Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 29 (2026), No. 6, pp. 1805–1817
DOI : 10.47974/JIM-2316

Positivity preserving zipper rational quintic fractal interpolation function

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Abstract

A new class of zipper rational quintic fractal interpolation function (ZRQFIF) characterized by two shape parameters has been introduced. Further, we discuss the convergence of this function and to ensure the shape preservation, the constraints imposed on scaling factor as well as on shape parameter are studied. Examples have been provided to support the theoretical part. The proposed function represents a significant development in the shape preservation field by offering a powerful tool for visualizing the data.

Subject Classification: *28A80, 26A18, 41A05, 41A29.*

Keywords: *Fractal, IFS, FIF, Zipper, Positivity.*

1. Introduction

The interpolation techniques are plays an important role in visualization of the discrete data. The functions with irregular data sets can be approximated using classical interpolation as the derivative of these interpolants are smooth globally and some of them are smooth piecewise, but the hidden shape preserving properties cannot be preserved using this interpolation. But the fractal interpolation is the ideal method for irregularity. The word fractal was coined by the mathematician Mandelbrot in 1975 and the fractal interpolation function was given by

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Barnsley [1] in 1986. Fractal interpolation is used widely in many real world applications, like: biology, finance and others. The k-times differentiable FIF was introduced by Barnsley and Harrington [2] and Navascués and Sebastian [3] gave smooth FIF in 2006. The shape preserving properties of FIFs are the growing interest in this field of research, as it can be used in many scientific and economic applications. The rational splines are more flexible as compared to others and this property makes them more usable, for detailed review see [4]–[7]. The local non-affine -FIF is introduced by Banerjee, Akhtar, and Navascués [8]. In the past two to three decades, many researchers have worked on fractal dimension, fractal surface and their applications, see [9]–[16].

The flexible method to construct a fractal is using the iterated function system known as Zipper [17], [18] and this flexibility is provided by the presence of signature.

The purpose of this paper is to construct a ZRQFIF and to discuss it positivity property. The construction of ZRQFIF is given in this paper and the paper is organized as follow: a short review zipper-FIF is given in the second section, the construction of ZRQFIF is given, here we have consider the rational function $\frac{U_{l,\varepsilon}(q)}{V_{l,\varepsilon}(q)}$, here $Q_{l,\varepsilon}(q)$ a linear polynomial having two shape parameters. In the fourth section, the convergence of ZRQFIF is discussed and in the fifth section, the positivity conditions of ZRQFIF are given.

2. Basics of Zipper Fractal Interpolation Functions

Definition: Consider a non-surjective map $\mu_l;\ l \in \wedge = 1,2,..., M-1$ on a complete metric space X. Then the system $\Im = \{X, \mu_l; l \in \wedge\}$ is known as the zipper iterated function system with the vertices $(v_1, v_2, \ldots, v_M)$ and signature is a binary-valued parameter defined as $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{M-1}) \in \{0,1\}^{M-1}$, where each ε_l takes value either 0 or 1, if $\mu_l(v_1) = v_{l+\varepsilon_l}$, $\mu_l(v_M) = v_{l+1-\varepsilon_l}\ \forall\ l \in \wedge$. And $A \subset X$ is the compact set which satisfies

$$A = \bigcup_{j=1}^{M-1} \mu_j(A)$$

and is known as an attractor associated with the zipper $\Im$. The review of zipper FIF is based on [1], [18] yi.

Consider the dataset $\{(q_m, z_m) \in R^2 : m \in \wedge^* = 1,2,...,M\}$ with $y_1 < y_2 < \ldots < y_M$. Consider $\ell = [q_1, q_M]$ & $\ell_l = [q_l, q_{l+1}]\ \forall\ l \in \wedge$. Let $L_l : \ell \to \ell_l$ be an affine map such that

$$L_l(q_1) = q_{l+\varepsilon_l}, L_l(q_M) = q_{l+1-\varepsilon_l} \tag{1}$$

For $l \in \wedge$, let the function $F_l : K^* := \ell \times [c,d] \to R$ of the form

$$F_l(q,z) = \delta_l z + Q_l(q)$$

δ_l is the V.S.F. and $Q_l : \ell \to R$ be the continuous function, so as $|\delta_l| = \max\{|\delta_l| : l \in \wedge\} < 1$ & following condition holds:

$$F_l(q_1, z_1) = z_{l+\varepsilon_l}, F_l(q_M, z_M) = z_{l+1-\varepsilon_l}, l \in \wedge. \tag{2}$$

Define the map $\mu_l : K^* \to \ell_l \times R$ by

$$\mu_l(q,z) = (L_l(q), F_l(q,z)) \forall (q,z) \in K^*$$

The system $\Im = \{K^*; \mu_l : l \in \wedge\}$ is the zipper IFS with signature $\varepsilon \in \{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{M-1}\}$ and vertices $\{(q_m, z_m), m \in \wedge^*\}$. The following result can be seen in [18].

