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On the properties of cones and polar cones

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Abstract

The concepts of cones, polar cones, double polar cones, and pointed cones are introduced. Some properties of these cones are proved, and the topic of representation of convex cones is discussed.

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1. Introduction

Definition 1: The set $K \subseteq \mathbb{R}^n$ is called a *cone* (*with vertex at* $\mathbf{0}$) if from $\mathrm{x} \in K$, $t \geq 0$ it follows $t\mathrm{x} \in K$, i.e., if $\mathrm{x} \in K$, then K also contains the following ray $\Lambda = \{t\mathrm{x}: t \geq 0\}$.

Each ray Λ with an origin at the vertex $\mathbf{0}$ of the cone K and which ray is contained in K is said to be *ray* of the cone K.

Convex cone is a set which is convex and which is a cone.

Examples of cones.

1) The whole space $\mathbb{R}^n$ is a cone by Definition 1 because $\mathbb{R}^n$ contains all points, including the points of the form tx with $t \geq 0$.
2) Octants in $\mathbb{R}^n$, formed by the coordinate axes, are cones with vertices at $\mathbf{0}$. Similarly, 2^n −ants in $\mathbb{R}^n$ are cones with vertices at $\mathbf{0}$.
3) Each linear subspace of $\mathbb{R}^n$ is a cone.
4) The sets $\{\mathrm{x} : A\mathrm{x} \leq 0\}$, $\{\mathrm{y} : \mathrm{y} = A\mathrm{x}, \mathrm{x} \geq 0\}$, $\{\mathrm{x} : A\mathrm{x} = 0\ ,\ \mathrm{x} \geq 0\}$ are convex closed cones.

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The related topics are studied in reference titles [1]-[17], etc.

In Section 2, basic properties of the cones are stated. In Section 3, the polar cone and the dual polar cone are defined, and some properties are stated and proved. Section 4 is devoted to representation of convex cones. In Section 5, an open problem for research is formulated.

2. Basic properties of cones

Theorem 1: *A necessary and sufficient condition for the set $K \subseteq \mathbb{R}^n$ to be a convex cone (with vertex at $\mathbf{0}$) is that $\mathrm{x}_1, \mathrm{x}_2 \in K$ and $\alpha \geq 0, \beta \geq 0$ imply $\alpha \mathrm{x}_1 + \beta \mathrm{x}_2 \in K$.*

Proof: *Necessity.* Assume that K is a convex cone, and $\mathrm{x}_1, \mathrm{x}_2 \in K$ and $\alpha \geq 0$, $\beta \geq 0$ (hence $\alpha + \beta \geq 0$).

Case 1: If $\alpha = \beta = 0$ (hence $\alpha + \beta = 0$), then $\alpha \mathrm{x}_1 + \beta \mathrm{x}_2 \equiv 0 \in K$ because K is a cone (with vertex at **0**) by assumption.

Case 2: If $\alpha + \beta > 0$ (that is, if $\alpha \geq 0$, $\beta \geq 0$ and at least one of α, β is different from 0) then

$$\alpha \mathrm{x}_1 + \beta \mathrm{x}_2 \equiv (\alpha + \beta)\left(\frac{\alpha}{\alpha+\beta}\mathrm{x}_1 + \frac{\beta}{\alpha+\beta}\mathrm{x}_2\right) \in K$$

according to Definition 1 because $\alpha + \beta > 0$ by assumption,

$$\frac{\alpha}{\alpha+\beta}\mathrm{x}_1 + \frac{\beta}{\alpha+\beta}\mathrm{x}_2 \in K$$

(this is satisfied because $\mathrm{x}_1, \mathrm{x}_2 \in K$, $\frac{\alpha}{\alpha+\beta} \geq 0$, $\frac{\beta}{\alpha+\beta} \geq 0$, $\frac{\alpha}{\alpha+\beta} + \frac{\beta}{\alpha+\beta} = 1$ and K is convex by assumption).

