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On totally geodesic submanifolds of Lorentzian Sasakian manifolds

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Abstract

In this paper, properties of totally geodesic submanifolds of Lorentzian Sasakian manifolds are discussed.

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1. Introduction

The concept of geodesics dates back to the work of mathematicians like Gauss and Riemann in the 19th century, who laid the groundwork for differential geometry. They explored the properties of curves and surfaces, leading to the understanding of geodesics as the shortest paths between points on a manifold.

The 20th century saw the rise of Lorentzian geometry, particularly with the advent of general relativity. Einstein's theory required a new understanding of spacetime, which is modeled by Lorentzian manifolds. These manifolds have a signature that distinguishes between time-like

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and space-like intervals, fundamentally changing the approach to geometry.

In recent years, research has focused on the properties and classifications of totally geodesic submanifolds in various types of manifolds, including Lorentzian Sasakian manifolds. This research article discussed conditions under which a submanifold can be considered totally geodesic, and provides proofs and theorems related to these concepts.

A Lorentzian Sasakian manifold ($\mathbb{M}$, J, ξ, η, g) satisfies [1, 2, 3, 4]

$$J^2(A)=-A+\eta(A)\xi,\ \eta(\xi)=1,\ J\xi=0,\ \eta o\xi=0. \tag{1.1}$$

$$g(JA,\ JB)=g\ (A,B)-\varepsilon\eta(A)\ \eta(B),\ g(\xi,\xi)=\varepsilon=-1. \tag{1.2}$$

A Sasakian manifold [5, 6] satisfies

$$(D_A J)B=g(A,B)\xi-\eta\ (B)A. \tag{1.3}$$

On a Lorentzian Sasakian manifold [7, 8, 9, 10, 11, 12, 13] we have

$$S(A,\ \xi)=\ 2n\ \ \eta(A)-\ J\ (A). \tag{1.4}$$

Gauss and Weingarten formula is

$$\overline{\nabla}_A B=D_A B+\sigma(A,\ B) \tag{1.5}$$

$$\overline{\nabla}_A V=-E_V A+D_A^{\perp} V \tag{1.6}$$

where σ and E are related by

$$g(E_V A,\ B)=g(\sigma(A,B),\ V) \tag{1.7}$$

The Tachibana operator Q (S, T) is defined as [6]

$$\begin{aligned} Q(S,T)(A_1,A_2\ ...A_K,A,B)\ &=-\Gamma((A\Lambda_E B)A_1,A_2,...A_K)... \\ &\quad -\Gamma(A_1,A_2,...(A\Lambda_E B)A_K) \end{aligned} \tag{1.8}$$

2. Lorentzian Sasakian manifold

If $J(TM)\subset(TM)$ then submanifold $\mathcal{M}$ of $\mathbb{M}$ is invariant.

On submanifold $\mathcal{M}$ we have

$$\sigma(A,\xi)=E_V\xi=0. \tag{2.1}$$

Theorem 1: *On submanifold $\mathcal{M}$ of $\mathbb{M}$, if $Q(E\sigma)=0$, then $\mathcal{M}$ is totally geodesic.*

Proof: As $Q(E\sigma) = 0$ so

$$Q(E\sigma)(U, V; A, B) = \sigma((A \wedge_E B) U, V) + \sigma(V, (A \wedge_E B) U$$
$$= \sigma(E(B, U)A - E(A, U) B, V) + \sigma(U, E(B, V) A - E(A, V) B)$$
$$= 0 \tag{2.2}$$

Replacing A and V by ξ and with (1.4), we have

$$-2n\sigma(U, B) = 0,$$

proving our result.

Theorem 2: *On submanifold $\mathcal{M}$ of $\mathcal{M}$, if $Q(g, \sigma) = 0$, then $\mathcal{M}$ is totally geodesic.*

Proof: $Q(g, \sigma) = 0$ *is equivalent to*

$$Q(g, \sigma)(U, V; A, B) = \sigma((A \wedge_g B) U, V) + \sigma(U, A \wedge_g B) V) = 0 \tag{2.3}$$

Replacing B and U by ξ gives $\sigma(A, V) = 0$, proving our result.