Theorem 1: *For the IFS* $\{\ell \times K^*; \mu_l : l \in \wedge\}$, *the following conditions hold:*

- There exist a compact set $H \subset \ell \times K^*$, which is unique, so as $H = \bigcup_{l=1}^{M-1} \mu_l(H)$.
- *H be the graph of* $g_\varepsilon^\delta : \ell \to R$, *which interpolates the data* $\{(q_m, z_m) \in R^2 : m \in \wedge^*\}$ *and that is* $H = \{(q, g_\varepsilon^\delta(q)) : q \in \ell\}$ & $g_\varepsilon^\delta(q_l) = z_l$.

3. Construction of Zipper RQFIF

Consider zipper rational IFS, where the rational function is $Q_{l,\varepsilon}(q) = \frac{U_{l,\varepsilon}(q)}{V_{l,\varepsilon}(q)}$,

$$U_{l,\varepsilon}(q) = A_{l,\varepsilon}(1-\theta)^5 + B_{l,\varepsilon}(1-\theta)^4\theta + C_{l,\varepsilon}(1-\theta)^3\theta^2 + D_{l,\varepsilon}(1-\theta)^2\theta^3 + E_{l,\varepsilon}(1-\theta)\theta^4 + F_{l,\varepsilon}\theta^5, V_l(q) = s_l(1-\theta) + t_l\theta$$

for $0 \le \theta = \frac{q-q_1}{q_M - q_1} \le 1$, s_l & t_l are the non-negative shape parameters. Consider a subspace $C^*(\ell) = \{g : \ell \to R,\ g(q_1) = z_1,\ g(q_M) = z_M\}$ of $C(\ell)$ and is complete w.r.t uniform norm. Let $\varepsilon_l \in \{0,1\}^{M-1}$ is fixed then R-B operator is defined as $T_\varepsilon^\delta : C^*(\ell) \to C^*(\ell)$:

$$T_\varepsilon^\delta(L_l(q)) = F_l(q, g(q))$$

$$= \delta_l g(q) + \frac{1}{s_l(1-\theta) + t_l\theta} \begin{pmatrix} A_{l,\varepsilon}(1-\theta)^5 + B_{l,\varepsilon}(1-\theta)^4\theta + C_{l,\varepsilon}(1-\theta)^3\theta^2 \\ + D_{l,\varepsilon}(1-\theta)^2\theta^3 + E_{l,\varepsilon}(1-\theta)\theta^4 + F_{l,\varepsilon}\theta^5 \end{pmatrix}$$

T_ε^δ be a contraction on $(C^*(\ell), \|.\|_\infty)$, then according to the theorem given by Banach for fixed points , T_ε^δ will possess fixed point S_ε^δ which satisfy:

$$S_\varepsilon^\delta(L_l(q)) = \delta_l S_\varepsilon^\delta(q) + Q_{l,\varepsilon}(q) \tag{3}$$

$$= \delta_l S_\varepsilon^\delta(q) + \frac{1}{s_l(1-\theta)+t_l\theta}\begin{pmatrix} A_{l,\varepsilon}(1-\theta)^5 + B_{l,\varepsilon}(1-\theta)^4\theta + C_{l,\varepsilon}(1-\theta)^3\theta^2 \\ +D_{l,\varepsilon}(1-\theta)^2\theta^3 + E_{l,\varepsilon}(1-\theta)\theta^4 + F_{l,\varepsilon}\theta^5 \end{pmatrix}$$

The conditions given in equation (2) can be used for calculating the coefficients in above expression and it is similar to derive it from $S_\varepsilon^\delta(q_l) = q_{l+\varepsilon_l}$ & $S_\varepsilon^\delta(q_{l+1}) = q_{l+1-\varepsilon_l}$. As stated in [2] the scaling factor is selected as $|\delta_l| < a_l^2 \ \forall\ l \in \wedge$ for a differentiable FIF. The following conditions are needed for constructing the C^2 ZRQFIF:

$$S_\varepsilon^\delta(L_l(q_1)) = z_{l+\varepsilon_l}, S_\varepsilon^\delta(L_j(q_M)) = z_{l+1-\varepsilon_l},$$

$$(S_\varepsilon^\delta)^{(1)}(L_l(q_1)) = d_{l+\varepsilon_l}, (S_\varepsilon^\delta)^{(1)}(L_l(q_M)) = d_{l+1-\varepsilon_l}$$

$$\&(S_\varepsilon^\delta)^{(2)}(L_l(q_1)) = D_{l+\varepsilon_l}, (S_\varepsilon^\delta)^{(2)}(L_l(q_M)) = D_{l+1-\varepsilon_l}.$$

