

$$\sum_{j=1}^{\infty}\left(\frac{1}{j}\sum_{i=1}^{j}A(i)\right)^{\alpha} \leq \left(\frac{\alpha}{\alpha-1}\right)^{\alpha}\sum_{j=1}^{\infty}A^{\alpha}(j), \qquad \alpha>1 \tag{1}$$

where, $A(j)\geq 0$ for $j\geq 1$.

Integral form of equation (1) is given by [1]

$$\int_0^{\infty}\left(\frac{1}{t}\int_0^t G(s)\,ds\right)^{\alpha} dt \leq \left(\frac{\alpha}{\alpha-1}\right)^{\alpha}\int_0^{\infty}G^{\alpha}(t)\,dt, \quad \alpha>1 \tag{2}$$

where, G is a non-negative integrable function over any finite interval (0, t) and G^{α} is integrable and convergent over the open interval $(0,\infty)$.

The research on Hardy's inequalities, across both the discrete and continuous settings, has mainly focused on formulating weighted variants, these weighted forms play an important role in analysis because they involve optimal weight classes and satisfy rich structural conditions [2]. With applications in spectral theory, PDEs, and ODEs, this area has fostered important connections across mathematics and inspired numerous publications. In recent decades, substantial effort has been devoted to refining and expanding these inequalities, motivating the developments presented in this paper. In 1964, Levinson [3] extended equation (1) applying Jensen's equality under certain conditions and proved that

$$\int_0^{\infty}\Psi\left(\frac{1}{\Omega(y)}\int_0^y g(t)\chi(t)\,dt\right)dy \leq \mathcal{K}^p\int_0^{\infty}\Psi(g(y))\,dy \tag{3}$$

where χ and g are positive functions, $\mathcal{K}$ is a positive constant, $\Psi(x)$ is a real-valued non-negative convex function and $\Omega(y)=\int_0^y \omega(z)dz$ and $p>1$. The further development of equation (3) has been studied in [3], [4], [5], and [6].

For completeness, key results in dynamic inequalities that motivate the topic of this article are also discussed. The time scale form of equation (1) is given by [7], which states that, for $p > 1$ and a positive function g such that $\int_{\alpha}^{\infty}(g(t))^p\Delta t>\infty$,

$$\int_{\alpha}^{\infty}\frac{1}{(\breve{\sigma}(\tau)-\alpha)^p}\left(\int_{\alpha}^{\breve{\sigma}(\tau)} g(t)\Delta t\right)^p \Delta\tau \leq \left(\frac{p}{p-1}\right)^p\int_{\alpha}^{\infty}g^p(t)\Delta t. \tag{4}$$

Further, [8] obtained the time scale version of equation (4), as in,

$$\int_0^\infty \frac{1}{\tau^\rho}\left(\int_0^{\breve{\sigma}(\tau)} g(t)\Delta t\right)^p \Delta\tau \le \left(\frac{p\mathcal{K}^\rho}{\rho-1}\right)^p \int_0^\infty \frac{g^p(\tau)}{\tau^{\rho-p}}\Delta\tau \tag{5}$$

where $\rho > 1$ and $p > 1$ with a constant $\mathcal{K} > 0$.

The authors in [9] extended inequality equation (5) and demonstrated that

$$\int_\alpha^\infty \frac{1}{(\breve{\sigma}(\tau)-\alpha)^\rho}\left(\int_\alpha^{\breve{\sigma}(\tau)} g(t)\,\Delta t\right)^p \Delta\tau \le \left(\frac{p}{\rho-1}\right)^p \int_\alpha^\infty g^p(\tau)\frac{(\breve{\sigma}(\tau)-\alpha)^{(p-1)\rho}}{(\tau-\alpha)^{p(\rho-1)}}\Delta\tau \tag{6}$$

Weighted inequalities play a crucial role in analyzing and optimizing dynamic systems, particularly those that exhibit time-dependent or heterogeneous behaviour. In time-scale calculus, these inequalities provide deeper insights into stability, control, and estimation across diverse fields like signal processing and communication systems, population dynamics and epidemiology, financial mathematics and risk analysis, engineering and mechanics, computational and numerical analysis etc.

Recent advancements in dynamical systems and control theory have enhanced the analysis and stability of complex systems. Control approaches based on linear and nonlinear models, along with feedback mechanisms and Lyapunov stability analysis, have proven effective in ensuring robustness and efficiency in engineering applications [10]. In addition, the integration of machine learning techniques has improved the study of nonlinear and chaotic systems. Models such as support vector machines, random forests, and recurrent neural networks effectively capture complex patterns, enabling better prediction and deeper understanding of system dynamics [11].

