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Use of homogeneous coordinates in the study of graves triad and related results

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Abstract

A matrix of order three is used to represent a triangle for real projective space of dimension two, here vertices of the triangle's with respect to a frame of reference are represented as rows of a matrix. Suppose there are three triangles such that the first is circumscribing the second, the second is circumscribing the third, and the third is circumscribing the first, forming a cyclic order and satisfying the Pappus configuration; then the triad of triangles is called a graves triad. The three triangles forming graves' triad give rise to three pairs of perspective triangles, called the perspective graves triad. This triad contributes to various results. The existence of the graves triad was already proven. In this paper, the existence of the perspective graves triad is established by using homogeneous coordinates. The related results are interesting because here a well-defined system of reference has been used. Some results of the perspective graves triad are discussed in relation to co-linearity.

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Keywords: *Homogeneous coordinates, Graves triad, Perspective graves triad, Simplex.*

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1. Introduction

A triangle is said to be non-degenerate if its matrix representation is non-singular. In homogeneous coordinates, a one-dimensional vector space is represented by a point, a two-dimensional vector space by a line, and a three-dimensional vector space by a plane. A point (a_1, a_2) in two dimension is represented as (a_1w, a_2w, w) for a non-zero w. When a scalar is multiplied in homogeneous coordinates, the point remains unaltered. All the entries of the homogeneous coordinates cannot be zero, so the homogeneous coordinates (a_i) and (ta_i) represent the same points for $t \neq 0$. [6] If D be a diagonal matrix which is non-singular, then the matrices U and V are said to be equivalent when $V = DU$. An equivalence class is obtained from this equivalence relation; here, a specific class represents the vertices of a triangle with respect to other triangle in matrix's rows.

In this paper, a 3×3 matrix represents a simplex $(B) = B_1B_2B_3$ in a real projective space P_2 of dimension two with reference to a triangle $(A) = A_1A_2A_3$. Here, the i^{th} row of the matrix denotes the coordinate of the vertex B_i for $i = 1, 2, 3$. [1], [2] Triad means a collection of three objects. The set of projective triads contains $3p - 2$ points, p of them is located on each of the three intersecting lines, and the intersecting point also lies in that set. [3], [5] Moreover, if we consider a point A on one of the lines and point B on another line, then the intersection point of AB with the third line also remains in the same set. If there are three triangles such that the first is circumscribing the second, the second is circumscribing the third, and the third is circumscribing the first, forming a cyclic order and satisfying the Pappus configuration, then the triad of these triangles is called the graves triad of triangles. Since the algebraic expressions are easy to understand, we express the results in algebraic form using homogeneous coordinates [12]. Guinand [4] has studied the existence of the graves triad and some properties related to the perspective graves triad using analytical methods. He obtained that if two of the triangles are perspective, then the other two pairs will also be. In this work it has been established that the graves triad of triangles in two dimensional projective space exists and some relevant properties in P_2 where P_2 is a projective space of dimension two over a field F, where the field F is formally real (A field F is called formally real if $x_i \in F$ and $\sum_{i=1}^{n} {x_i}^2 = 0$ then $x_i = 0$ for $i = 1,2, \ldots n$).

In matrix form triangle B with respect to A is represented by $(B)_A = ((b_{ij}))$ and C with respect to B as $(C)_B = ((c_{ij}))$ then we have C with respect to A as $(C)_A = ((d_{ij})) = ((c_{ij}))((b_{ij}))$. [7] Moreover $(B)_A = S = ((b_{ij}))$ is used to represent (B) referenced to (A). If a point P is expressed as (x_i) with respect to (A), (y_i) with respect to (B) and $(B)_A = ((b_{ij}))$ then

$$\sum_{j=1}^{n+1} x_j A_j = \sum_{i=1}^{n+1} y_i B_i = \sum_{i=1}^{n+1} y_i \sum_{j=1}^{n+1} b_{ij} A_j = \sum_{j=1}^{n+1} \left(\sum_{i=1}^{n+1} b_{ij} y_i\right) A_j$$

$$\Longrightarrow x_j = \sum_{i=1}^{n+1} b_{ij} y_i \text{ where } j = 1,2, \ldots., n+1.$$