All the constants in $U_{l,\varepsilon}$ are calculated by using the above conditions:

$$A_{l,\varepsilon} = s_l(z_{l+\varepsilon_l} - \delta_l z_1) \,\&\, F_{l,\varepsilon} = t_l(z_{l+1-\varepsilon_l} - \delta_l z_M)$$

$$B_{l,\varepsilon} = (4s_l + t_l)(z_{l+\varepsilon_l} - \delta_l z_1) + s_l(h_l d_{l+\varepsilon_l} - \delta_l(q_M - q_1)),$$

where, $h_l = q_{l+1-\varepsilon_l} - q_{l+\varepsilon_l}$

$$E_{l,\varepsilon} = (s_l + 4t_l)(z_{l+1-\varepsilon_l} - \delta_l z_M) - t_l(h_l d_{l+1-\varepsilon_l} - \delta_l(q_M - q_1)d_M)$$

$$C_{l,\varepsilon} = (6s_l + 4t_l)\begin{pmatrix} z_{l+\varepsilon_l} \\ -\delta_l z_1 \end{pmatrix} + (3s_l + t_l)\begin{pmatrix} h_l d_{l+\varepsilon_l} \\ -\delta_l(q_M - q_1)d_1 \end{pmatrix} + \frac{1}{2}s_l\begin{pmatrix} h_l^2 D_{l+\varepsilon_l} \\ -\delta_l(q_M - q_1)^2 D_1 \end{pmatrix}$$

$$D_{l,\varepsilon} = (4s_l + 6t_l)\begin{pmatrix} z_{l+1-\varepsilon_l} \\ -\delta_l z_M \end{pmatrix} - (s_l + 3t_l)\begin{pmatrix} h_l d_{l+1-\varepsilon_l} - \delta_l \\ (q_M - q_1)d_M \end{pmatrix} + \frac{1}{2}t_l\begin{pmatrix} h_l^2 D_{l+1-\varepsilon_l} - \delta_l \\ (q_M - q_1)^2 D_M \end{pmatrix}.$$

Hence we get the required Zipper RQFIF.

Remark: If $\varepsilon_l = 0 = \delta_l$, then the ZRQFIF reduces to the RQ interpolant [19], and can be written as

$$P(q) = \frac{u_l(\theta)}{v_l(\theta)}, \theta = \frac{q - q_1}{q_M - q_1}, q \in \ell. \tag{5}$$

$$u_l(\theta) = s_l z_l (1-\theta)^5 + (4s_l + t_l) z_l + s_l \hbar_l d_l (1-\theta)^4 \theta + (6s_l + 4t_l) z_l + (3s_l + t_l) \hbar_l d_l$$
$$+ \frac{1}{2} s_l \hbar_l^2 D_l (1-\theta)^3 \theta^2 + (4s_l + 6t_l) z_l - (s_l + 3t_l) \hbar_l d_{l+1} + \frac{1}{2} t_l \hbar_l^2 D_{l+1} (1-\theta)^2 \theta^3$$
$$+ (s_l + 4t_l) z_{l+1} - s_l \hbar_l d_{l+1} . (1-\theta)\theta^4 + t_l z_{l+1} \theta^5 \;\&\; v_l(\theta) = s_l (1-\theta) + t_l \theta.$$

4. Convergence

Theorem 2: [19] *Let* $f \in C^3[q_1, q_M]$ *and* $\{(q_m, z_m) \in R^2 : m \in \wedge^*\}$ *be the data set. Let P is the classical quintic rational spline associated with f, then the error between f & P in every subinterval* $[q_l, q_{l+1}]$ *is,*

$$| f(q) - P(q) | \le e \hbar_l^3 \, \| f^{(3)}(q) \| \tag{6}$$

where,

$$e = \max_{1 \le l \le M-1}, e_l = \max_{0 \le \theta \le 1} \sigma(s_l, t_l, \theta)$$

$$\sigma(s_l, t_l, \theta) = \left\{ \begin{array}{l} \max \sigma_1(s_l, t_l, \theta) \text{ for } 0 \le \theta \le \dfrac{2s_l + t_l}{3(s_l + t_l)} \\ \max \sigma_2(s_l, t_l, \theta) \text{ for } \dfrac{2s_l + t_l}{3(s_l + t_l)} \le \theta \le 1 \end{array} \right\}.$$

Theorem 3: *Let* $f \in C^3[q_1, q_M]$ *and* $\Delta = \{q_1, q_2, ..., q_M\}$ *be the partition* ℓ *satisfying the condition* $q_1 < q_2 < ... < q_M$. *Let* S^δ *be the RQFIF, interpolating the values of f on the points of* Δ, *then*

$$\| f - S^\delta \|_\infty \le e \hbar_l^3 \, \| f^{(3)} \|_\infty + \frac{| \delta |_\infty}{1 - | \delta |_\infty} \{ \Xi(\hbar) + \Xi^*(\hbar) \}$$

where,

$$\Xi(\hbar) = \| f \|_\infty + \hbar_l \{ \theta(1-\theta) + \theta^2 (1-\theta)^2 \} | d |_\infty + \frac{1}{2} \hbar_l^2 (\theta^2 (1-\theta)^2) | D |_\infty$$

$$\Xi^*(\hbar) = \| z \|_{1,l} + \hbar_l \{\theta(1-\theta) + \theta^2(1-\theta)^2\} |d|_{1,l} + \frac{1}{2}\hbar_l^2(\theta^2(1-\theta)^2)|D|_{1,l}$$

and

$$e_l = \max_{0\le\theta\le1} \sigma(s_l, t_l, \theta); 0 \le \theta \le 1, |d|_\infty = \max\{|d|_l; 1 \le l \le M\},$$

$$|D|_\infty = \max\{|D|_m; 1 \le m \le M\}, |d|_{1,M} = \max\{|d|_1, |d|_M, \} |D|_{1,M} = \max\{|D|_1, |D|_M\}.$$