This paper extends classical Hardy-type inequalities by incorporating weight functions and the Diamond-alpha derivative in time-scale calculus. We establish conditions, generalize results, derive sharper bounds, and identify optimal constants, with applications to dynamic equations in both discrete and continuous cases. The manuscript starts with preliminaries on nabla and Diamond-alpha calculus, followed by main results covering a range of established inequalities.

1. Preliminaries

A time scale $\mathbb{T}$ is defined as a non-empty closed subset of the real numbers endowed with the standard topology inherited from $\mathbb{R}$. The purpose of this section is to establish the foundational notation and essential properties of time scale calculus that will be employed in the subsequent analysis of dynamic inequalities on arbitrary time scales. The comprehensive treatments of these preliminaries can be found in [12] and [13]. The details of delta and nabla calculus id discussed in Table 1: Difference between Delta and Nabla Calculus [13].

Diamond-Alpha Calculus: -

Let $\phi(s)$ be delta as well as nabla differentiable on $\mathbb{T}$. Then, for $s \in \mathbb{T}$, $\phi^{\Diamond_\alpha}(s)$, which is the α -diamond derivative, is given by

$$\phi^{\Diamond_\alpha}(s) = \alpha\phi^{\Delta}(s) + (1-\alpha)\phi^{\nabla}(s),\ 0 \le \alpha \le 1. \tag{7}$$

Thus, ϕ is $\Diamond_\alpha$ differentiable iff ϕ is Δ and ∇ differentiable. For $\alpha = 1$, $\Diamond_\alpha$ -derivative reduces to Δ -derivative and ∇ -derivative for $\alpha = 0$.

- For two $\Diamond_\alpha$ differentiable function, the product and quotient rule are defined respectively,

$$(\phi\cdot\psi)^{\Diamond_\alpha}(s) = \phi^{\Diamond_\alpha}(s)\psi(s) + \alpha\phi^{\sigma}(s)\psi^{\Delta}(s) + (1-\alpha)\phi^{\rho}(s)\psi^{\nabla}(s) \tag{8}$$

$$\left(\frac{\phi}{\psi}\right)^{\Diamond_\alpha}(s) = \frac{\phi^{\Diamond_\alpha}(s)\psi^{\sigma}(s)\psi^{\rho}(s) - \alpha\phi^{\sigma}(s)\psi^{\Delta}(s) - (1-\alpha)\phi^{\rho}(s)\psi^{\sigma}(s)\psi^{\nabla}(s)}{\psi(s)\psi^{\sigma}(s)\psi^{\rho}(s)} \tag{9}$$

- For a time scale or measure chain, $\mathbb{T}$ let us consider $c, s \in \mathbb{T}$ and ϕ be a function defined from $\mathbb{T}$ to $\mathbb{R}$. Next, we can represent the diamond-α integral form in the following manner: [13]

$$\int_c^s \phi(t)\Diamond_\alpha t = \alpha\int_c^s \phi(t)\Delta t + (1-\alpha)\int_c^s \phi(t)\nabla t,\ 0 \le \alpha \tag{10}$$

  assuming the presence of the delta and nabla integrals of ϕ on $\mathbb{T}$..
- Generally, [13] $\left(\int_c^s \phi(t)\Diamond_\alpha t\right)^{\Diamond_\alpha} \ne \phi(s)$, for $s \in \mathbb{T}$.
- The monotonicity of a continuous function refers to its behaviour in terms of being consistently increasing or decreasing over an interval. A function can be classified as monotonic if it preserves the order of inputs, meaning that it does not switch between increasing and decreasing within a given domain.

Table 1

Difference between Delta and Nabla Calculus [13]