Sufficiency. Assume that for every $\mathrm{x}_1, \mathrm{x}_2 \in K$ and $\alpha \geq 0$, $\beta \geq 0$ we have $\alpha \mathrm{x}_1 + \beta \mathrm{x}_2 \in K$. Since $\alpha \mathrm{x}_1 + \beta \mathrm{x}_2$ is an arbitrary linear combination with nonnegative coefficients of x_1, x_2 then, in particular, K also contains the convex combinations of every two points of K because the convex combinations are special case of the linear combinations with nonnegative coefficients, that is, K is a *convex set* by definition.

Furthermore, $\alpha \mathrm{x}_1 + \beta \mathrm{x}_2 \in K$ implies that K also contains the product of each point of K with a nonnegative number (indeed, for $\alpha = 0$ we have $\beta \mathrm{x}_2 \in K$, and for $\beta = 0$ we have $\alpha \mathrm{x}_1 \in K$), that is, K is a *cone* by Definition 1.

From the two conclusions, drawn above, it follows that K is a convex cone by Definition 1.

It can be proved that the following three theorems hold true.

Theorem 2: *If K is a cone, then its closure $\overline{K}$ also is a cone.*

Theorem 3: *The intersection of convex closed cones is also a closed convex cone.*

Theorem 4: *If K_1, K_2 are convex cones in $\mathbb{R}^n$ then $K_1 + K_2$ also is a convex cone.*

3. Polar cones and dual polar cones

Definition 2: Let $X \subseteq \mathbb{R}^n$, $X \neq \emptyset$. The set

$$X^* = \{\mathrm{x}^* \in \mathbb{R}^n : \langle \mathrm{x}^*, \mathrm{x} \rangle \leq 0 \quad \forall \mathrm{x} \in X\} \tag{1}$$

is said to be the *polar cone* of X.

According to Definition 2, X^* is the totality of all vectors (points) which form non-acute angle with the vectors with endpoints which belong to X.

Remark 1: The concept of a ``polar cone" is defined for an arbitrary set X but this concept is important when X is a cone.

Examples of polar cones.

1) If $K = \{0\}$ then $K^* = \mathbb{R}^n$.
2) If $K^* = \mathbb{R}^n$ then $K = \{0\}$.
3) Let $X = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq x_2, x_2 \geq 0\}$ (the set X is a closed convex cone).

According to Definition 2,

$$X^* = \{\mathrm{x}^* \in \mathbb{R}^2 : \langle \mathrm{x}^*, \mathrm{x} \rangle \leq 0 \quad \forall \mathrm{x} \in X\},$$

that is, $\langle \mathrm{x}^*, \mathrm{x} \rangle = \langle (x_1^*, x_2^*), (x_1, x_2) \rangle = x_1^* x_1 + x_2^* x_2 \leq 0$, hence using $x_1 \geq x_2$, we get $(x_1^* + x_2^*) x_2 \leq 0$, and from $x_2 \geq 0$ it follows that $x_1^* + x_2^* \leq 0$, that is, $x_1^* \leq -x_2^*$. Furthermore, since $x_1 \geq x_2 \geq 0$ according to definition of X, $x_1^* \leq -x_2^*$ and $x_1^* x_1 + x_2^* x_2 \leq 0$, then $x_1^* \leq 0$.

Therefore

$$X^* = \{\mathrm{x}^* = (x_1^*, x_2^*) \in \mathbb{R}^2 : x_1^* \leq -x_2^*, x_1^* \leq 0\}.$$

Definition 3: The set

$$X^{**} = (X^*)^* = \{z \in \mathbb{R}^n : \langle \mathrm{z}, \mathrm{x}^* \rangle \leq 0 \quad \forall \mathrm{x}^* \in X^*\} \tag{2}$$

is said to be the *polar cone of X^** (*double polar cone of X*).

The following Lemma 1 presents some properties of cones, polar cones, and double polar cones.