Proof: Now $S^\delta(q) = \delta_l S^\delta(L_l^{-1}(q)) + P(L_l^{-1}(q), \delta_l)$ for all $l \in \wedge$. The R-B operator defined on $C(\ell)$ w.r.t. the scaling vector $\delta \neq 0$ is:

$$T_\delta(g^*(q)) = \delta_l g^*(L_l^{-1}(q)) + Q_l(L_l^{-1}(q), \delta_l), q \in \ell_l, l \in \wedge. \tag{7}$$

As with $\delta \neq 0$, T_δ has S^δ as its fixed point and in similar manner if the scaling vector is zero then T_δ has P as its fixed point $\delta = 0$, $|\delta|_\infty$ is the contractivity factor of the contraction map T_δ. Thus

$$\| T_\delta S^\delta - T_\delta P \|_\infty \le |\delta|_\infty \| S^\delta - P \|_\infty \tag{8}$$

by using the mean value theorem, we get

$$\| T_\delta P(q) - T_0 P(q) \|_\infty \le | \delta_l P(L^{-1}(q)) + Q_l(L^{-1}(q), \delta_l) - Q_l(L^{-1}(q), 0)|_\infty$$

$$\| T_\delta P(q) - T_0 P(q) \|_\infty \le \left| \delta_l P(L^{-1}(q)) + \frac{u_l(\delta_l, s_l, t_l, L^{-1}(q))}{v_l(s_l, t_l, L^{-1}(q))} - \frac{u_l(0, s_l, t_l, L^{-1}(q))}{v_l(s_l, t_l, L^{-1}(q))} \right|_\infty$$

$$\le |\delta|_\infty \left(\| P \|_\infty + \left| \frac{\partial \left\{ \frac{u_l(\tau_l, s_l, t_l, L^{-1}(q))}{v_l(s_l, t_l, L^{-1}(q))} \right\}}{\partial \delta_l} \right| \right); |\tau_l| \in (0, \delta_l) \tag{9}$$

from the classical RQIF (5) we observe that $P(q)$ can be rewrite as:

$$\begin{aligned} P(L_l(q)) = {} & \omega_1(s_l, t_l, \theta) z_l + \omega_2(s_l, t_l, \theta) z_{l+1} + \omega_3(s_l, t_l, \theta) d_l \\ & - \omega_4(s_l, t_l, \theta) d_{l+1} + \omega_5(s_l, t_l, \theta) D_l + \omega_6(s_l, t_l, \theta) D_{l+1} \end{aligned} \tag{10}$$

where

$$\omega_1(s_l, t_l, \theta) = \frac{s_l(1-\theta)^5 + (4s_l + t_l)(1-\theta)^4\theta + (6s_l + 4t_l)(1-\theta)^3\theta^2}{s_l(1-\theta) + t_l\theta} \ge 0$$

$$\omega_2(s_l,t_l,\theta)=\frac{t_l\theta^5+(s_l+4t_l)(1-\theta)\theta^4+(4s_l+6t_l)(1-\theta)^2\theta^3}{s_l(1-\theta)+t_l\theta}\geq 0$$

$$\omega_3(s_l,t_l,\theta)=\frac{s_l\hbar_l(1-\theta)^4\theta+(3s_l+t_l)\hbar_l(1-\theta)^3\theta^2}{s_l(1-\theta)+t_l\theta}\geq 0$$

$$\omega_4(s_l,t_l,\theta)=-\left(\frac{t_l\hbar_l(1-\theta)\theta^4+(s_l+3t_l)\hbar_l(1-\theta)^2\theta^3}{s_l(1-\theta)+t_l\theta}\right)\geq 0$$

$$\omega_5(s_l,t_l,\theta)=\frac{\frac{1}{2}s_l\hbar_l^2(1-\theta)^3\theta^2}{s_l(1-\theta)+t_l\theta}\geq 0 \;\;\&\;\; \omega_6(s_l,t_l,\theta)=\frac{\frac{1}{2}t_l\hbar_l^2(1-\theta)^2\theta^3}{s_l(1-\theta)+t_l\theta}\geq 0$$

We see that $\omega_1(s_l,t_l,\theta)+\omega_2(s_l,t_l,\theta)=1$ and also for $s_l>0\,\&\,t_l>0$. We have

$$\omega_3(s_l,t_l,\theta)-\omega_4(s_l,t_l,\theta)=\hbar_l\{\theta(1-\theta)+\theta^2(1-\theta)^2\}$$

$$\omega_5(s_l,t_l,\theta)+\omega_6(s_l,t_l,\theta)=\frac{1}{2}\hbar_l^2\theta^2(1-\theta)^2$$

so,

$$P(q)\leq \max_{m=l,l+1}\{|z_m|\}+\hbar_l\{\theta(1-\theta)+\theta^2(1-\theta)^2\}\max_{m=l,l+1}\{|d_m|\}$$

$$+\frac{1}{2}\hbar_l^2\theta^2(1-\theta)^2\max_{m=l,l+1}\{|D_m|\}$$

$$\leq |z|_{1,M}+\hbar_l\{\theta(1-\theta)+\theta^2(1-\theta)^2\}|d|_\infty+\frac{1}{2}\hbar_l^2\theta^2(1-\theta)^2|D|_\infty .$$

