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Special triad of tetrahedrons and σ-transformation in a three-dimensional projective space

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Abstract

Given a pair of perspective tetrahedron (A) and (B) where the vertex B_i is on the opposite face of A_i the center and plane of perspectivity are called the pole and polar of each other respectively for the reference tetrahedron (A). Konully studied a pair of mutually self polar tetrahedrons and Sanyal [9] studied the polar tetrahedrons (C) of (B) for some reference (A). Misra and Sanyal [7] studied a conjugate pair where each of the pair is the polar tetrahedron of the other for (A). In the present study a triad of double perspective tetrahedrons is obtained which under special condition form a special triad where every pair is a conjugate pair for the third. Using a transformation called σ-transformation a family of special triad is obtained, which is fixed for a fixed reference (A) and a point P not on the face of (A). The different centers of perspectivities of the double perspective pairs of this family form desmic systems.

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1. Introduction

Let P_3 be a 3-dimensional projective space over a field F with characteristic not equal to two. Let $(X) = X_1X_2X_3X_4$, $X = A, B$ be two tetrahedrons in P_3. The homogeneous coordinates of the vertices of (B) referred to (A) along with some unit point form the rows of a 4 × 4 matrix M [9, 12]. In symbol it is written as $(B)_A$ to represent (B) with reference to (A) [7, 11]. So if $(B)_A = ((b_{ij}))$, $(A)_C = ((a_{ij}))$ then $(B)_C = ((d_{ij})) = ((b_{ij}))((a_{ij}))$. Also if $(B)_A = ((b_{ij}))$ then $(A)_B = ((b_{ij}))^{-1}$. Hence any matrix N for which $MN = NM = \lambda I$, $(\lambda \neq 0) \in F$ serves the purpose of M^{-1} [7, 14].

Two tetrahedrons (A) and (B) are called perspective if the lines A_iB_i (line joining the vertices A_i and B_i) are concurrent at some point P [1]. In that case the lines common to corresponding faces are on a plane p. P and p are called the center of perspectivity and plane of perspectivity respectively [3, 4, 13]. It is denoted as $(A)\frac{P}{\hat{p}}(B)$ [2].

Here a perspective pair if tetrahedrons is denoted using the following special notation.

$$(B)_A = \begin{bmatrix} a_1 & p_2 & p_3 & p_4 \\ p_1 & a_2 & p_3 & p_4 \\ p_1 & p_2 & a_3 & p_4 \\ p_1 & p_2 & p_3 & a_4 \end{bmatrix} = \frac{(a_1, a_2, a_3, a_4)}{(p_1, p_2, p_3, p_4)} \tag{1.1}$$

Where the parameters $a_1, a_2, a_3, a_4 \in F$ and the coordinates of P for (A) are (p_1, p_2, p_3, p_4) [8].

Let $(A)\frac{P}{\hat{p}}(B)$. If $B_i \in a^i$, the opposite plane of A_i, $\forall i \in \{1,2,3,4\}$ then p is called harmonic polar (or polar) of P and P is called the harmonic pole (or pole) of p for (A) [5]. For two points P and Q, if the polar of P for (A) passes through Q and vice versa then P and Q are called a pair of conjugate points for (A) [7].

If B_i and B_k, $i \neq k$, $k = 1,2,3,4$ are conjugate points for (A) then (B) is called self polar for (A) and (B) are called mutually self polar tetrahedrons [4]. Let (A), (B) and (C) be three tetrahedrons where the faces of (C) are the polar of the vertices of (B) for (A) then (C) is called the polar tetrahedron of (B) for (A) [9]. Let C_k, $i \neq k$, (B) and (C) be three tetrahedrons where $(B)_A = ((b_{ij}))$, $(C)_A = ((c_{ij}))$. (B) and (C) are called a conjugate pair for (A) if B_i and C_k, $i \neq k$ are conjugate point for $(B)_A = \frac{(a_1, a_2, a_3, a_4)}{(p_1, p_2, p_3, p_4)}$. In that case (as in [7])

$$((b_{ij}))\,[((c_{ij}^{\;-1}))]^T = diag(m_1, m_2, m_3, m_4), m_i \neq 0$$

and $$((c_{ij}))\,[((b_{ij}^{-1}))]^T = diag(n_1, n_2, n_3, n_4), n_i \neq 0 \tag{1.2}$$

Two tetrahedrons (A) and (B) are said to be desmic if they are four fold perspective [1, 6, 10].

