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On the separation of sets

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Abstract

The concepts of separability, proper separability, and strict separability and related concepts are considered in this paper. Some properties of these sets are proved, and applications are considered.

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1. Introduction

Let X, Y be sets in the plane $\mathbb{R}^2$.

The following question arises: When a straight line can be drawn between the sets X, Y such that X is entirely on the one side of this line and Y is entirely on the other side of this line.

If such a straight line exists for the sets X, Y then these sets are called *separable*.

Similarly, if X, Y are sets in $\mathbb{R}^3$, the question about the existence of a plane which separates X and Y is considered, and so on, analogous question arises about the sets in $\mathbb{R}^n$.

Definition 1: The set

$$H = \{\mathrm{x} \in \mathbb{R}^n : \langle v, x\rangle = \alpha\}, \tag{1}$$

where $\mathrm{v} \in \mathbb{R}^n$, $\mathrm{v} \neq 0$ is a vector with constant components, $\alpha \in \mathbb{R}$, is called a *hyperplane*, and α is the distance from the hyperplane H to the origin 0.

Definition 2: The hyperplane H, defined by (1), determines

$$H_+ = \{\mathrm{x} \in \mathbb{R}^n : \langle \mathrm{v}, \mathrm{x}\rangle \geq \alpha\}, \quad H_- = \{\mathrm{x} \in \mathbb{R}^n : \langle \mathrm{v}, \mathrm{x}\rangle \leq \alpha\} \tag{2}$$

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and

$$H_+ = \{x \in \mathbb{R}^n : \langle v, x\rangle > \alpha\}, \quad H_- = \{x \in \mathbb{R}^n : \langle v, x\rangle < \alpha\},$$

closed half-spaces and open half-spaces, respectively.

When $\alpha = 0$, H is a hyperplane through 0.

When $n = 2$, H is a straight line, and H_+ and H_- are half-planes.

When $n = 3$, H is a plane, and H_+ and H_- are half-spaces in $\mathbb{R}^3$.

Definition 3: The sets X, Y are called *separable* if $\exists$ a vector $v \neq 0$, a number α with

$$\langle v, x\rangle \leq \alpha \leq \langle v, y\rangle \quad \forall x \in X, \forall y \in Y. \tag{3}$$

The sets X and Y are called *properly separable* if, moreover, $X \cup Y \nsubseteq H$.

The sets X, Y are called *strictly separable* if $\exists$ a vector $v \neq 0$ and numbers α_1 and α_2, $\alpha_1 \neq \alpha_2$, such that

$$\langle v, x\rangle \leq \alpha_1 < \alpha_2 \leq \langle v, y\rangle \quad \forall x \in X, \forall y \in Y. \tag{4}$$

In the three cases of Definition 3, the sets X, Y can be *separated*, *properly separated*, and *strictly separated*, respectively.

The inequalities (4), which define strictly separable sets, can also be expressed equivalently by using only one parameter α instead of two parameters α_1, α_2 as follows

$$\langle v, x\rangle \leq \alpha < \langle v, y\rangle \text{ for all } x \in X, \text{ for all } y \in Y, \tag{4'}$$

or

$$\langle v, x\rangle < \alpha \leq \langle v, y\rangle \text{ for all } x \in X, \text{for all } y \in Y, \tag{4''}$$

or

$$\langle v, x\rangle < \alpha < \langle v, y\rangle \text{ for all } x \in X, \text{for all } y \in Y. \tag{4'''}$$

The concepts, discussed in the paper, and similar to them, are considered in reference titles [1]-[18], etc.

Some main properties of separable (strictly separable) sets are stated and proved in Section 2. In Section 3, a problem for future research is formulated.

2. Properties of separable sets

Theorem 1: *Let the set $X \subseteq \mathbb{R}^n$, $X \neq \emptyset$ be closed and convex, $y \notin X$. Then X, $\{y\}$ are strictly separable.*

Proof: The set X is nonempty, closed and convex, and $y \notin X$ by assumption. Therefore $\exists$ a single projection $p \in X$ of the point y onto the set X, with

$$\langle x - p, y - p\rangle \leq 0 \quad \forall x \in X. \tag{5}$$

Set $v = p - y$, $\alpha = \langle v, p\rangle$. Because $y \notin X$, $p \in X$, then $v \neq 0$. Hence (5) implies

$$\langle x - p, -v\rangle \leq 0, \quad \text{that is,} \quad \langle v, x\rangle \geq \langle v, p\rangle = \alpha \quad \forall x \in X. \tag{6}$$

However,

$$\alpha - \langle v, y\rangle = \langle v, p\rangle - \langle v, y\rangle = \langle v, p - y\rangle = \langle v, v\rangle = \|v\|^2 > 0 \quad (7)$$

because $v \neq 0$, hence $\|v\|^2 > 0$ strictly.

Therefore

$$\langle v, y\rangle \overset{(7)}{<} \alpha \overset{(6)}{\leq} \langle v, x\rangle \text{ for all } x \in X,$$

which means that X, $\{y\}$ may be strictly separated by definition (see $(4')$, $(4'')$).

Remark 1: Theorem 1 can be stated equivalently as follows: If the set $X \neq \emptyset$ is convex in $\mathbb{R}^n$ and $y \notin \overline{X}$, then $\overline{X}$ and $\{y\}$ are strictly separable.