As this is true for m, so

$$\|P\|_\infty\leq \Xi(\hbar)=\|f\|_\infty+\hbar_l\{\theta(1-\theta)+\theta^2(1-\theta)^2\}|d|_\infty+\frac{1}{2}\hbar_l^2(\theta^2(1-\theta)^2)|D|_\infty . \tag{11}$$

Since $v_l(q)$ is independent of δ, from equation (9),

$$\frac{\partial\left\{\dfrac{u_l(\tau_l,s_l,t_l,L^{-1}(q))}{v_l(s_l,t_l,L^{-1}(q))}\right\}}{\partial\delta_l}=\omega_1(s_l,t_l,\theta)z_1+\omega_2(s_l,t_l,\theta)z_l+\omega_3(s_l,t_l,\theta)d_1$$
$$-\omega_4(s_l,t_l,\theta)d_l+\omega_5(s_l,t_l,\theta)D_1+\omega_6(s_l,t_l,\theta)D_l$$

via same arguments we obtain,

$$\frac{\partial\left\{\dfrac{u_l(\tau_l,s_l,t_l,L^{-1}(q))}{v_l(s_l,t_l,L^{-1}(q))}\right\}}{\partial\delta_l}\le\Xi^*(\hbar)$$
$$=\|z\|_{1,M}+\hbar_l\{\theta(1-\theta)+\theta^2(1-\theta)^2\}|d|_{1,M}+\frac{1}{2}\hbar_l^2(\theta^2(1-\theta)^2)|D|_{1,M} \tag{12}$$

by substituting the values from equations (11) and (12) in equation (9), we get

$$\|T_\delta P(q)-T_0P(q)\|_\infty\le|\delta|_\infty(\Xi(\hbar)+\Xi^*(\hbar));q\in[q_1,q_M]. \tag{13}$$

Now by using (8) & (13)

$$\|S^\delta-P\|_\infty=\|T_\delta S^\delta(q)-T_0P(q)\|_\infty\le\|T_\delta S^\delta(q)-T_\delta P(q)\|_\infty+\|T_\delta P(q)-T_0P(q)\|_\infty$$

$$\le|\delta|_\infty\|S^\delta-P\|_\infty+|\delta|_\infty(\Xi(\hbar)+\Xi^*(\hbar))\Rightarrow\|S^\delta-P\|_\infty\le\frac{|\delta|_\infty(\Xi(\hbar)+\Xi^*(\hbar))}{1-|\delta|_\infty}$$

we have

$$\|f-S^\delta\|_\infty\le\|f-p\|_\infty+\|p-S^\delta\|_\infty$$

so by using the Theorem 2,

$$\|f-S^\delta\|_\infty\le e\hbar_l^3\|f^{(3)}\|_\infty+\frac{|\delta|_\infty}{1-|\delta|_\infty}\{\Xi(\hbar)+\Xi^*(\hbar)\}.$$

Theorem 4: *Let* S_ε^δ *and* S^δ *are the ZRQFIF and RQFIF respectively for the data set* $\{(q_m,z_m)\in R^2:m\in\wedge^*\}$ *having vertical scaling* $|\delta_l|<a_l^2$, *then for fixed signature* $\varepsilon\in(0,1)^M$,

$$\|S_\varepsilon^\delta-S^\delta\|_\infty\le\frac{|\varepsilon|_\infty}{\lambda(1-|\delta|_\infty)}\left\{(|s|_\infty+|t|_\infty)\left(16\omega(f,\hbar)+5h\omega(f',\hbar)+\frac{1}{2}h^2\omega(f'',\hbar)\right)\right\}$$

where, $\hbar = \max\{h_l; l \in \wedge\}$ $\omega(f,\hbar) = \sup_{|q_1-q_2|\le\hbar}\{f(q_1)-f(q_2); q_1, q_2 \in \ell\}$

$$\lambda_l = \min\{V_{l,\varepsilon}(\theta): 0 \le \theta \le 1\}, \lambda = \min\{\lambda_l : l \in \wedge\} \;\&\; |\varepsilon|_\infty = \begin{cases} 0, \text{ if } \varepsilon_l = 0 \\ 1, \text{ if } \varepsilon_l \ne 0. \end{cases}$$