DELTA (Δ) - Calculus	**NABLA (∇) - Calculus**
Forward Jump Operator: $\sigma(s) := \inf\{\tau > s : \tau \in \mathbb{T}\}, \forall s \in \mathbb{T}$	Backward Jump Operator: $\rho(s) := \sup\{\tau < s : \tau \in \mathbb{T}\}, \forall s \in \mathbb{T}$
Forward Graniness function: $\mu(t) = \sigma(t) - t$	**Backward Graniness function:** $v(s) = s - \rho(s)$
Right Scattered point: $\sigma(s) > s$	**Left Scattered point:** $\rho(s) > s$
Right Dense: $\sigma(s) = s$	**Left Dense:** $\rho(s) = s$
Right Continuous Function: Let $\phi : \mathbb{T} \to \mathbb{R}$. This function is said to be rd-continuous if it is right continuous over the entire class of rd-points $s \in \mathbb{T}$ and $\lim_{t\to s} \phi(t)$ exists and is bounded over entire ld-points.	**Left Continuous Function:** Let $\phi : \mathbb{T} \to \mathbb{R}$. This function is said to be ld-continuous if it is left continuous over the entire class of rd-points $s \in \mathbb{T}$ and $\lim_{t\to s} \phi(t)$ exists and is bounded over entire rd-points.
Δ **– Differentiation** $\phi^{\Delta}(s) = \dfrac{\phi(\sigma(s)) - \phi(s)}{\mu(s)}$	∇ **–Differentiation** $\phi^{\nabla}(s) = \dfrac{\phi(s) - \phi(\rho(s))}{v(s)}$
Composition Operator: $\phi \circ \sigma = \phi(\sigma(s)) = \phi^{\sigma}$ $= \phi(s) + \mu(s)\phi^{\Delta}(s)$	**Composition Operator:** $\phi \circ \rho = \phi(\rho(s)) = \phi^{\rho}$ $= \phi(s) - v(s)\phi^{\nabla}(s)$
Product Rule: $(\phi\psi)^{\Delta}(s) = \phi^{\Delta}(s)\psi(s) + \phi^{\sigma}(s)\psi^{\Delta}(s)$ $= \phi(s)\psi^{\Delta}(s) + \phi^{\Delta}(s)\psi^{\sigma}(s)$	**Product Rule:** $(\phi\psi)^{\nabla}(s) = \phi^{\nabla}(s)\psi(s) + \phi^{\rho}(s)\psi^{\nabla}(s)$ $= \phi(s)\psi^{\nabla}(s) + \phi^{\nabla}(s)\psi^{\rho}(s)$
Quotient Rule: $\left(\dfrac{\phi}{\psi}\right)^{\Delta}(s) = \dfrac{\phi^{\Delta}(s)\psi(s) - \phi(s)\psi^{\Delta}(s)}{\psi(s)\psi^{\sigma}(s)}$	**Quotient Rule:** $\left(\dfrac{\phi}{\psi}\right)^{\nabla}(s) = \dfrac{\phi^{\nabla}(s)\psi(s) - \phi(s)\psi^{\nabla}(s)}{\psi(s)\psi^{\rho}(s)}$
Cauchy Δ – Integral: $\int_c^d \phi(s)\Delta(s) = \Phi(d) - \Phi(c), \forall s \in \mathbb{T}$ Where, $\Phi^{\Delta}(s) = \phi(s)$.	**Cauchy ∇ – Integral:** $\int_c^d \phi(s)\nabla(s) = \Phi(d) - \Phi(c), \forall s \in \mathbb{T}$ Where, $\Phi^{\nabla}(s) = \phi(s)$.
Relationship between Δ and ∇ Calculus: $\phi^{\nabla}(s) = \phi^{\Delta}(\rho(s))$ OR $\phi^{\Delta}(s) = \phi^{\nabla}(\sigma(s))$	

Main Results

Initially, we demonstrate new Hardy-type inequalities for nabla (∇) -conformable functions on time scale. Keeping the delta form of the inequality in mind [14], we generalized the result of diamond-α Hardy-type inequality. For the further development of the result, it will be convenient to consider $0.\infty=0$ and $0/0=0$ [14].

Theorem 1: *Suppose* $\mathfrak{H},\mathfrak{F}$ *and* $\mathfrak{G}$ *be left dense continuous monotonic non-decreasing functions on the interval* $(\infty,c]_{\mathbb{T}}$ *and* $\chi>1,\ \ m>n,$ *where m and n are positive real numbers. For a non-negative constant* $\mathbb{K}$ *with the property mentioned below,*

$$\gamma(\omega)-\left(\frac{m}{n(\chi-1)}\right)\frac{\mathfrak{H}^{\nabla}\Upsilon(\omega)}{\mathfrak{H}(\omega)}\geq\frac{1}{\mathbb{K}}>0, \tag{11}$$

Then

$$\int_c^\infty \Upsilon^{-\chi}(\omega)[\varphi^{\rho}(\omega)]^{\frac{m}{n}}\nabla\omega\leqslant\left[\frac{m\mathbb{K}}{n(\chi-1)}\right]^{\frac{m}{n}}\int_c^\infty \Upsilon^{\frac{m}{n}-\chi}(\omega)[\mathfrak{H}^{\rho}(\omega)\,\mathfrak{F}(\omega)\mathfrak{G}(\omega)]^{\frac{m}{n}}\nabla\omega. \tag{12}$$

where $\Upsilon(\omega)=\int_c^{\omega}\gamma(\zeta)\nabla\zeta$ and $\varphi(\omega)=\mathfrak{H}(\omega)\int_c^{\infty}\mathfrak{G}(\zeta)\,\mathfrak{F}(\zeta)\nabla\zeta,$ for every $\omega\in(\infty,c]_{\mathbb{T}}.$

Proof: Without loss of generality, let us define ϑ for any $\omega\in(\infty,c]_{\mathbb{T}}$ as

$$\vartheta(\omega)=\int_{\omega}^{\infty}\Upsilon^{-\chi}(\zeta)\,\gamma(\zeta)\,\nabla\zeta. \tag{13}$$