This is due to the fact that if we do not assume that X is a closed set, in order that statement of Theorem 1 be true, we must require $y \notin \overline{X}$ instead of $y \notin X$.

Theorem 2: The sets X, Y may be separated (separated strictly, respectively) $\Leftrightarrow$ the sets $X - Y$ and $\{0\}$ can be separated (strictly separated, respectively).

Proof: $\Rightarrow$) Let the sets X, Y be separable (strictly separable). Then $\exists\, v \in \mathbb{R}^n$, $v \neq 0$, and a scalar $\alpha \in \mathbb{R}$, such that

$$\langle v, x\rangle \leq \alpha \leq \langle v, y\rangle \text{ for all } x \in X, \text{ for all } y \in Y$$
$$(\langle v, x\rangle < \alpha < \langle v, y\rangle \text{ for all } x \in X, \text{ for all } y \in Y).$$

Therefore for every $z \overset{\text{def}}{=} x - y$, $z \in X - Y$ for $x \in X$ and $y \in Y$, the next relations are fulfilled, respectively,

$$\langle v, z\rangle = \langle v, x - y\rangle = \langle v, x\rangle - \langle v, y\rangle \leq 0 \equiv \langle v, 0\rangle \quad (8)$$
$$(\langle v, z\rangle = \langle v, x - y\rangle = \langle v, x\rangle - \langle v, y\rangle < 0 \equiv \langle v, 0\rangle). \quad (8')$$

Relations (8) and $(8')$ show that the sets $X - Y$ and $\{0\}$ are separable (strictly separable, respectively).

$\Leftarrow$) Let the sets $X - Y$, $\{0\}$ be separable (strictly separable). Hence, the inequality (8) (inequality $(8')$, respectively) is satisfied. Furthermore, the following inequality holds true, respectively,

$$\sup_{x \in X} \langle v, x\rangle - \inf_{y \in Y} \langle v, y\rangle \leq 0$$
$$(\sup_{x \in X} \langle v, x\rangle - \inf_{y \in Y} \langle v, y\rangle < 0).$$

Hence,

$$\langle v, x\rangle \leq \sup_{x \in X} \langle v, x\rangle \overset{\leq}{(<)} \inf_{y \in Y} \langle v, y\rangle \leq \langle v, y\rangle \quad \forall\, x \in X \text{ and } \forall\, y \in Y. \quad (9)$$

Denote

$$\sup_{x \in X} \langle v, x\rangle = \alpha_1, \quad \inf_{y \in Y} \langle v, y\rangle = \alpha_2.$$

Then inequalities (9) imply

$$\langle v, x\rangle \leq \alpha_1 \overset{\leq}{(<)} \alpha_2 \leq \langle v, y\rangle \quad \forall\, x \in X \text{ and } \forall\, y \in Y. \quad (9')$$

This means that the sets X and Y are separable (strictly separable, respectively) by Definition 3.

Theorem 3: *If the sets X, $Y \neq \emptyset$ are closed and convex, $X \cap Y = \emptyset$, and, e.g., the set Y is bounded, then the sets X, Y can be strictly separated.*

Proof: From the properties of Minkowski sum of sets it follows that the set $X - Y$ is closed and convex. Obviously, $0 \notin X - Y$; if we assume the contrary, that $0 \in X - Y$, then $0 = x - y$, $x \in X$, $y \in Y$, hence $x = y$, where $x \in X$ and $y \in Y$, that is, X and Y have a common point, a contradiction with the assumption of Theorem 3. Therefore Theorem 1 implies that the sets $X - Y$, $\{0\}$ can be separated strictly, and then sufficiency of Theorem 2 implies that X, Y are strictly separable sets.

The next result is a consequence of the separation theorems.

Theorem 4: *Let the sets X, $Y \in \mathbb{R}^n$ be compact and convex, and*

$$\sup_{z \in X} \langle d, z \rangle = \sup_{z \in Y} \langle d, z \rangle \qquad \forall d \in \mathbb{R}^n.$$

Then $X = Y$.

Proof: Suppose the contrary, i.e., $\exists$ some $z_0 \in X$ with $z_0 \notin Y$. Then $\exists$ a vector v in $\mathbb{R}^n$, $v \neq 0$, and a real constant a with

$$\langle v, y \rangle \geq a > \langle v, z_0 \rangle \quad \forall y \in Y$$

in accordance with Theorem 1. Denote $d = -v \neq 0$, we obtain

$$\langle d, y \rangle < \langle d, z_0 \rangle \quad \forall y \in Y.$$

Because $z_0 \in X$, we get

$$\langle d, z_0 \rangle \leq \sup_{z \in X} \langle d, z \rangle.$$

Then

$$\sup_{y \in Y} \langle d, y \rangle < \langle d, z_0 \rangle \leq \sup_{z \in X} \langle d, z \rangle$$

a contradiction with the hypothesis of Theorem 4. Hence, assumption was incorrect, therefore $X = Y$.

3. Conclusion

In this paper, the concepts of separable sets and related concepts are considered. Some properties of these sets are proved, and applications are considered.

Open problem for future research is to establish other separability properties of convex sets.

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