Proof: By using the self-referential equation for S_ε^δ & S^δ, $\forall y \in \ell_l$, we have

$$\le |\delta|| S_\varepsilon^\delta(q) - S^\delta(q)||_\infty + \frac{\left|s_l\left(z_{l+\varepsilon_l} - z_l\right)\right|}{\left|V_{l,\varepsilon}(\theta)\right|} + \frac{\left|\left(4s_l + t_l\right)\left(z_{l+\varepsilon_l} - z_l\right)\right| + \left|s_l h_l\left(d_{l+\varepsilon_l} - d_l\right)\right|}{\left|V_{l,\varepsilon}(\theta)\right|}$$

$$+ \frac{\left|\left(6s_l + 4t_l\right)\left(z_{l+\varepsilon_l} - z_l\right)\right| + \left|\left(3s_l + t_l\right)h_l\left(d_{l+\varepsilon_l} - d_l\right)\right| + \left|\frac{1}{2}s_l h_l^2\left(D_{l+\varepsilon_l} - D_l\right)\right|}{\left|V_{l,\varepsilon}(\theta)\right|}$$

$$+ \frac{\left|\left(4s_l + 6t_l\right)\left(z_{l+1-\varepsilon_l} - z_{l+1}\right)\right| + \left|\left(s_l + 3t_l\right)h_l\left(d_{l+1-\varepsilon_l} - d_{l+1}\right)\right| + \left|\frac{1}{2}t_l h_l^2\left(D_{l+1-\varepsilon_l} - D_{l+1}\right)\right|}{\left|V_{l,\varepsilon}(\theta)\right|}$$

$$+ \frac{\left|\left(s_l + 4t_l\right)\left(z_{l+1-\varepsilon_l} - z_{l+1}\right)\right| - \left|t_l h_l\left(d_{l+1-\varepsilon_l} - d_{l+1}\right)\right|}{\left|V_{l,\varepsilon}(\theta)\right|} + \frac{\left|t_l\left(z_{l+1-\varepsilon_l} - z_{l+1}\right)\right|}{\left|V_{l,\varepsilon}(\theta)\right|}$$

$$\le |\delta|_\infty || S_\varepsilon^\delta - S^\delta ||_\infty + \left|\varepsilon_l\right| \frac{1}{\lambda} \begin{pmatrix} |s|_\infty \left|z_{l+1} - z_l\right| + 4|s|_\infty \left|z_{l+1} - z_l\right| + |t|_\infty \left|z_{l+1} - z_l\right| \\ +\hbar|s|_\infty \left|d_{l+1} - d_l\right| + 6|s|_\infty \left|d_{l+1} - d_l\right| + 4|t|_\infty \left|z_{l+1} - z_l\right| \\ +3\hbar|s|_\infty \left|d_{l+1} - d_l\right| + \hbar|t|_\infty \left|d_{l+1} - d_l\right| + \frac{1}{2}\hbar^2|s|_\infty \left|D_{l+1} - D_l\right| \\ +4|s|_\infty \left|z_{l+1} - z_l\right| + 6|t|_\infty \left|z_{l+1} - z_l\right| + \hbar|s|_\infty \left|d_{l+1} - d_l\right| \\ +3\hbar|t|_\infty \left|d_{l+1} - d_l\right| + \frac{1}{2}\hbar^2|t|_\infty \left|D_{l+1} - D_l\right| + |s|_\infty \left|z_{l+1} - z_l\right| \\ +4|t|_\infty \left|z_{l+1} - z_l\right| + \hbar|t|_\infty \left|d_{l+1} - d_l\right| \end{pmatrix}$$

$$\le |\delta|_\infty || S_\varepsilon^\delta - S^\delta ||_\infty + |\varepsilon|_\infty \frac{1}{\lambda} \begin{pmatrix} 6|s|_\infty \left|z_{l+1} - z_l\right| + 16|t|_\infty \left|z_{l+1} - z_l\right| \\ +5\hbar|s|_\infty \left|d_{l+1} - d_l\right| + 5\hbar|t|_\infty \left|d_{l+1} - d_l\right| \\ +\frac{1}{2}\hbar^2|s|_\infty \left|D_{l+1} - D_l\right| + \frac{1}{2}\hbar^2|t|_\infty \left|D_{l+1} - D_l\right| \end{pmatrix}$$

$$\leq |\delta|_\infty \| S_\varepsilon^\delta - S^\delta \|_\infty + |\varepsilon|_\infty \frac{1}{\lambda}\begin{pmatrix} 16|s|_\infty \omega(f,\hbar) + 16|t|_\infty \omega(f,\hbar) + 5\hbar|s|_\infty \omega(f',\hbar) \\ +5\hbar|t|_\infty \omega(f',\hbar) + \frac{1}{2}\hbar^2|s|_\infty \omega(f'',h) + \frac{1}{2}\hbar^2|t|_\infty \omega(f'',h) \end{pmatrix}$$

$$\leq |\delta|_\infty \| S_\varepsilon^\delta - S^\delta \|_\infty + |\varepsilon|_\infty \frac{1}{\lambda}\begin{pmatrix} 16\left(|s|_\infty + |t|_\infty\right)\omega(f,\hbar) + 5\left(|s|_\infty + |t|_\infty\right)\hbar\omega(f',\hbar) \\ +\frac{1}{2}\left(|s|_\infty + |t|_\infty\right)\hbar^2\omega(f'',\hbar) \end{pmatrix}.$$

Theorem 5: *Let* S_ε^δ *be the ZRQFIF for the data set* $\{(q_m, z_m) \in R^2 : m \in \wedge^*\}$ *with the fixed signature* $\varepsilon \in (0,1)^M$, *then*

$$\| f - S_\varepsilon^\delta \|_\infty \leq e_l \hbar_l^3 \| f^{(3)} \|_\infty + \frac{|\delta|_\infty}{1-|\delta|_\infty}\{\Xi(\hbar) + \Xi^*(\hbar)\}$$

$$+\frac{|\varepsilon|_\infty}{\lambda\left(1-|\delta|_\infty\right)}\left\{\left(|s|_\infty + |t|_\infty\right)\left(16\omega(f,\hbar) + 5\hbar\omega(f',\hbar) + \frac{1}{2}\hbar^2\omega(f'',\hbar)\right)\right\}.$$

Proof: By using the theorem 3 and theorem 4 in following inequality

$$\| f - S_\varepsilon^\delta \|_\infty \leq \| f - S^\delta \|_\infty + \| S_\varepsilon^\delta - S^\delta \|_\infty$$

we'll get our required result.