If $(B)_A = \frac{(-a,b,c,d)}{(a,-b,-c,-d)}$ then (A) is perspective with each of $B_1B_2B_3B_4$, $B_2B_1B_4B_3$, $B_3B_4B_1B_2$, and $B_4B_3B_2B_1$. The four centers of these four perceptivities form the vertices of a third tetrahedron (C) and which is given as $(C)_A = \frac{(-a,-b,-c,-d)}{(a,b,c,d)}$.. ($A$), ($B$) and ($C$) form a desmic system of tetrahedrons.

Let P, Q, R and S are four collinear points where R and S are expressed in terms of P and Q as follows. $R = \lambda P + \mu Q$ and $S = \lambda' P + \mu' Q$. Then the cross ratio of these four points denoted by (PQ, RS) is defined as $(PQ, RS) = \frac{\lambda\mu'}{\lambda'\mu}$.

2. Doubly perspective tetrahedrons

Let s be a permutation of order 2 on the set {1, 2, 3, 4}. It may be of the type $(1i)(jk)$, $i \neq j \neq k$ or of the type $(i)(j)(k1)$, $i \neq j \neq k \neq 1$. If $(X) = X_1X_2X_3X_4$ and $s = (12)(34)$ then $(X_s) = X_2X_1X_4X_3$. In the present study all the results are proved taking $s = (12)(34)$. But it can be proved similarly for any permutation of order two on the set {1, 2, 3, 4}.

Definition 2.1: If $(A)\frac{p_1}{\hat{p}_1}(B)$ and $(A)\frac{p_2}{\hat{p}_2}(B_s)$ then (A) and (B) are called a pair of doubly perspective tetrahedrons.

Let $P_1 = c$ be a point not on the face of (A) and (A) and (B) are perspective from the center P_1. Then $(B)_A = \frac{(a_1,a_2,a_3,a_4)}{(p_1,p_2,p_3,p_4)}$. The parametric expressions of the lines A_1B_2 and A_2B_1 are $\lambda A_1 + B_2$ and $\mu A_2 + B_1$ respectively for λ and $\mu \in F$. So, they meet at (a_1, a_2, p_3, p_4). Similarly, A_3B_4 and A_3B_4 intersect at (p_1, p_2, a_3, a_4). If (A) and (B_s) are perspective from some center P_2 then both these points (a_1, a_2, p_3, p_4) and (p_1, p_2, a_3, a_4) has to be equal to P_2. This gives $a_1 = p_1k$, $a_2 = p_2k$, $a_3 = p_3k^{-1}$, $a_4 = p_4k^{-1}$ for some non zero $k \in F$. So

$$(B)_A = \frac{(p_1k,\ p_2k,\ p_3k^{-1},\ p_4k^{-1})}{(p_1,\ p_2,\ p_{3,}\ p_4)} \tag{2.1}$$

and $$(B_S)_A = \frac{(p_1k^{-1},\ p_2k^{-1},\ p_3,\ p_4)}{(p_1,\ p_2,\ p_3k^{-1},\ p_4k^{-1})} \tag{2.2}$$

gives $P_2 = (p_1, p_2, p_3k^{-1}, p_4k^{-1})$. Now let $P_1P_2 \,.\, A_1A_2 = A_{12}$ and $P_1P_2 \,.\, A_3A_4 = A_{34}$ then $A_{12} = (p_1, p_2, 0, 0)$ and $A_{34} = (0, 0, p_3, p_4)$. Let Q_1 and Q_2 be two other points so that the cross ratios $(A_{12}A_{34}, P_2Q_1) = (A_{12}A_{34}, P_1Q_2) = -1$ then $Q_1 = (p_1, p_2, -p_3k^{-1}, -p_4k^{-1})$ and $Q_2 = (p_1, p_2, -p_3, -p_4)$. If (C) be a tetrahedron such that $(A)\frac{Q_1}{\pi}(C)$ and $(A)\frac{Q_2}{\pi}(C_s)$ then

$$(C)_A = \frac{(p_1k^{-1}, p_2k^{-1}, -p_3, -p_4)}{(p_1, p_2, -p_3k^{-1}, -p_4k^{-1})} \tag{2.3}$$

Using (1.2) it can be observed that (B) and (C) are a conjugate pair for (A). Hence, we have the following theorem.