Hence,

$$(y_i) = (x_i)((b_{ij}))^{-1}$$

Therefore, $(y_i) = (x_i)((B_{ji}))$ because multiplication of $((b_{ij}))^{-1}$ by a scalar cannot effect the transformation. Here B_{ij} is the co-factor of b_{ij} in $det((b_{ij}))$. We have $((B_{ij})) = det((b_{ij}))[((b_{ij}))^{-1}]$ so,

$$(B)_A = ((b_{ij}))$$

$$((B)_A)^{-1} = ((b_{ij}))^{-1}$$

$$((B)_A)^{-1} = ((B_{ji})) = (A)_B$$

$$\Rightarrow ((B)_A)^{-1} = (A)_B.$$

If the vertex B_1 of triangle (B) locates in front of the side of the vertex A_1 in the triangle (A), that is the side A_2A_3, then the homogeneous coordinate of B_1 is the linear combination of A_2 and A_3. Since A_1 is absent from the linear combination of A_2, A_3 with respect to triangle (A), so first coordinate becomes 'zero' [9], [10]. Therefore the obtained matrix for the triangle (B) with respect to triangle (A) possesses a zero diagonal [8], [11].

2. Existence of Graves Triad in Two Dimensional Projective Space

Here, the existence of the graves triad is established using homogeneous coordinates system, which will open up various new aspects to the application in other fields.

Theorem 1: *If triangle $(A) = A_1A_2A_3$ circumscribes triangle $(B) = B_1B_2B_3$ and triangle (B) circumscribes triangle $(C) = C_1C_2C_3$, then (C) circumscribes (A), thus forming a cyclic triad of circumscribing triangles called the graves triad of triangles. Also, $\{C_2, C_3, A_1\}$, $\{C_1, C_3, A_2\}$, and $\{C_1, C_2, A_3\}$ are sets of co-linear points in the Projective space of dimension two.*

Proof: Let $(A) = A_1A_2A_3$, $(B) = B_1B_2B_3$ and $(C) = C_1C_2C_3$ be a traid of triangles forming graves triad, where $(A) = A_1A_2A_3$ is the triangle of reference then we have

$$(A)_A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

If $(A) = A_1A_2A_3$ be a triangle which circumscribes triangle $(B) = B_1B_2B_3$. Then A_2A_3 passes through B_1. Hence, the point B_1 is represented as

$$B_1 = b_{12}A_2 + b_{13}A_3 = b_{12}(0,1,0) + b_{13}(0,0,1) = (0, b_{12}, b_{13}).$$

Since, A_1A_3 passes through B_2 and A_1A_2 passes through B_3 the points B_2 and B_3 can be expressed as follows

$$B_2 = b_{21}A_1 + b_{23}A_3 = b_{21}(1,0,0) + b_{23}(0,0,1) = (b_{21}, 0, b_{23})$$

and $B_3 = b31A_1 + b_{32}A_2 = b_{31}(1,0,0) + b_{32}(0,1,0) = (b_{31}, b_{32}, 0).$

In matrix form the triangle (B) with reference to (A) is represented as

$$(B)_A = \begin{pmatrix} 0 & b_{12} & b_{13} \\ b_{21} & 0 & b_{23} \\ b_{31} & b_{32} & 0 \end{pmatrix}$$

Further if triangle $(B) = B_1B_2B_3$ circumscribes $(C) = C_1C_2C_3$, then B_2B_3 passes through C_1. Therefore, the point C_1 can be written as

$$C_1 = c_{12}B_2 + c_{13}B_3 = c_{12}(b_{21}, 0, b_{23}) + c_{13}(b_{31}, b_{32}, 0).$$
$$= (c_{12}b_{21} + c_{13}b_{31}, c_{13}b_{32}, c_{12}b_{23})$$