Lemma 1: *Let X, Y be nonempty sets in $\mathbb{R}^n$. Then the following properties are satisfied.*

1) *X^* is a convex closed cone*;
2) $X \subseteq X^{**}$;
3) *if $X \subseteq Y$ then $Y^* \subseteq X^*$.*

Proof:

1) a) Let $x^* \in X^*$ and $t \geq 0$. Therefore $\langle x^*, x\rangle \leq 0\ \forall\ x \in X$ by Definition 2. Hence

$$\langle t x^*, x\rangle = t\langle x^*, x\rangle \leq 0 \quad \forall x \in X$$

because $t \geq 0$ and $\langle x^*, x\rangle \leq 0$ for $x^* \in X^*$. Therefore $t x^* \in X^*$ by Definition 2, hence X^* is a cone according to Definition 1.

The proof that X^* is a cone justifies the name of the concept of the *polar cone* in Definition 2.

b) Let $x_1^*, x_2^* \in X^*$, and $\lambda \in [0,1]$. Therefore

$$\langle x_1^*, x\rangle \leq 0, \langle x_2^*, x\rangle \leq 0 \quad \forall x \in X$$

and

$$\langle \lambda x_1^* + (1-\lambda)x_2^*, x\rangle = \lambda\langle x_1^*, x\rangle + (1-\lambda)\langle x_2^*, x\rangle \leq 0,$$

where we have used that $\lambda \geq 0$, $\langle x_1^*, x\rangle \leq 0$ for $x_1^* \in X^*$; $1-\lambda \geq 0$, $\langle x_2^*, x\rangle \leq 0$ for $x_2^* \in X^*$. Hence $\lambda x_1^* + (1-\lambda)x_2^* \in X^*$, i.e., the set X^* is convex by definition.

Remark 2: Using Theorem 1, the property that X^* is a convex cone (items a) and b)) can be proved simultaneously. Let x_1^*, $x_2^* \in X^*$ (hence $\langle x_1^*, x\rangle \leq 0$ and $\langle x_2^*, x\rangle \leq 0$ for every $x \in X$) and let $\alpha \geq 0$, $\beta \geq 0$. Therefore

$$\langle \alpha x_1^* + \beta x_2^*, x\rangle = \alpha\langle x_1^*, x\rangle + \beta\langle x_2^*, x\rangle \leq 0 \quad \forall x \in X,$$

where we have used that $\alpha \geq 0$, $\langle x_1^*, x\rangle \leq 0$ for $x_1^* \in X^*$ and $\beta \geq 0$, $\langle x_2^*, x\rangle \leq 0$ for $x_2^* \in X^*$. Hence $\alpha x_1^* + \beta x_2^* \in X^*$ according to Definition 2, and from sufficiency of Theorem 1 it follows that X^* is a convex cone.

c) The scalar (inner) product $\langle .,. \rangle$ is continuous function of its arguments. Therefore if $x_k^* \in X^*$, $x_k^* \to x_0^*$ for $k \to \infty$, then $\langle x_k^*, x\rangle \leq 0$ for all $x \in X$ according to Definition 2, and

$$\langle x_k^*, x\rangle \leq 0 \overset{k\to\infty}{\to} \langle x_0^*, x\rangle \leq 0 \quad \forall x \in X,$$

i.e., $x_0^* \in X^*$ by Definition 2. We have proved that $x_k^* \in X^*$ and $x_k^* \to x_0^*$ imply $x_0^* \in X^*$. Therefore, by definition X^* is a closed set.

2) Let $x \in X$. Therefore $\langle x^*, x\rangle \leq 0$ for every $x^* \in X^*$ according to Definition 2, hence $\langle x, x^*\rangle \leq 0\ \forall\ x^* \in X^*$ (symmetry of the scalar product in $\mathbb{R}^n$). Therefore $x \in X^{**}$ in accordance with Definition 3 of X^{**}. We have shown that $x \in X$ implies $x \in X^{**}$. However, $x \in X$ has been arbitrarily chosen. Hence $X \subseteq X^{**}$.