Theorem 2.1*:* *Let* $(A)\frac{P_1, P_2}{\pi}(B)$ *and* $A_{12} = P_1P_2 \,.\, A_1A_2$, $A_{34} = P_1P_2 \,.\, A_3A_4$. *If* (C) *be a third tetrahedron such that* $(A)\frac{Q_1, Q_2}{\pi}(C)$ *where* Q_1 *and* Q_2 *are the harmonic conjugate of* P_2 *and* P_1 *respectively with respect to* A_{12} *and* A_{34} *then* (B) *and* (C) *form a pair of conjugate tetrahedrons for* (A).

The validity of the following proposition is immediate.

Proposition 2.1: If $(B)\frac{A_{12}}{\pi}C_1C_2C_4C_3$ and $(B)\frac{A_{34}}{\pi}C_2C_1C_3C_4$, then (A), (B) and (C) form a triad of mutually doubly perspective tetrahedrons and is denoted as $\{(A), (B), (C)\}$.

Let $B_{ij} = B_iB_j.P_1P_2$ and $C_{ij} = C_iC_j.Q_1Q_2$. Then

$$B_{12} = ((1+k)p_1, (1+k)p_2, 2p_3, 2p_4)),\ B_{34} = (2kp_1, 2kp_2, (1+k)p_3, (1+k)p_4)$$
$$C_{12} = ((1+k)p_1, (1+k)p_2, -2p_3, -2p_4)), C_{34} = (2kp_1, 2kp_2, -(1+k)p_3, -(1+k)p_4) \tag{2.4}$$

where
$$(A_{12}A_{34}, B_{34}C_{34}) = (A_{12}A_{34}, B_{12}C_{12}) = -1 \tag{2.5}$$

$$(A_{12}A_{34}, P_2P_1) = (B_{12}B_{34}, P_2P_1) = (A_{12}A_{34}, Q_1Q_2) = (C_{12}C_{34}, Q_1Q_2) = k \tag{2.6}$$

and
$$(B_{12}B_{34}, A_{12}C_{12}) = (B_{12}B_{34}, C_{34}A_{34}) = (C_{12}C_{34}, A_{12}B_{12}) = (C_{12}C_{34}, B_{34}A_{34})$$

$$= \frac{2(k+1)^2}{k^2+6k+1} \tag{2.7}$$

Note: For $k = -1$, (A) and (B) form a desmic pair and (C) coincides with (B).

3. Special triad of tetrahedrons

Definition 3.1: A set of three tetrahedrons are said to form a special triad if any two of them are conjugate for the third. Any two of these tetrahedrons fix the third and hence are called the generators of the third.

Theorem 3.1: *Let* $(A)\xrightarrow[\pi]{P_1,P_2}(B)$ *and* $A_{12}=P_1P_2.A_1A_2, A_{34}=P_1P_2.A_3A_4$. (A) *and* (B) *generate a special triad if* $(A_{12}A_{34},P_1P_2)=-3$ *or* $\frac{-1}{3}$.

Proof: Starting with the tetrahedrons (A) and (B) of article 2 where (C) is so constructed that (B) and (C) are doubly perspective and form a conjugate pair for (A). It can be shown that (A), (B) and (C) form a special triad if the value of k in their expression is equal to –3 or $\frac{-1}{3}$ (as $k^{-1}=(A_{12}A_{34},P_1P_2)$). This will prove the theorem. Here the proof is done for $k=-3$. The other case follows similarly.

For $k=-3$

$$P_1=(p_1,p_2,p_3,p_4),\ P_2=(3p_1,3p_2,-p_3,-p_4),$$
$$Q_1=(3p_1,3p_2,p_3,p_4),\ Q_2=(p_1,p_2-p_3,-p_4).$$

From (2.4) it is observed that for $k=-3$, $P_1=C_{12}$, $P_2=C_{34}$, $Q_1=B_{34}$, $Q_2=B_{12}$. Hence from (2.7)

$$(B_{12}B_{34},A_{12}C_{12})=(B_{12}B_{34},C_{34}A_{34})=(C_{12}C_{34},A_{12}B_{12})=(C_{12}C_{34},B_{34}A_{34})=-1$$

i.e., $(Q_2Q_1,A_{12}P_1)=(Q_2Q_1,P_2A_{34})=(P_1P_2,A_{12}Q_2)=(P_1P_2,Q_1A_{34})=-1$

i.e., $$(Q_1Q_2,P_1A_{12})=(Q_1Q_2,P_2A_{34})=-1$$

and $$(P_1P_2,Q_2A_{12})=(P_1P_2,Q_1A_{34})=-1 \tag{3.1}$$

At this stage $(B)\xrightarrow[\pi]{P_1,P_2}(A)$, $(B)\xrightarrow[\pi]{A_{34},A_{12}}$(C), $B_{12}=P_1P_2\ .\ B_1B_2=Q_2$, $B_{34}=P_1P_2\ .\ B_3B_4=Q_2$ and $(Q_1Q_2,\ P_1A_{12})$ = $(Q_1Q_2,\ P_2A_{34})$ = -1, So applying theorem 2.1 it is obtained that (A) and (C) are conjugate for (B).