Theorem 3: *On a submanifold $\mathcal{M}$ of $\mathcal{M}$, if $Q(E, \overline{\nabla}.\sigma) = 0$ then $\mathcal{M}$ is totally geodesic.*

Proof: $Q(E, \overline{\nabla}.\sigma) = 0$ gives

$$(\overline{\nabla}(A \wedge_E \xi) U\sigma)(V, \xi) + (\overline{\nabla}_U \sigma)(A \wedge_E \xi) V, \xi)$$
$$+ (\overline{\nabla}_U \sigma)(V, (A \wedge_E \xi) \xi = 0 \tag{2.4}$$

Replacing V by ξ and by (1.4), (2.1) and (2.4), we get

$$-2n \ \phi\sigma(A, U) = 0. \tag{2.5}$$

Apply ϕ to (2.5) and with (2.1), we have

$$-2n \ \sigma(A, U) = 0 \tag{2.6}$$

With (2.4) and (2.6), we have

$$-2n \ \sigma(U, V) = 0,$$

proving our result.

Theorem 4: *On submanifold $\mathcal{M}$ of $\mathcal{M}$ if $Q(g, \overline{\nabla}.\sigma) = 0$ then $\mathcal{M}$ is totally geodesic.*

Proof: $Q(g, \overline{\nabla}.\sigma) = 0$ gives

$$(\overline{\nabla}_U\ \sigma)((A\Lambda_g B)V, C)+(\overline{\nabla}_U\sigma)(V, (A\ \Lambda_g B)C=0.$$

Replacing B, V and C by ξ gives

$$\sigma(D_U\xi, A)-\sigma(\phi U, A) = -2\sigma(U, A)=0. \tag{2.7}$$

Applying ϕ to (2.7) and with (2.1), we get

$$\phi\ \sigma(U, A)=0. \tag{2.8}$$

By (2.7) and (2.8), we have $\sigma(U, A)=0$, proving our result.

Theorem 5: *On a submanifold $\mathcal{M}$ of $\mathcal{M}$ if $Q(g,\overline{K}.\sigma)=0$ then $\mathcal{M}$ is totally geodesic.*

Proof: $Q(g,\overline{K}.\sigma)=0$ *gives*

$$(\overline{K}(A,B).\sigma)(U\Lambda_g V)C,W)+(\overline{K}(A,B).\sigma)(C,U\Lambda_g V)C)=0.$$

Replacing A, C, U and W by ξ and with (1.1), (1.4) and (2.1), we get

$$\sigma(V,B)=0 \tag{2.9}$$

proving our result.

Theorem 6: *On a submanifold $\mathcal{M}$ of $\mathcal{M}$ if $Q(E,\overline{K}.\sigma)=0$ then $\mathcal{M}$ is totally geodesic.*

Proof: $Q(E,\overline{K}.\sigma)=0$ gives

$$\begin{aligned}Q(E,\overline{K}.\sigma)(U,V,C,W;A,B) &= (\overline{K}(A,B).\sigma)(U\Lambda_E V)C,W)\\ &\quad+(\overline{K}(A,B).\sigma)(C,U\Lambda_E V)W)\\ &=0.\end{aligned}$$

Replacing A, U and W by ξ gives

$$(\overline{K}(\xi,B).\sigma)(E(V,C)\xi-E(\xi,C)V,\xi)+(\overline{K}(\xi,B).\sigma)(C,E(V,\xi)\xi-E(\xi,\xi)V)=0.$$

Here

$$\begin{aligned}E(\xi,C)\sigma(V)K(\xi,B)\xi)-E(V,\xi)\sigma(C,K(\xi,B)\xi)-K^{\perp}(\xi,B)E(\xi,\xi)\sigma(V,C)\\ +E(\xi,\xi)\sigma(K(\xi,B)V,C)+S(\xi,\xi)\sigma(K(\xi,B)C,V)=0.\end{aligned}$$

Replacing C by ξ and with (1.7) and (2.3), we have

$$2n(\sigma(V,B)-\xi\phi\sigma(V,B))=0.$$

Proving our result.

Conclusion

In the present work, we have investigated the geometric aspects of geodesic submanifolds in context of Lorentzian Sasakian manifolds. We first established that, under suitable conditions, the considered submanifolds are totally geodesic and inherit significant geometric properties from the ambient Lorentzian Sasakian structure. We derived several necessary conditions under which a submanifold is totally geodesic. The obtained results contribute to a deeper understanding of the interaction between the geometric structure of the manifold and the behavior of time-like and space-like directions.

It is expected that the results presented in this paper will provide a useful foundation for further investigations of submanifold theory, curvature properties, and their applications in differential geometry and related areas of mathematical physics.

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