5. Positivity of ZRQFIF

Theorem 6: *Let* $\{(q_m, z_m) \in R^2 : m \in \wedge^*\}$ *be the considered positive data set, then* S_ε^δ *be positive only if the constraints given below holds for scaling factor and shape parameters.*

$$\frac{h_l\left(d_{l+1-\varepsilon_l} - \frac{\delta_l}{h_l}(q_M - q_1)d_M\right)}{z_{l+1-\varepsilon_l} - \delta_l z_M} < \frac{s_l}{t_l} < \frac{h_l^2\left(D_{l+1-\varepsilon_l} - \frac{\delta_l}{h_l^2}(q_M - q_1)^2 D_M\right)}{2\left(d_{l+1-\varepsilon_l} - \frac{\delta_l}{h_l}(q_M - q_1)d_M\right)}$$

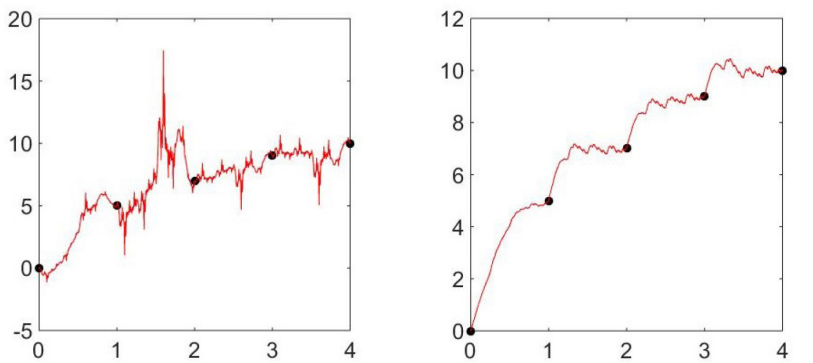
$$0 \leq \delta_l < \min\left\{\frac{z_{l+\varepsilon_l}}{z_1}, \frac{z_{l+1-\varepsilon_l}}{z_M}\right\}$$

Proof: To prove that the ZRQFIF S_ε^δ is an positive function, it is sufficient to demonstrate that $S_\varepsilon^\delta(L_l(q)) \geq 0$. By the hypothesis of IFS, we conclude that $S_\varepsilon^\delta(L_l(q))$ is positive if $U_{l,\varepsilon}(\theta) \geq 0 \forall \theta \in [0,1]$. Therefore $S_\varepsilon^\delta(L_l(y)) \geq 0$ if $A_{l,\varepsilon}$, $B_{l,\varepsilon}$, $C_{l,\varepsilon}$, $D_{l,\varepsilon}$, $E_{l,\varepsilon}$ & $F_{l,\varepsilon} > 0$, by simplifying these inequalities, we get the conditions over δ_l and s_l, t_l as follow and by using these conditions a nonnegative ZRQFIF can be constructed.

$$\frac{h_l\left(d_{l+1-\varepsilon_l} - \frac{\delta_l}{h_l}(q_M - q_1)d_M\right)}{z_{l+1-\varepsilon_l} - \delta_l z_M} < \frac{s_l}{t_l} < \frac{h_l^2\left(D_{l+1-\varepsilon_l} - \frac{\delta_l}{h_l^2}(q_M - q_1)^2 D_M\right)}{2\left(d_{l+1-\varepsilon_l} - \frac{\delta_l}{h_l}(q_M - q_1)d_M\right)}$$

$$0 \leq \delta_l < \min\left\{\frac{z_{l+\varepsilon_l}}{z_1}, \frac{z_{l+1-\varepsilon_l}}{z_M}\right\}.$$

Example 1: Consider the interpolating points as $(0,0)$, $(1,5)$, $(2,7)$, $(3,9)$, $(4,10)$. In Figure 1 (a), we took some random values for shape parameters $s_l = (2,7,2,9)$ & $t_l = (9,8,5,4)$ and the scaling vector and signature are δ = (0.2, 0.9, –0.35, 0.4), $\varepsilon = ((1,1,1,1)$, the graph corresponding to these values is negative. Then the values of s_l, t_l are calculated by using the theorem 6. In Figure 1 (b), the shape parameter is taken as $s_l = (87, 20,$



(**a**) Non-Positive ZRQFIF with $\varepsilon = ((1,1,1,1), \delta = (0.2, 0.9, -0.35, 0.4)$ $s_l = (2,7,2,9)$ $t_l = (9,8,5,4)$

(**b**) Positive ZRQFIF with $\varepsilon = (0,0,0,0), \delta = (0.12, 0.29, 0.3, 0.34)$ $s_l = (87,20,0.2,0.9)$ $t_l = (289,88,45,34)$

Figure 1

Non-positive and positive ZRQFIF S_ε^δ.