Applying the integration by parts formula for nabla-differentiable functions and considering $\varphi(c)=0$ and $\vartheta(\infty)=0,$ we obtain

$$\int_c^\infty \gamma(\omega)\Upsilon^{-\chi}(\omega)[\varphi^{\rho}(\omega)]^{\frac{m}{n}}\nabla\zeta=\int_c^\infty\left(\varphi^{\frac{m}{n}}(\omega)\right)^{\nabla}\vartheta(\omega)\nabla\zeta. \tag{14}$$

Now by using the chain rule for the nabla version, we get

$$\left(\varphi^{\frac{m}{n}}(\omega)\right)^{\nabla}=\frac{m}{n}\varphi^{\nabla}(\omega)\,\varphi^{\frac{m}{n}-1}(\beta)\quad\text{ for }\beta\in[\rho(\omega),\omega] \tag{15}$$

and

$$\left(\Upsilon^{1-\chi}(\omega)\right)^{\nabla}=(1-\chi)\,\Upsilon^{-\chi}(\beta)\,\Upsilon^{\nabla}(\omega)=(1-\chi)\,\Upsilon^{-\chi}(\beta)\,\gamma(\omega),\quad \beta\in[\rho(\omega),\omega] \tag{16}$$

Using the nabla derivative formula of the product formula on $\varphi(\omega)$, we obtain

$$\varphi(\omega)=\mathfrak{H}^{\rho}(\omega)\,\mathfrak{F}(\omega)\,\mathfrak{G}(\omega)+\frac{(\varphi\,\mathfrak{H}^{\nabla})(\omega)}{\mathfrak{H}(\omega)}. \tag{17}$$

Now implementing the theory of $\mathfrak{H}$ being non-decreasing, we infer that $\varphi^{\nabla}(\omega)\geqslant 0,$ and hence

$$\left(\varphi^{\frac{m}{n}}(\omega)\right)^{\nabla}\leqslant\frac{m}{n}\Big[\varphi^{\rho}(\omega)\Big]^{\frac{\vartheta m}{n}-1}\varphi^{\nabla}(\omega). \tag{18}$$

Also, as $\Upsilon^{\nabla}(\omega)=\gamma(\omega)>0,$ we have

$$\left(\Upsilon^{1-\chi}(\omega)\right)^{\nabla}\leqslant(1-\chi)\,\gamma(\omega)\,\Upsilon^{-\chi}(\omega), \tag{19}$$

Integrating both sides results to

$$\int_{\omega}^{\infty}\left(\Upsilon^{1-\chi}(\tau)\right)^{\nabla}\nabla\tau\leqslant(1-\chi)\int_{\omega}^{\infty}\Upsilon^{-\chi}(\tau)\,\gamma(\tau)\nabla\tau. \tag{20}$$

Therefore, we've got

$$\begin{aligned}\vartheta(\omega)\quad&\leqslant\frac{1}{1-\chi}\int_{\omega}^{\infty}\left(\Upsilon^{1-\chi}(\tau)\right)^{\nabla}\nabla\tau=\frac{1}{1-7}\left(\Upsilon^{1-\chi}(\infty)-\Upsilon^{1-\chi}(\omega)\right)\\&=\frac{-1}{1-\chi}\left(\Upsilon^{1-\chi}(\omega)-\Upsilon^{1-\chi}(\infty)\right)\leqslant\frac{\Upsilon^{1-\chi}(\omega)}{\chi-1}\end{aligned} \tag{21}$$

By unifying equation (19), (20) and (21), we get

$$\begin{aligned}&\int_{c}^{\infty}\Upsilon^{-\chi}(\omega)\Big[\varphi^{\rho}(\omega)\Big]^{\frac{m}{n}}\left(\gamma(\omega)-\frac{m}{n(\chi-1)}\frac{\Upsilon(\omega)\mathfrak{H}^{\nabla}(\omega)}{\mathfrak{H}(\omega)}\right)\nabla\zeta\\&\leqslant\frac{m}{n(\chi-1)}\int_{c}^{\infty}\Upsilon^{1-\chi}(\omega)[\varphi^{\rho}(\omega)]^{\frac{m}{n}-1}\mathfrak{H}^{\rho}(\omega)\,\mathfrak{F}(\omega)\,\mathfrak{G}(\omega)\nabla\zeta\end{aligned} \tag{22}$$

Employing the property (8), we conclude

$$\begin{aligned}&\int_{c}^{\infty}\Upsilon^{-\chi}(\omega)[\varphi^{\rho}(\omega)]^{\frac{m}{n}}\frac{1}{\mathbb{K}}\nabla\zeta\\&\leqslant\frac{m}{n(\chi-1)}\int_{c}^{\infty}\left[\Upsilon^{-\frac{\chi(m-n)}{m}(\omega)}[\varphi^{\rho}(\omega)]^{\frac{m-n}{n}}\right]\left[\Upsilon^{1-\frac{\chi n}{m}}(\omega)\,\mathfrak{G}(\omega)\,\mathfrak{F}(\omega)\,\mathfrak{H}^{\rho}(\omega)\right]\nabla\zeta.\end{aligned} \tag{23}$$