Similarly, since B_1B_3 passes through C_2 and B_1B_2 passes through C_3 the points C_2 and C_3 can be expressed as follows

$$C_2 = c_{21}B_1 + c_{23}B_3 = c_{21}(0, b_{12}, b_{13}) + c_{23}(b_{31}, b_{32}, 0)$$
$$= (c_{23}b_{31}, c_{21}b_{12} + c_{23}b_{32}, c_{21}b_{13}) \text{ and}$$
$$C_3 = c_{31}B_1 + c_{32}B_2 = c_{31}(0, b_{12}, b_{13}) + c_{32}(b_{21}, 0, b_{23})$$
$$= (c_{32}b_{21}, c_{31}b_{12}, c_{31}b_{13} + c_{32}b_{23}).$$

In matrix form the triangle (C) with reference to (A) is given by

$$(C)_A = \begin{pmatrix} c_{12}b_{21} + c_{13}b_{31} & c_{13}b_{32} & c_{12}b_{23} \\ c_{23}b_{31} & c_{21}b_{12} + c_{23}b_{32} & c_{21}b_{13} \\ c_{32}b_{21} & c_{31}b_{12} & c_{31}b_{13} + c_{32}b_{23} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & c_{12} & c_{13} \\ c_{21} & 0 & c_{23} \\ c_{31} & c_{32} & 0 \end{pmatrix} \begin{pmatrix} 0 & b_{12} & b_{13} \\ b_{21} & 0 & b_{23} \\ b_{31} & b_{32} & 0 \end{pmatrix}$$
$$\Rightarrow (C)_A = \begin{pmatrix} 0 & c_{12} & c_{13} \\ c_{21} & 0 & c_{23} \\ c_{31} & c_{32} & 0 \end{pmatrix} (B)_A$$

Right multiplying $[(B)_A]^{-1}$ on both sides of the above equation we have

$$(C)_A \; [(B)_A]^{-1} = \begin{pmatrix} 0 & c_{12} & c_{13} \\ c_{21} & 0 & c_{23} \\ c_{31} & c_{32} & 0 \end{pmatrix} (B)_A \; [(B)_A]^{-1}$$
$$\Rightarrow (C)_A \; [(B)_A]^{-1} = \begin{pmatrix} 0 & c_{12} & c_{13} \\ c_{21} & 0 & c_{23} \\ c_{31} & c_{32} & 0 \end{pmatrix}$$
$$\Rightarrow (C)_B = \begin{pmatrix} 0 & c_{12} & c_{13} \\ c_{21} & 0 & c_{23} \\ c_{31} & c_{32} & 0 \end{pmatrix}.$$

Since $(C) = C_1C_2C_3$ circumscribes $(A) = A_1A_2A_3$ and C_2C_3 passes through A_1 we may express A_1 as follows

$$A_1 = a_{12}C_2 + a_{13}C_3.$$

Similarly, since C_1C_3 passes through A_2 and C_1C_2 passes through A_3, so A_2 and A_3 can be expressed as follows,

$$A_2 = a_{21}C_1 + a_{23}C_3 \text{ and } A_3 = a_{31}C_1 + a_{32}C_2.$$

In matrix form the triangle (A) with reference to (A) is represented as

$$(A)_A = \begin{pmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix}$$

$$\Rightarrow (A)_A = \begin{pmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix} (C)_A$$

$$\Rightarrow (A)_A = \begin{pmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix} \begin{pmatrix} 0 & c_{12} & c_{13} \\ c_{21} & 0 & c_{23} \\ c_{31} & c_{32} & 0 \end{pmatrix} \begin{pmatrix} 0 & b_{12} & b_{13} \\ b_{21} & 0 & b_{23} \\ b_{31} & b_{32} & 0 \end{pmatrix}$$

$$\Rightarrow (A)_A = \begin{pmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix} (C)_B (B)_A$$

Right multiplying $[(B)_A]^{-1}$ on both sides of the above equation, we obtain

$$(A)_A[(B)_A]^{-1} = \begin{pmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix} (C)_B (B)_A [(B)_A]^{-1}$$