3) Let $X \subseteq Y$ and $y^* \in Y^*$. Hence $\langle y^*, y\rangle \leq 0$ for every $y \in Y$. However, $X \subseteq Y$, hence $\langle y^*, x\rangle \leq 0$ for every $x \in X$. Hence $y^* \in X^*$ according to Definition 2 of X^*. We proved that $y^* \in Y^*$ implies $y^* \in X^*$. Because $y^* \in Y^*$ was arbitrarily chosen, then $Y^* \subseteq X^*$.

Theorem 5: *If K is a nonempty convex cone in $\mathbb{R}^n$, then $K^{**} = \overline{K}$.*

Proof: According to Lemma 1, $K \subseteq K^{**}$ and K^{**} is a closed set. Therefore $\overline{K} \subseteq K^{**}$.

Concerning the converse inclusion, suppose that $\mathrm{x} \in K^{**}$. Assume that $\mathrm{x} \notin \overline{K}$. Then the point x and the set $\overline{K}$ are strictly separable. Therefore there exists a vector $\mathrm{v} \neq 0$, such that

$$\langle \mathrm{v}, \mathrm{x} \rangle > \langle \mathrm{v}, \mathrm{y} \rangle \quad \forall \mathrm{y} \in \overline{K}. \tag{3}$$

Since $\overline{K}$ is a cone according to Theorem 2, then for every $\lambda > 0$, $\mathrm{y} \in \overline{K}$,

$$\langle \mathrm{v}, \mathrm{x} \rangle > \lambda \langle \mathrm{v}, \mathrm{y} \rangle,$$

that is,

$$\frac{1}{\lambda} \langle \mathrm{v}, \mathrm{x} \rangle > \langle \mathrm{v}, \mathrm{y} \rangle \quad \forall \mathrm{y} \in \overline{K},$$

hence $\langle \mathrm{v}, \mathrm{y} \rangle \leq 0$ for each $\mathrm{y} \in \overline{K}$. Therefore from Definition 2 and Lemma 1 we get $\mathrm{v} \in \overline{K}^{*} \subseteq K^{*}$ because $K \subseteq \overline{K}$ and Lemma 1, part 3) imply $\overline{K}^{*} \subseteq K^{*}$. However, from $0 \in \overline{K}^{*}$ and taking $\mathrm{y} = 0 \in \overline{K}$ in (3) we have $\langle \mathrm{v}, \mathrm{x} \rangle > 0$, which is a contradiction with the condition that $\mathrm{x} \in K^{**}$ because from Definition 3 of K^{**} we have that $\langle \mathrm{v}, \mathrm{x} \rangle \leq 0$ for $\mathrm{v} \in K^{*}$, $\mathrm{x} \in K^{**}$. Therefore the assumption $\mathrm{x} \notin \overline{K}$ is wrong, i.e., $\mathrm{x} \in \overline{K}$. We have proved that $\mathrm{x} \in K^{**}$ implies $\mathrm{x} \in \overline{K}$. Therefore $K^{**} \subseteq \overline{K}$.

From the two opposite inclusions it follows that $K^{**} = \overline{K}$.

Remark 3: An equivalent statement of Theorem 5 is as follows.

If K is a nonempty, closed and convex cone (that is, $K = \overline{K}$ due to the closedness of K), then $K^{**} = K$.

Note that if K is not a cone or if K is not a convex set, then statement of Theorem 5 may not hold.

Theorem 6: *If S is a subspace then $S^{*} = S^{\perp}$, where $S^{\perp}$ is the orthogonal complement of S.*