Under the application of the nabla Holder's inequality for the specific exponent pair $\frac{m}{n}$ and $\frac{m}{m-n}$, we have got,

$$\int_c^\infty \Upsilon^{-\chi}(\omega)\left[\varphi^\rho(\omega)\right]^{\frac{m}{n}} \nabla\omega$$

$$\leqslant \frac{m\mathbb{K}}{n(\chi-1)}\left[\int_c^\infty \Upsilon^{\chi}(\omega)\left[\varphi^\rho(\omega)\right]^{\frac{m}{n}} \nabla\omega\right]^{\frac{m-n}{n}}$$

$$\times\left[\int_c^\infty \Upsilon^{\frac{m}{n}-\chi}(\omega)\left(\mathfrak{H}^\rho\,\mathfrak{G}(\omega)\,\mathfrak{F}(\omega)\right)^{\frac{m}{n}} \nabla\omega\right]^{\frac{n}{m}}$$

The above Inequality produces the required result

$$\int_c^\infty \Upsilon^{-\chi}(\omega)[\varphi^\rho(\omega)]^{\frac{m}{n}} \nabla\omega \leqslant \left[\frac{m\mathbb{K}}{n(\chi-1)}\right]^{\frac{m}{n}} \int_c^\infty \Upsilon^{\frac{m}{n}-\chi}(\omega)[\mathfrak{H}^\rho(\omega)\,\mathfrak{F}(\omega)\,\mathfrak{G}(\omega)]^{\frac{m}{n}} \nabla\omega. \tag{24}$$

This completes the proof.

Remark 1: Consider $\mathbb{T}=\mathbb{R}$. In Theorem 1, put $\mathfrak{H}(\omega) = \gamma(\omega) = \mathfrak{G}(\omega) = 1$, $n=1$, $\mathbb{K}=1$, and $c=0$, results in inequality (1).

Remark 2: By substituting $\mathfrak{H}(\omega) = \gamma(\omega) = \mathfrak{G}(\omega) = 1$, $n=1$, and $\chi=m$ in equation (12), we attain

$$\int_c^\infty (\omega-c)^{-m}\left[\int_c^{\rho(\omega)} \mathfrak{F}(\zeta)\nabla\zeta\right]^m \nabla\omega \leqslant \left[\frac{m\mathbb{K}}{m-1}\right]^m \int_c^\infty \mathfrak{F}^m(\omega)\nabla\omega. \tag{25}$$

As, $\omega \leqslant \rho(\omega)$, then $(\omega-c)^{-m} \geqslant (\rho(\omega)-c)^{-\alpha}$, that means, with the previous inequality that

$$\int_c^\infty\left[\frac{1}{\rho(t)-c}\int_c^{\rho(t)} \mathfrak{F}(\omega)\nabla\omega\right]^m \nabla t \leqslant \left[\frac{m\mathbb{K}}{m-1}\right]^m \int_c^\infty \mathfrak{F}^m(\omega)\nabla\omega. \tag{26}$$

Taking $\mathbb{K}=1$, we derive inequality (6).

In the next theorem, we gave the Diamond-α form of the Hardy-type inequality. The Diamond-alpha Hardy-type inequality has broad applications in dynamic systems, boundary value problems, function spaces, and optimization on time scales. It aids in analyzing stability and oscillation in dynamic equations and provides integral estimates in boundary problems. In Sobolev spaces, it enables norm estimates and supports interpolation. It's also valuable in control theory for hybrid systems, unifying discrete-continuous dynamics, and in population models and epidemiology, where it bounds growth rates under periodic influences. This versatility makes it a key tool for connecting discrete and continuous models in various fields.

Theorem 2: Let $\mathfrak{H},\mathfrak{F}$ and $\mathfrak{G}$ be a *rd-continuous* as well as ld-continuous monotonic non-decreasing functions on $\mathbb{T}$ and $\chi > 1$, $m > n$, where m and n are positive real numbers. For a non-negative constant $\mathbb{K}$ with the property given below,

$$\gamma(\omega) - \left(\frac{m}{n(\chi-1)}\right)\frac{\left(\alpha\mathfrak{H}^{\Delta} + (1-\alpha)\mathfrak{H}^{\nabla}\right)\Upsilon(\omega)}{\mathfrak{H}(\omega)} \geqslant \frac{1}{\mathbb{K}} > 0, \tag{27}$$