Again, right multiplying $[(C)_B]^{-1}$ on both sides of the above equation we have

$$(A)_A[(B)_A]^{-1}[(C)_B]^{-1} = \begin{pmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix} (C)_B [(C)_B]^{-1}$$

$$\Rightarrow (A)_A[(B)_A]^{-1}[(C)_B]^{-1} = \begin{pmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix}$$

Hence

$$(A)_A (A)_B (B)_C = \begin{pmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix}$$

$$\Rightarrow (A)_C = \begin{pmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix}$$

As we have

$$\begin{pmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix} \begin{pmatrix} 0 & c_{12} & c_{13} \\ a_{21} & 0 & c_{23} \\ a_{31} & c_{32} & 0 \end{pmatrix} \begin{pmatrix} 0 & b_{12} & b_{13} \\ b_{21} & 0 & b_{23} \\ b_{31} & b_{32} & 0 \end{pmatrix} = (A)_A = \mathrm{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore, we say that

$$(A)_C (C)_B (B)_A = (A)_A = I$$

$$\Rightarrow (A)_C [(B)_A]^{-1} [(C)_B]^{-1}$$

$$\Rightarrow (A)_C = \begin{pmatrix} -b_{23}b_{32} & b_{13}b_{32} & b_{12}b_{23} \\ b_{23}b_{31} & -b_{13}b_{31} & b_{13}b_{21} \\ b_{21}b_{32} & b_{12}b_{31} & -b_{12}b_{21} \end{pmatrix} \begin{pmatrix} -c_{23}c_{32} & c_{13}c_{32} & c_{12}c_{23} \\ c_{23}c_{31} & -c_{13}c_{31} & c_{13}c_{21} \\ c_{21}c_{32} & c_{12}c_{31} & -c_{12}c_{21} \end{pmatrix}$$

where

$$[(B)_A]^{-1} = \begin{pmatrix} -b_{23}b_{32} & b_{13}b_{32} & b_{12}b_{23} \\ b_{23}b_{31} & -b_{13}b_{31} & b_{13}b_{21} \\ b_{21}b_{32} & b_{12}b_{31} & -b_{12}b_{21} \end{pmatrix},$$

$$[(C)_B]^{-1} = \begin{pmatrix} -c_{23}c_{32} & c_{13}c_{32} & c_{12}c_{23} \\ c_{23}c_{31} & -c_{13}c_{31} & c_{13}c_{21} \\ c_{21}c_{32} & c_{12}c_{31} & -c_{12}c_{21} \end{pmatrix}$$

and

$$(A)_C = \begin{pmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix}.$$

Hence, by comparison we have

$$b_{23}b_{32}c_{23}c_{32} + b_{13}b_{32}c_{23}c_{31} + b_{12}b_{23}c_{21}c_{32} = 0$$
$$b_{23}b_{31}c_{13}c_{32} + b_{13}b_{31}c_{13}c_{31} + b_{13}b_{21}c_{12}c_{31} = 0$$
$$b_{21}b_{32}c_{12}c_{23} + b_{12}b_{31}c_{13}c_{21} + b_{12}b_{21}c_{12}c_{21} = 0$$

and when each element of $(A)_C$ is expressed in term of b_{ij} and c_{ij} of $(B)_A$ and $(C)_B$ so, we have

$$a_{12} = b_{12}b_{23}c_{12}c_{31} - b_{13}b_{32}c_{13}c_{31} - b_{23}b_{32}c_{13}c_{32}$$
$$a_{13} = b_{13}b_{32}c_{13}c_{21} - b_{12}b_{23}c_{12}c_{21} - b_{23}b_{32}c_{12}c_{23}$$
$$a_{21} = b_{13}b_{21}c_{21}c_{32} - b_{13}b_{31}c_{23}c_{31} - b_{23}b_{31}c_{23}c_{32}$$
$$a_{23} = b_{23}b_{31}c_{12}c_{23} - b_{13}b_{31}c_{13}c_{21} - b_{13}b_{21}c_{12}c_{21}$$
$$a_{31} = b_{12}b_{31}c_{23}c_{31} - b_{12}b_{21}c_{21}c_{32} - b_{21}b_{32}c_{23}c_{32}$$
$$a_{32} = b_{21}b_{32}c_{13}c_{32} - b_{12}b_{31}c_{13}c_{31} - b_{12}b_{21}c_{12}c_{31}.$$

Therefore, we conclude that if the above system of equations is satisfied, then there exists a cyclic triad of triangles known as the graves triad.