$0.2,\ 0.9)$, $t_l = (289,\ 88,\ 45,\ 34)$ and signature, scaling vector are taken as $\varepsilon = (0,0,0,0)$ $\delta = (0.12,\ 0.29,\ 0.3,\ 0.34)$ respectively.

5. Conclusion

This paper have studied the construction of the C^2 ZRQFIF with linear denominator having two shape parameters. If we take signature and scaling factor zero then, the ZRQFIF will become the RQFIF. The uniform convergence of ZRQFIF to the data generating original function is discussed. The graphs of ZRQFIF are also generated with the different approaches of signature, scaling vector and the shape parameters. If we choose the range of scaling vector and shape parameters as stated in theorem 6, then the proposed ZRQFIF preserves positivity. For future work, to construct and to study the ZRQFIF, constant in scaling vector can be replaced with variable.

References

[1] M. F. Barnsley, "Fractal functions and interpolation," *Constructive approximation*, vol. 2, no. 1, pp. 303-329 (1986).

[2] M. F. Barnsley and A. N. Harrington, "The calculus of fractal interpolation function," *Journal of Approximation Theory*, vol. 57, no. 1, pp. 14-34 (1989).

[3] M. A. Navascués and M. V. Sebastián, "Smooth fractal interpolation," *Journal of Inequalities and Applications*, vol. 2006, no. 1, pp. 1-20 (2006).

[4] B. Prasad, K. Katiyar, and B. Singh, "Trigonometric quadratic fractal interpolation functions," *International Journal of Applied Engineering Research*, vol. 10, no. 16, pp. 37290-37302 (2015).

[5] S. Sharma and K. Katiyar, "Preserving convexity through C^2–rational quintic fractal interpolation function," *In AIP Conference Proceedings*, vol. 2735, no. 1, p. 040015 (2023).

[6] S. Sharma and K. Katiyar, "Alpha fractal rational quintic spline with shape preserving properties," *International Journal of Computational and Applied Mathematics & Computer Science*, vol. 3, pp. 113-121 (2023).

[7] S. Sharma, K. Katiyar, G. Sudhamsu, M. K. Wratch, and R. Salgotra, "Positivity preserving rational cubic fractal interpolation function together with its zipper form," *Axioms*, vol. 13, no. 9, pp. 584 (2024).

[8] A. Banerjee, M. N. Akhtar, and M. A. Navascués, "Local α-fractal interpolation function," *The European Physical Journal Special Topics*, vol. 232, no. 7, pp. 1043-1050 (2023).

[9] M. N. Akhtar, M. G. P. Prasad, and M. A. Navascués, "Box dimensions of α-fractal functions," *Fractals*, vol. 24, no. 3, pp. 1650037 (2016).

[10] A. Hossain, M. N. Akhtar, and M. A. Navascués, "Fractal interpolation on the real projective plane," *Numerical Algorithms*, vol. 96, no. 2, pp. 557-582 (2024).

[11] M. A. Navascués, M. N. Akhtar, and R. Mohapatra, "Fractal frames of functions on the rectangle," *Fractal and Fractional*, vol. 5, no. 2, pp 42 (2021).

[12] C. H. He and C. Liu, "Fractal dimensions of a porous concrete and its effect on the concrete's strength," *Facta Universitatis, Series: Mechanical Engineering*, vol. 21, no. 1, pp. 137-150 (2023).

[13] M. A. Navascués, "Fractal curves on Banach algebras," *Fractal and Fractional*, vol. 6, no. 12, pp. 1-17 (2022).

[14] A. Gowrisankar, D. Easwaramoorthy, R. Valarmathi, P. E. Jeevaraj, C. Ananth, and I. Vasiliev, "An integrated perspective of fractal time series analysis for infected cases of COVID-19," *In Fractal Signatures in the Dynamics of an Epidemiology*, pp. 103-117 (2023).

[15] T. Gopalakrishnan and H. Z. Almngoshi, "Fractal image compression using digital cone metric space," *Journal of Interdisciplinary Mathematics*, vol. 28, no. 2, pp. 647–658 (2025).

[16] J. Hattab, "Mixed multifractal dimensions of overlaps of self-similar sets and measures," *Journal of Interdisciplinary Mathematics*, vol. 28, no. 7, pp. 2533-2553 (2025).

[17] V. V. Aseev, A. V. Tetenov, and A. S. Kravchenko, "On selfsimilar Jordan curves on the plane," *Siberian Mathematical Journal*, vol. 44, no. 3, pp. 379-386 (2003).

[18] V. V. Aseev, "On the regularity of self-similar zippers," *Materials*, pp. 24-30 (2002).

[19] M. Hussain, M. Z. Hussain, and R. J. Cripps, "C^2 rational quintic function," *Journal of Prime Research in Mathematics*, vol. 5, no. 1, pp. 115-126 (2009).

Received March, 2025