then

$$\int_{\mathbb{T}} \Upsilon^{-\chi}(\omega)[\varphi^{\rho}(\omega)]^{\frac{m}{n}} \lozenge_{\alpha}\omega$$
$$\leqslant \left[\frac{m\mathbb{K}}{n(\chi-1)}\right]^{\frac{m}{n}} \int_{\mathbb{T}} \Upsilon^{\frac{m}{n}-\chi}(\omega)[\left(\alpha\mathfrak{H}^{\sigma} + (1-\alpha)\mathfrak{H}^{\rho}\right)(\omega)\,\mathfrak{F}(\omega)\mathfrak{G}(\omega)]^{\frac{m}{n}} \lozenge_{\alpha}\omega. \tag{28}$$

where $\Upsilon(\omega) = \int_{\mathbb{T}} \gamma(\zeta) \lozenge_{\alpha}\zeta$ and $\varphi(\omega) = \mathfrak{H}(\omega)\int_{\mathbb{T}} \mathfrak{G}(\zeta)\mathfrak{F}(\zeta)\lozenge_{\alpha}\zeta$, for every $\omega \in \mathbb{T}$.

Proof: Without loss of generality, let us define ϑ for any $\omega \in \mathbb{T}$ as

$$\vartheta(\omega) = \int_{\mathbb{T}} \Upsilon^{-\chi}(\zeta)\gamma(\zeta)\lozenge_{\alpha}\zeta. \tag{29}$$

Applying the integration by parts formula for $\lozenge_{\alpha}$-differentiable functions and considering, we obtain

$$\int_{\mathbb{T}} \gamma(\omega)\Upsilon^{-\chi}(\omega)[\varphi^{\rho}(\omega)]^{\frac{m}{n}} \lozenge_{\alpha}\zeta = \int_{\mathbb{T}} \left(\varphi^{\frac{m}{n}}(\omega)\right)^{\lozenge_{\alpha}} \vartheta(\omega)\lozenge_{\alpha}\zeta. \tag{30}$$

Now by using the chain rule for the Diamond-α version, we get

$$\left(\varphi^{\frac{m}{n}}(\omega)\right)^{\lozenge_{\alpha}} = \frac{m}{n}\left(\alpha\varphi^{\Delta} + (1-\alpha)\varphi^{\nabla}\right)(\omega)\,\varphi^{\frac{m}{n}-1}(\beta) \quad \text{for } \beta \in \mathbb{T} \tag{31}$$

and

$$\left(\Upsilon^{1-\chi}(\omega)\right)^{\lozenge_{\alpha}} = (1-\chi)\Upsilon^{-\chi}(\beta)\Upsilon^{\lozenge_{\alpha}}(\omega) = (1-\chi)\Upsilon^{-\chi}(\beta)\gamma(\omega),\ \beta \in [\rho(\omega),\sigma(\omega)] \tag{32}$$

Using the $\lozenge_{\alpha}$-derivative formulation of the product rule on $\varphi(\omega)$, we derive

$$\varphi(\omega)=(\alpha\mathfrak{H}^{\sigma}+(1-\alpha)\mathfrak{H}^{\rho})(\omega)\mathfrak{F}(\omega)\mathfrak{G}(\omega)+\frac{(\varphi\mathfrak{H}^{\Diamond_\alpha})(\omega)}{\mathfrak{H}(\omega)}. \tag{33}$$

Now implementing the theory of $\mathfrak{H}$ being non-decreasing, we infer that $\varphi^{\Diamond_\alpha}(\omega)\geqslant 0$, and hence

$$\left(\varphi^{\frac{m}{n}}(\omega)\right)^{\Diamond_\alpha}\leqslant\frac{m}{n}\left[\alpha\varphi^{\sigma}+(1-\alpha)\varphi^{\rho}\right]^{\frac{\vartheta m}{n}-1}(\omega)\,\varphi^{\Diamond_\alpha}(\omega). \tag{34}$$

Also, as $\Upsilon^{\Diamond_\alpha}(\omega)=\gamma(\omega)>0$, we have

$$\left(\Upsilon^{1-\chi}(\omega)\right)^{\Diamond_\alpha}\leqslant(1-\chi)\,\gamma(\omega)\,\Upsilon^{-\chi}(\omega), \tag{35}$$

Integrating both sides results to

$$\int_{\mathbb{T}}\left(\Upsilon^{1-\chi}(\tau)\right)^{\Diamond_\alpha}\Diamond_\alpha\tau\leqslant(1-\chi)\int_{\mathbb{T}}\Upsilon^{-\chi}(\tau)\,\gamma(\tau)\,\Diamond_\alpha\tau. \tag{36}$$