Finally we have to show that if (A) circumscribes (B) and (B) circumscribes (C) then these three sets of points $\{C_2, A_1, C_3\}$, $\{C_1, A_2, C_3\}$ and $\{C_1, A_3, C_2\}$ are co-linear.

Since, we have (B) with reference to (A) as

$$(B)_A = \begin{pmatrix} 0 & b_{12} & b_{13} \\ b_{21} & 0 & b_{23} \\ b_{31} & b_{32} & 0 \end{pmatrix}$$

$$(C)_B = \begin{pmatrix} 0 & c_{12} & c_{13} \\ c_{21} & 0 & c_{23} \\ c_{31} & c_{32} & 0 \end{pmatrix}$$

$$(A)_A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence, $(C)_A = (C)_B(B)_A$

$$(C)_A = \begin{pmatrix} c_{12}b_{21} + c_{13}b_{31} & c_{13}b_{32} & c_{12}b_{23} \\ c_{23}b_{31} & c_{21}b_{12} + c_{23}b_{32} & c_{21}b_{13} \\ c_{32}b_{21} & c_{31}b_{12} & c_{31}b_{13} + c_{32}b_{23} \end{pmatrix}.$$

From the above matrices, we got the homogeneous coordinates of C_1, C_2, C_3 are

$$C_1 = (c_{12}b_{21} + c_{13}b_{31}, c_{13}b_{32}, c_{12}b_{23})$$
$$C_2 = (c_{23}b_{31}, c_{21}b_{12} + c_{23}b_{32}, c_{21}b_{13})$$
$$C_3 = (c_{32}b_{21}, c_{31}b_{12}, c_{31}b_{13} + c_{32}b_{23})$$

and the homogeneous coordinates of A_1, A_2, A_3 are

$$A_1 = (1,0,0)$$
$$A_2 = (0,1,0)$$
$$A_3 = (0,0,1).$$

For the co-linearity of C_2, A_1, C_3, it is required that the determinant of C_2, A_1, C_3 is zero

$$\begin{vmatrix} 1 & 0 & 0 \\ c_{23}b_{31} & c_{21}b_{12} + c_{23}b_{32} & c_{21}b_{13} \\ c_{32}b_{21} & c_{31}b_{12} & c_{31}b_{13} + c_{32}b_{23} \end{vmatrix}$$
$$= b_{23}b_{32}c_{23}c_{32} + b_{13}b_{32}c_{23}c_{31} + b_{12}b_{23}c_{21}c_{32} = 0$$

Next, for the co-linearity of C_1, A_3, C_2 it is required that determinant of C_1, A_3, C_2 is zero

$$\begin{vmatrix} 0 & 1 & 0 \\ c_{12}b_{21} + c_{13}b_{32} & c_{13}b_{32} & c_{12}b_{23} \\ c_{32}b_{21} & c_{31}b_{12} & c_{31}b_{13} + c_{32}b_{23} \end{vmatrix}$$
$$= b_{23}b_{31}c_{13}c_{32} + b_{13}b_{31}c_{13}c_{31} + b_{13}b_{21}c_{12}c_{31} = 0.$$

Finally, for the co-linearity of C_1, A_3, C_2 it is required that determinant of C_1, A_3, C_2 is zero

$$\begin{vmatrix} 0 & 0 & 1 \\ c_{12}b_{21} + c_{13}b_{31} & c_{13}b_{32} & c_{12}b_{23} \\ c_{23}b_{31} & c_{21}b_{12} + c_{23}b_{32} & c_{21}b_{13} \end{vmatrix}$$
$$= b_{21}b_{32}c_{12}c_{23} + b_{12}b_{31}c_{13}c_{21} + b_{12}b_{21}c_{12}c_{21} = 0.$$

Since the determinants are zero, the sets of points $\{C_2, A_1, C_3\}$, $\{C_1, A_2, C_3\}$, and $\{C_1, A_3, C_2\}$ are co-linear.