Similarly as $(C)\xrightarrow[\pi]{Q_1,Q_2}(A)$, $(C)\xrightarrow[\pi]{A_{34},A_{12}}(B)$, $C_{12}=Q_1Q_2\ .\ C_1C_2=P_1$, $C_{34}=Q_1Q_2\ .\ C_3C_4=P_2$ and (P_1P_2,Q_2A_{12}) = (Q_1Q_2,Q_1A_{34}) = -1, So applying theorem 2.1 it is obtained that (A) and (B) are conjugate for (C).

Again as $(B)\xrightarrow[\pi]{A_{34}}C_2C_1C_3C_4$ and $(C)\xrightarrow[\pi]{A_{12}}B_1B_2B_4B_3$ it is obtained that (A) and $C_2C_1C_3C_4$ are conjugate in this order for (B) and (A) and $B_1B_2B_4B_3$ are conjugate in this order for (C).

4. σ - Transformation

Let

$$D_1 = C_1C_4.B_2B_4, D_2 = C_2C_3.B_1B_3, D_3 = C_2C_3.B_2B_4, D_4 = C_1C_4.B_1B_3$$

$$E_1 = B_1B_4.C_2C_4, E_2 = B_2B_3.C_1C_3, E_3 = B_2B_3.C_2C_4, E_4 = B_1B_4.C_1C_3 \tag{4.1}$$

where

$$D_1 = (p_1, -kp_2, p_3, -p_4),\ D_2 = (kp_1, -p_2, p_3, -p_4),$$

$$D_3 = (p_1, -p_2, p_3, -p_4k^{-1}),\ D_4 = (p_1, -p_2, p_3k^{-1}, -p_4),$$

$$E_1 = (kp_1, -p_2, -p_3, p_4), E_2 = (p_1, -kp_2, -p_3, p_4),$$

$$E_3 = (p_1, -p_2, -p_3k^{-1}, p_4), E_4 = (p_1, -p_2, -p_3, p_4k^{-1}).$$

$$(D)_A = \frac{(p_1k^{-1}, -p_2k^{-1}, p_3, -p_4)}{(p_1, -p_2, p_3k^{-1}, -p_4k^{-1})}, (E)_A = \frac{(p_1k, -p_2k, -p_3k^{-1}, p_4k^{-1})}{(p_{1,} - p_2, -p_3, p_4)}.$$

So $(A)\frac{R_1}{\pi}(D), (A)\frac{R_2}{\pi}(D_S), (A)\frac{S_1}{\pi}(E)$ and $(A)\frac{S_2}{\pi}(E_S)$ and (A), (D) and (E) form a triad of mutually doubly perspective tetrahedrons and (D) and (E) are conjugate for (A). As in proposition 2.1 here (D) and (E) also perspective from the centers

$$A'_{12} = (p_1, -p_2, 0, 0), A'_{34} = (0, 0, p_3, -p_4)$$

In (4.1) the vertices D_i and E_i, hence the tetrahedrons (D) and (E) are generated from (B) and (C). This process is denoted as a transformation in the following definition.

Definition 4.1: The transformation which transforms the pair $[(B),(C)]$ to $[(D),(E)]$ as in (4.1) is called σ - transformation and denoted as $[(B),(C)]\sigma = [(D),(E)]$.

It can be observed that the order of σ is two.

Note: The expressions of D_i and E_i in terms in terms of B_i and C_i as given in (4.1) depend on the way the tetrahedrons (B) and (C) are doubly perspective. Here (B) is perspective with $C_2C_1C_3C_4$ and $C_1C_2C_4C_3$. If (B) will perspective with $C_2C_1C_4C_3$ then the expressions will be different i.e.

σ - transformation depends on the way the original tetrahedrons are doubly perspective.