Therefore, we've got

$$\begin{aligned}\vartheta(\omega)&\leqslant\frac{1}{1-\chi}\int_{\mathbb{T}}\left(\Upsilon^{1-\chi}(\tau)\right)^{\Diamond_\alpha}\Diamond_\alpha\tau=\frac{1}{1-\chi}\left(\Upsilon^{1-\chi}(\inf\mathbb{T})-\Upsilon^{1-\chi}(\sup\mathbb{T})\right)\\&=\frac{-1}{1-\chi}\left(\Upsilon^{1-\chi}(\sup\mathbb{T})-\Upsilon^{1-\chi}(\inf\mathbb{T})\right)\leqslant\frac{\Upsilon^{1-\chi}(\omega)}{7-1}\end{aligned} \tag{37}$$

By unifying equation (35), (36) and (37), we get

$$\int_{\mathbb{T}}\Upsilon^{-\chi}(\omega)\left[(\alpha\varphi^{\sigma}+(1-\alpha)\varphi^{\rho})(\omega)\right]^{\frac{m}{n}}\left(\gamma(\omega)-\frac{m}{n(\chi-1)}\frac{\Upsilon(\omega)(\alpha\mathfrak{H}^{\Delta}+(1-\alpha)\mathfrak{H}^{\nabla}(\omega)}{\mathfrak{H}(\omega)}\right)\Diamond_\alpha\zeta$$

$$\leqslant\frac{m}{n(\chi-1)}\int_{\mathbb{T}}\Upsilon^{1-\chi}(\omega)[(\alpha\varphi^{\sigma}+(1-\alpha)\varphi^{\rho})(\omega)]^{\frac{m}{n}-1}\left(\alpha\mathfrak{H}+(1-\alpha)\mathfrak{H}^{\rho}\right)(\omega)\,\mathfrak{F}(\omega)\mathfrak{G}(\omega)\,\Diamond_\alpha\zeta$$

Employing the property (26), we conclude

$$\int_{\mathbb{T}}\Upsilon^{-\chi}(\omega)[(\alpha\varphi^{\sigma}+(1-\alpha)\varphi^{\rho})(\omega)]^{\frac{m}{n}}\frac{1}{\mathbb{K}}\Diamond_\alpha\zeta$$

$$\begin{aligned}&\leqslant\frac{m}{n(\chi-1)}\int_{\mathbb{T}}\left[\Upsilon^{-\frac{\chi(m-n)}{m}(\omega)}[(\alpha\varphi^{\sigma}+(1-\alpha)\varphi^{\rho})(\omega)]^{\frac{m-n}{n}}\right]\\&\quad\times\left[\Upsilon^{1-\frac{\chi n}{m}}(\omega)\,\mathfrak{G}(\omega)\,\mathfrak{F}(\omega)(\alpha\mathfrak{H}^{\sigma}+(1-\alpha)\mathfrak{H}^{\rho})(\omega)\right]\Diamond_\alpha\zeta.\end{aligned}$$

Imposing the $\Diamond_\alpha$ version of Holder's inequality with exponents pair $\frac{m}{n}$ and $\frac{m}{m-n}$, we have

$$\int_{\mathbb{T}} \Upsilon^{-\chi}(\omega)\left[(\alpha\varphi^{\sigma}+(1-\alpha)\varphi^{\rho})(\omega)\right]^{\frac{m}{n}}\Diamond_\alpha\omega$$

$$\leqslant \frac{m\mathbb{K}}{n(\chi-1)}\left[\int_{\mathbb{T}} \Upsilon^{\chi}(\omega)\left[(\alpha\varphi^{\sigma}+(1-\alpha)\varphi^{\rho})(\omega)\right]^{\frac{m}{n}}\Diamond_\alpha\omega\right]^{\frac{m-n}{n}}$$

$$\times\left[\int_{\mathbb{T}} \Upsilon^{\frac{m}{n}-\chi}(\omega)\Big((\alpha\mathfrak{H}^{\sigma}+(1-\alpha)\mathfrak{H}^{\rho})(\omega)\mathfrak{G}(\omega)\mathfrak{F}(\omega)\Big)^{\frac{m}{n}}\Diamond_\alpha\omega\right]^{\frac{n}{m}}$$

The above inequality produces the required result

$$\int_{\mathbb{T}} \Upsilon^{-\chi}(\omega)[\varphi^{\rho}(\omega)]^{\frac{m}{n}}\Diamond_\alpha\omega$$

$$\leqslant\left[\frac{m\mathbb{K}}{n(\chi-1)}\right]^{\frac{m}{n}}\int_{\mathbb{T}} \Upsilon^{\frac{m}{n}-\chi}(\omega)[(\alpha\mathfrak{H}^{\sigma}+(1-\alpha)\mathfrak{H}^{\rho})(\omega)\mathfrak{F}(\omega)\mathfrak{G}(\omega)]^{\frac{m}{n}}\Diamond_\alpha\omega.$$

Remark 3: Consider $\mathbb{T}=\mathbb{R}$. In Theorem 2, put $\mathfrak{H}(\omega) = \gamma(\omega) = \mathfrak{G}(\omega)=1$, $n=1$, $\mathbb{K}=1$, $c=0$ results in inequality (1).