3. Perspective Graves Triad of Triangles

Definition 1: Two simplexes (A) and (B) are called perspective if the lines A_iB_i (lines joining the vertices A_i and B_i) are concurrent at some point P. P is called as center of perspectivity. Here

$$(B)_A = \begin{pmatrix} a_1 & p_2 & p_3 \\ p_1 & a_2 & p_3 \\ p_1 & p_2 & a_3 \end{pmatrix}$$

which, can be denoted as

$$(B)_A = \frac{(a_1, a_2, a_3)}{(p_1, p_2, p_3)}$$

where the parameters $a_1, a_2, a_3 \in F$ and the coordinates of P for (A) are (p_1, p_2, p_3) [1].

Let P be the center of perspective of triangles $(B) = B_1B_2B_3$ and $(A) = A_1A_2A_3$ and coordinates of P are $P = (p_1, p_2, p_3)$. Hence, triangle (B) is in perspective with triangle (A) from the center P, so the line through A_1B_1, A_2B_2, A_3B_3 meet at P. Therefore,

$$\begin{aligned} B_1 &= P + \lambda_1 A_1 \\ &= (P_1, P_2, P_3) + \lambda_1(1,0,0) \\ &= (P_1 + \lambda_1, P_2, P_3) \\ \text{Similarly, } B_2 &= (P_1, P_2 + \lambda_2, P_3) \\ \text{and } B_3 &= (P_1, P_2, P_3 + \lambda_3) \end{aligned}$$

which implies,

$$(B)_A = \begin{pmatrix} P_1 + \lambda_1 & P_2 & P_3 \\ P_1 & P_2 + \lambda_2 & P_3 \\ P_1 & P_2 & P_3 + \lambda_3 \end{pmatrix}.$$

But we have

$$(B)_A = \begin{pmatrix} 0 & b_{12} & b_{13} \\ b_{21} & 0 & b_{23} \\ b_{31} & b_{32} & 0 \end{pmatrix}$$

By row operation $R_2 \to \frac{b_{13}}{b_{23}} R_2$ on the matrices $(B)_A$ we have

$$(B)_A = \begin{pmatrix} 0 & b_{12} & b_{13} \\ \frac{b_{21}b_{13}}{b_{23}} & 0 & b_{13} \\ b_{31} & b_{32} & 0 \end{pmatrix}.$$

Again, by row operation $R_3 \to \frac{b_{21}b_{13}}{b_{23}b_{31}} R_3$ we have

$$(B)_A = \begin{pmatrix} 0 & b_{12} & b_{13} \\ \frac{b_{21}b_{13}}{b_{23}} & 0 & b_{13} \\ \frac{b_{21}b_{13}}{b_{23}} & \frac{b_{32}b_{21}b_{13}}{b_{23}b_{31}} & 0 \end{pmatrix}.$$

Hence, we say that triangle (B) and (A) are in perspective if and only if

$$\frac{b_{32}b_{21}b_{13}}{b_{23}b_{31}} = b_{12}$$

$$\Longrightarrow b_{32}b_{21}b_{13} = b_{12}b_{23}b_{31}.$$

Therefore, we have

$$p_1 = \frac{b_{21}b_{13}}{b_{23}}$$

$$p_2 = b_{12}$$

$$p_3 = b_{13}.$$

So, the coordinates of P are

$$P(\frac{b_{21}b_{13}}{b_{23}}, b_{12}, b_{13}).$$

Theorem 2: *Let A_1, A_2, A_3 be points on sides B_2B_3, B_1B_3, B_1B_2 respectively of triangle $(B) = B_1B_2B_3$ distinct from the vertices, such that B_1A_1, B_2A_2, B_3A_3 are concurrent and let L, M, N be points such that L is the intersection of sides A_2A_3 and B_2B_3, M is the intersection of sides A_1A_3 and B_1B_3, N is the intersection of the sides A_1A_2 and B_1B_2, then the points L, M, N are col-linear.*