Theorem 4.1: *If* $\{(A),(B),(C)\}$ *is a special triad and if* $[(B),(C)]\sigma = [(D),(E)]$ *then* $\{(A),(D),(E)\}$ *is also a special triad.*

Proof: Given the point $P_1 = (p_1, p_2, p_3, p_4)$ one can construct uniquely the tetrahedrons (B) and (C) as constructed in article 2 where (A) is the reference tetrahedron. Since one can choose P_1 arbitrarily (in this case P_1 should not be on any face of (A)) its coordinates can be taken as $(p_1, -p_2, p_3, -p_4)$ and the tetrahedrons (D) and (E) are obtained instead of (B) and (C). So the relation among (A) , (B) and (C) is preserved in (A), (D) and (E). This implies that if {(A), (B), (C)} is a special triad then {(A), (D), (E)} is also a special triad.

Let us apply σ - transformation to the doubly perspective pairs [(A), (B)] and [(A), (C)] to obtain two new pairs [(K), (L)] and [(M), (N)] as follows.

$$K_1 = A_1A_4.B_2B_4 = B_2 - kB_3 = (p_1, 0, 0, p_4),$$

$$K_2 = A_2A_3.B_2B_3 = B_2 - B_3 = (0, kp_2, p_3, 0),$$

$$K_3 = A_1A_4.B_1B_4 = B_1 - B_4 = (kp_1, 0, 0, p_4),$$

$$K_4 = A_2A_3.B_1B_4 = B_1 - kB_4 = (0, p_2, p_3, 0),$$

$$L_1 = A_1A_3.B_2B_4 = B_2 - kB_4 = (p_1, 0, p_3, 0),$$

$$L_2 = A_2A_4.B_2B_4 = B_2 - B_4 = (0, kp_2, 0, p_4),$$

$$L_3 = A_1A_3.B_1B_3 = B_1 - B_3 = (kp_1, 0, p_3, 0),$$

$$L_4 = A_2A_4.B_1B_3 = B_1 - kB_3 = (0, p_2, 0, p_4),$$

$$M_1 = A_1A_3.C_2C_4 = -kC_2 + C_4 = (kp_1, 0, -p_3, 0),$$

$$M_2 = A_2A_4.C_2C_4 = C_2 - C_4 = (0, p_2, 0, -p_4),$$

$$M_3 = A_1A_3.C_1C_3 = C_1 - C_3 = (p_1, 0, -p_3, 0),$$

$$M_4 = A_2A_4.C_1C_3 = kC_1 - C_3 = (0, kp_2, 0, -p_4),$$

$$N_1 = A_1A_4.C_2C_3 = kC_2 - C_3 = (kp_1, 0, 0, -p_4),$$

$$N_2 = A_2A_3.C_2C_3 = C_2 - C_3 = (0, p_2, -p_3, 0),$$

$$N_3 = A_1A_4.C_1C_4 = C_1 - C_4 = (p_1, 0, 0, -p_4),$$

$$N_4 = A_2A_3.C_1C_4 = kC_1 - C_4 = (0, kp_2, -p_3, 0).$$

As $[(A),(B)]\sigma = [(K),(L)]$ and $[(A),(C)]\sigma = [(M),(N)]$ applying theorem 4.1 one gets the following.

Theorem 4.2: {(C), (K), (L)} *and* {(B), (M), (N)} *are special triads if* {(A), (B), (C)} *is a special triads.*

The vertices of (K) *and* (M) *can be expressed in the following way also.*

$$K_1 = A_1A_4.E_3E_4, K_2 = A_2A_3.E_2E_3, K_3 = A_1A_4.E_1E_4, K_4 = A_2A_3.E_1E_4,$$

$$M_1 = A_1A_3.E_1E_3,\ M_2 = A_2A_4.E_1E_3, M_3 = A_1A_3.E_2E_4, M_4 = A_2A_4.E_2E_4.$$

Which show that $[(A),(E)]\sigma = [(K),(M)]$. Similarly, $[(A),(D)]\sigma = [(L),(N)]$. Hence

Theorem 4.3: *If* {(A), (B), (C)} *is a special triad then* {(D), (K), (M)} *and* {(E), (L), (N)} *are also special triads.*

The nine tetrahedrons (A), (B), (C), (D), (E), (K), (L), (M) *and* (N) *obtained so far can be arranged in the following matrix scheme as*

$$\begin{bmatrix} A & B & C \\ D & M & K \\ E & N & L \end{bmatrix} \tag{4.2}$$

where tetrahedrons of each row or column form a special triad if the value of k is -3 *or* $\frac{-1}{3}$.