Remark 4: By substituting $\mathfrak{H}(\omega)=\gamma(\omega)=\mathfrak{G}(\omega)=1$, $n=1$, and $\chi=m$ in equation (28), we attain

$$\int_c^\infty(\omega-c)^{-m}\left[\int_c^{\rho(\omega)}\mathfrak{F}(\zeta)\Diamond_\alpha\zeta\right]^m\Diamond_\alpha\omega\leqslant\left[\frac{m\mathbb{K}}{m-1}\right]^m\int_c^\infty\mathfrak{F}^m(\omega)\Diamond_\alpha\omega.$$

Also,

$$\int_c^\infty\left[\frac{1}{\rho(t)-c}\int_c^{\rho(t)}\mathfrak{F}(\omega)\Diamond_\alpha\omega\right]^m\Diamond_\alpha t\leqslant\left[\frac{m\mathbb{K}}{m-1}\right]^m\int_c^\infty\mathfrak{F}^m(\omega)\Diamond_\alpha\omega.$$

Taking $\mathbb{K}=1$, we derive inequality (6).

2. Conclusion

In conclusion, we have developed new weighted Hardy-type inequalities using the Diamond-alpha derivative within the framework of time scales calculus, providing a unified approach for discrete and continuous analysis. The introduction of weight functions extends classical Hardy-type inequalities, offering sharper bounds and revealing the optimal conditions for their validity. Through applications to dynamic equations and illustrative examples on various time scales, we demonstrated the practical utility of these results. This work opens new avenues for further exploration of weighted inequalities and their applications in mathematical analysis, especially in the study of dynamic systems on arbitrary time scales.

Acknowledgement(s)

We would like to thank the reviewers for their constructive feedbacks.

Funding

This research received no external funding.

Competing interests

The authors acknowledge that this research was conducted without external fundings.

Authors' contributions:

All authors contributed equally to the preparation of this manuscript. All authors have reviewed and approved the final version for publication.

References

[1] G. H. Hardy, J. E. Littlewood, G. Pólya, and D. E. Littlewood, *Inequalities*. Cambridge, U.K.: Cambridge Univ. Press (1952).

[2] R. P. Agarwal, D. O'Regan, and S. H. Saker, *Hardy Type Inequalities on Time Scales*. Cham, Switzerland: Springer (2016).

[3] N. Levinson, "Generalization of an inequality of Hardy," *Duke Math. J.*, vol. 31, no. 3, pp. 389–394 (1964).

[4] W. Cheung, Z. Hanjš, and J. Pečarić, "Some Hardy-type inequalities," *J. Math. Anal. Appl.*, vol. 250, no. 2, pp. 621–634 (2000).

[5] E. T. Copson, "Some integral inequalities," *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, vol. 75, pp. 157–164 (1976), doi: 10.1017/S0308210500017868.

[6] G. S. Yang and D. Y. Hwang, "Generalizations of some reverse integral inequalities," *J. Math. Anal. Appl.*, vol. 233, no. 1, pp. 193–204 (1999).

[7] P. Řehák, "Hardy inequality on time scales and its application to half-linear dynamic equations," *J. Inequal. Appl.*, vol. 2005, no. 5, pp. 942–973 (2005).

[8] S. H. Saker and D. O'Regan, "Littlewood inequalities on time scales," *Bull. Malays. Math. Sci. Soc.*, vol. 39, no. 2, pp. 527–543 (2016).

[9] S. H. Saker, D. O'Regan, and R. P. Agarwal, "Littlewood and Bennett inequalities on time scales," *Mediterr. J. Math.*, vol. 12, no. 3, pp. 605–619 (2015).

[10] C. P. Selvan, A. Shrivastava, S. Ramaswamy, and K. Shipra, "Applying control theory in engineering using dynamical systems methods," *J. Interdiscip. Math.*, vol. 29, no. 3, pp. 627–635 (2026).

[11] G. Vaidya, S. M. U. Iqbal, M. I. Khan, and K. Gupta, "Analyzing nonlinear dynamical systems and chaos using machine learning in physical and biological systems," *J. Interdiscip. Math.*, vol. 29, no. 3, pp. 663–670 (2026).

[12] M. Bohner and A. C. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*. New York, USA: Springer (2001).

[13] M. Bohner and A. C. Peterson, *Advances in Dynamic Equations on Time Scales*. Boston, MA, USA: Springer (2002).

[14] A. Hamiaz, W. Abuelela, S. H. Saker, and D. Baleanu, "Some new dynamic inequalities with several functions of Hardy type on time scales," *J. Inequal. Appl.*, vol. 2021, no. 3 (2021).

Received November, 2024