Proof: We have B_1A_1, B_2A_2, B_3A_3 are concurrent; let them be concurrent at point P. Therefore, triangle $A_1A_2A_3$ and triangle $B_1B_2B_3$ are in perspective at center P, where coordinates of P are

$$P(\frac{b_{21}b_{13}}{b_{23}}, b_{12}, b_{13}).$$

Since L is the intersection of the side A_2A_3 and B_2B_3 means line through A_2A_3 and B_2B_3 meets at point L. let coordinate of L be: (l_1, l_2, l_3) then we have

$$\begin{aligned} A_3 &= (l_1, l_2, l_3) + k_1A_2 \\ \text{and } B_3 &= (l_1, l_2, l_3) + k_2B_2 \\ \Rightarrow (0,0,1) &= (l_1, l_2, l_3) + k_1(0,1,0) \\ \text{and} \left(\frac{b_{21}b_{13}}{b_{23}}, b_{12}, 0\right) &= (l_1, l_2, l_3) + k_2P\left(\frac{b_{21}b_{13}}{b_{23}}, 0, b_{13}\right) \\ \Rightarrow (l_1, l_2, l_3) &= (0, -k_1, 1) \\ \text{and } (l_1, l_2, l_3) &= ((1-k_2)\frac{b_{21}b_{13}}{b_{23}}, b_{12}, -k_2b_{13}). \end{aligned}$$

Now, equating both the above equations we have $k_1 = -b_{12}, k_2 = 1, b_{13} = -1$.

Hence, we have that $L = (0, b_{12}, 1)$, similarly

$$M = (\frac{b_{21}b_{13}}{b_{23}}, 0,1)$$

$$N = (\frac{b_{21}b_{13}}{b_{23}}, 1,0)$$

where $b_{12} = -1, b_{13} = -1$.

Now,

$$\begin{vmatrix} 0 & b_{12} & 1 \\ \frac{b_{21}b_{13}}{b_{23}} & 0 & 1 \\ \frac{b_{21}b_{13}}{b_{23}} & 1 & 0 \end{vmatrix}$$

By row reduction operation $R_2 \to R_2 - R_3$ we say

$$\begin{vmatrix} 0 & b_{12} & 1 \\ 0 & -1 & 1 \\ \frac{b_{21}b_{13}}{b_{23}} & 1 & 0 \end{vmatrix}$$

Here $b_{12} = -1, b_{13} = -1$

$$\begin{vmatrix} 0 & -1 & 1 \\ 0 & -1 & 1 \\ \frac{-b_{21}}{b_{23}} & 1 & 0 \end{vmatrix} = 0$$

As, determinant is zero so, the points L,M,N are co-linear.

Theorem 3: *Suppose triangle $(A) = A_1A_2A_3$ properly circumscribes triangle $(B) = B_1B_2B_3$ and triangle (B) properly circumscribes $(C) = C_1C_2C_3$. If two pairs of these three are in perspective, then the third pair is also in perspective.*

Proof: Since, triangle $(A) = A_1A_2A_3$ properly circumscribes triangle $(B) = B_1B_2B_3$ and both (A) and (B) are in perspective from center P then

$$(B)_A = \begin{pmatrix} 0 & b_{12} & b_{13} \\ \frac{b_{21}b_{13}}{b_{23}} & 0 & b_{13} \\ \frac{b_{21}b_{13}}{b_{23}} & \frac{b_{32}b_{21}b_{13}}{b_{23}b_{31}} & 0 \end{pmatrix}.$$

Both triangles are in perspective if and only if $b_{32}b_{21}b_{13} = b_{12}b_{23}b_{31}$.