5. The twin- family of special triad

In article 2 the process of generating a triad of mutually doubly perspective tetrahedrons started with (*A*) and a point p_1 not on the faces of (*A*). With the help of $\sigma-$ transformation we obtain two triad of mutually doubly perspective tetrahedrons {(*A*), (*B*), (*C*)} and {(*A*), (*D*), (*E*)} where (*A*) is the common tetrahedron and

$(A)\frac{P_1,P_2}{\pi}(B)$, $(A)\frac{Q_1,Q_2}{\pi}(C)$, $(A)\frac{R_1,R_2}{\pi}(D)$, $(A)\frac{S_1,S_2}{\pi}(E)$ where the tetrahedrons $(V)=P_1Q_2R_2S_1$ and $(W)=P_2Q_1R_1S_2$ are expressed as

$$(V)_A=\begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_1 & p_2 & -p_3 & -p_4 \\ p_1 & -p_2 & p_3 & -p_4 \\ p_1 & -p_2 & -p_3 & p_4 \end{bmatrix}=\frac{(-p_1,p_2,p_3,p_4)}{(p_1,-p_2,-p_3,-p_4)}$$

and $$(W)_A=\begin{bmatrix} p_1 & p_2 & p_3k^{-1} & p_4k^{-1} \\ p_1 & p_2 & -p_3k^{-1} & -p_4k^{-1} \\ p_1 & -p_2 & p_3k^{-1} & -p_4k^{-1} \\ p_1 & -p_2 & -p_3k^{-1} & p_4k^{-1} \end{bmatrix}=\frac{(-p_1,p_2,p_3k^{-1},p_4k^{-1})}{(p_1,-p_2,-p_3k^{-1},-p_4k^{-1})}$$

So (*V*) and (*A*) are desmic as well as (*W*) and (*A*) are. So there exists the third desmic tetrahedron of each pair, say, (V') and (W') respectively, where

$$(V')_A=\frac{(-p_1,-p_2,-p_3,-p_4)}{(p_1,p_2,p_3,p_4)} \text{ and } (W')_A=\frac{(-p_1,-p_2,-p_3k^{-1},-p_4k^{-1})}{(p_1,p_2,p_3k^{-1},p_4k^{-1})}.$$

As the vertices P_i,Q_i,R_i and $S_i(i=1,2)$ of the tetrahedrons (*V*) and (*W*) are the centers of perceptivities of the pairs [(*A*), (*B*)], [(*A*), (*C*)], [(*A*), (*D*)] and [(*A*), (*E*)] the vertices of (V') and (W') are also centers of perceptivities of the pairs $[(A),(B')]$, $[(A),(C')]$, $[(A),(D')]$ and $[(A),(E')]$ where

$$(B')_A=\frac{(-p_1k,p_2k,p_3k^{-1},p_4k^{-1})}{(-p_1,p_2,p_3,p_4)},(C')_A=\frac{(p_1k^{-1},-p_2k^{-1},p_3,p_4)}{(p_1,-p_2,p_3k^{-1},p_4k^{-1})},$$

$$(D')_A=\frac{(p_1k^{-1},p_2k^{-1},-p_3,p_4)}{(p_1,p_2,-p_3k^{-1},p_4k^{-1})},(E')_A=\frac{(p_1k,p_2k,p_3k^{-1},-p_4k^{-1})}{(p_1,p_2,p_3,-p_4)}.$$

Here $\{(A),(B'),(C')\}$ and $\{(A),(D'),(E')\}$ are special triads for $k=-3$ or $\frac{-1}{3}$. In the same way as in article 4 $(K'),(L'),(M')$ and (N') are generated applying σ - transformation to the perspective pairs and all these tetrahedrons can be arranged in the following scheme as

$$\begin{bmatrix} L' & N' & E' & & \\ K' & M' & D' & & \\ C' & B' & A & B & C \\ & & D & M & K \\ & & E & N & L \end{bmatrix}$$

In the above arrangement we have two families of special triads as in (4.2) where the tetrahedron (A) is common. It is called the twin family of special triads. Here every row or column of both the matrix in the arrangement forms a special triad. This twin family is fixed for a fixed reference tetrahedron (A) and a fixed point p_1 not on the face of (A).

Conclusion

A triad of double perspective tetrahedrons is obtained under some condition form a special triad where every pair is a conjugate pair for the third. Using a transformation called s-transformation a family of special triad is obtained, which is fixed for a fixed reference tetrahedron and a point not on the face of reference tetrahedron. Finally it has been observed that the different centers of perspectivities of the double perspective pairs for this family form desmic systems.

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