Similarly, we have triangle $(B) = B_1B_2B_3$ properly circumscribes triangle $(C) = C_1C_2C_3$. Therefore, we can say that

$$(C)_B = \begin{pmatrix} 0 & c_{12} & c_{13} \\ c_{21} & 0 & c_{23} \\ c_{31} & c_{32} & 0 \end{pmatrix}$$

By row operation $R_2 \to \frac{c_{13}}{c_{23}} R_2$ on matrix $(C)_B$ we have

$$(C)_B = \begin{pmatrix} 0 & c_{12} & c_{13} \\ \frac{c_{21}c_{13}}{c_{23}} & 0 & c_{13} \\ c_{31} & c_{32} & 0 \end{pmatrix}$$

Again, by row operation $R_3 \to \frac{c_{13}c_{21}}{c_{23}c_{31}} R_3$

$$(C)_B = \begin{pmatrix} 0 & c_{12} & c_{13} \\ \frac{c_{21}c_{13}}{c_{23}} & 0 & c_{13} \\ \frac{c_{21}c_{13}}{c_{23}} & \frac{c_{32}c_{21}c_{13}}{c_{23}c_{31}} & 0 \end{pmatrix}$$

Therefore, triangles (B) and (C) are in perspective iff

$$\frac{c_{32}c_{21}c_{13}}{c_{23}c_{31}} = c_{12}$$
$$\Longrightarrow c_{32}c_{21}c_{13} = c_{12}c_{23}c_{31}.$$

Now, it requires to prove triangles (C) and (A) are in perspective. Since, we have (C) with respect to (A)

$$(C)_A = \begin{pmatrix} c_{12}b_{21} + c_{13}b_{31} & c_{13}b_{32} & c_{12}b_{23} \\ c_{23}b_{31} & c_{21}b_{12} + c_{23}b_{32} & c_{21}b_{13} \\ c_{32}b_{21} & c_{31}b_{12} & c_{31}b_{13} + c_{32}b_{23} \end{pmatrix}$$

Applying row operations $R_2 \to \frac{c_{12}b_{23}}{c_{21}b_{13}} R_2$ and $R_3 \to \frac{c_{23}b_{31}c_{12}b_{23}}{c_{32}b_{21}c_{21}b_{13}} R_3$ we have $(C)_A$

$$(C)_A = \begin{pmatrix} c_{12}b_{21} + c_{13}b_{31} & c_{13}b_{32} & c_{12}b_{23} \\ \frac{c_{23}b_{31}c_{12}b_{23}}{c_{21}b_{13}} & \frac{(c_{21}b_{12}+c_{23}b_{32})(c_{12}b_{23})}{c_{21}b_{13}} & c_{21}b_{13} \\ \frac{c_{23}b_{31}c_{12}b_{23}}{c_{21}b_{13}} & \frac{c_{31}b_{12}c_{23}b_{31}c_{12}b_{23}}{c_{32}b_{21}c_{21}b_{13}} & \frac{(c_{31}b_{13}+c_{32}b_{23})(c_{23}b_{31}c_{12}b_{23})}{c_{32}b_{21}c_{21}b_{13}} \end{pmatrix}$$

So, triangles (C) and (A) are in perspective if and only if

$$\frac{c_{31}b_{12}c_{23}b_{31}c_{12}b_{23}}{c_{32}b_{21}c_{21}b_{13}} = c_{13}b_{32}$$

$$c_{31}b_{12}c_{23}b_{31}c_{12}b_{23} = c_{13}b_{32}c_{32}b_{21}c_{21}b_{13}.$$

Hence $c_{31}b_{12}c_{23}b_{31}c_{12}b_{23} = c_{12}b_{23}c_{32}b_{21}c_{21}b_{13}$ if and only if $b_{32}b_{21}b_{13} = b_{12}b_{23}b_{31}$ and $c_{32}c_{21}c_{13} = c_{12}c_{23}c_{31}$. In other way we say that triangles (C) and (A) are in perspective iff two pairs of triangle (B), (C) and (A), (B) are in perspective. This clearly shows that, if two pairs among the three are in perspective, then the third pair is also in perspective.

4. Conclusion

A set of three triangles are considered, in such a way that they are forming a cyclic order. In this paper homogeneous coordinate system is used to establish the existence of the perspective graves triad. Some of the related results of perspective grave triad are proved using a well-defined system of reference. Finally the perspectivites of triads of triangles are discussed with related to co-linearity.

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