Proof: According to definition of orthogonal complement,

$$S^{\perp} = \{\mathrm{x}_0 \in \mathbb{R}^n : \langle \mathrm{x}_0, \mathrm{x} \rangle = 0 \quad \forall \mathrm{x} \in S\}.$$

a) Let $\mathrm{x}^{*} \in S^{*}$. Therefore $\langle \mathrm{x}^{*}, \mathrm{x} \rangle \leq 0$ for each $\mathrm{x} \in S$ in accordance with Definition 2. Since S is a subspace, then $\mathrm{x} \in S$ implies $-\mathrm{x} \in S$. Hence, $\langle \mathrm{x}^{*}, -\mathrm{x} \rangle \leq 0$ for every $(-\mathrm{x}) \in S$. From the two opposite inequalities we obtain $\langle \mathrm{x}^{*}, \mathrm{x} \rangle = 0$ for each $\mathrm{x} \in S$, and from definition of $S^{\perp}$ it follows that $\mathrm{x}^{*} \in S^{\perp}$. Since x^{*} has been arbitrarily chosen in S^{*}, then $S^{*} \subseteq S^{\perp}$.

b) Let $\mathrm{x}_0 \in S^{\perp}$. Then $\langle \mathrm{x}_0, \mathrm{x} \rangle = 0$ for each $\mathrm{x} \in S$ according to definition of $S^{\perp}$. Therefore $\mathrm{x}_0 \in S^{*}$ because x_0 satisfies the nonstrict inequality of Definition 2 of S^{*} as an equality. Hence, $S^{\perp} \subseteq S^{*}$.

From the two opposite inclusions it follows that $S^{*} = S^{\perp}$.

4. Representation of convex cones

Definition 4: A convex closed cone K is called *pointed* if it has an extreme point.

If a convex closed cone has an extreme point, then this extreme point is unique and it is its vertex $\mathbf{0}$. The closed cone K contains both the points z and $-\mathrm{z}$ only when $\mathrm{z} = 0$.

According to Definition 4 and a known theorem of convex analysis, a convex closed cone is pointed if and only if it does not contain a straight line.

Theorem 7: *A closed and convex cone is a pointed cone if and only if there exists a supporting hyperplane, which passes through its vertex, and this hyperplane has no other points in common with the cone.*

Proof: *Necessity.* Let K be a closed convex pointed cone. Consider the set $X = \{\mathrm{x} \in K: \|\mathrm{x}\| = 1\}$. This set is compact, therefore $co\, X$ is also a compact set, and $0 \notin co\, X$ because $0 \notin X$ and $\mathbf{0}$ is an extreme point of the cone K. Then there exists a hyperplane H with defining equation $\langle \mathrm{v}, \mathrm{x} \rangle = 0$, such that $\langle \mathrm{v}, \mathrm{x} \rangle > 0$ for every $\mathrm{x} \in X \subseteq co\, X$. We will show that hyperplane H has no other points in common with the cone K. Let $\mathrm{y} \neq 0$ be an arbitrary point of K. Then $\mathrm{z} = \frac{\mathrm{y}}{\|\mathrm{y}\|} \in K$ and $\|\mathrm{z}\| = 1$, that is, $\mathrm{z} \in X$ in accordance with definition of the set X. Therefore $\langle \mathrm{v}, \mathrm{z} \rangle = \frac{1}{\|\mathrm{y}\|} \langle \mathrm{v}, \mathrm{y} \rangle > 0$, hence $\|\mathrm{y}\| > 0$ implies $\langle \mathrm{v}, \mathrm{y} \rangle > 0$, that is, $\mathrm{y} \notin H$.

Sufficiency. Conversely, if $\exists$ a supporting hyperplane $H = \{\mathrm{x}: \langle \mathrm{v}, \mathrm{x} \rangle = 0\}$ through the vertex of the convex closed cone K, such that H has no other points in common with K, then K does not contain a straight line and therefore K is a pointed cone according to the above discussion.

Definition 5: A ray of a convex closed cone K is said to be an *extreme ray* for K if each point of this ray, different from $\mathbf{0}$, cannot be expressed as a linear combination with positive coefficients of two distinct points of K, which do not lie on this ray.

An equivalent definition of an extreme ray for a convex closed cone is as follows.

Definition 6: A ray Λ is called *extreme* for the convex closed cone K if $K \backslash \Lambda$ is a convex set.

For example, each coordinate ray $O\mathrm{x}, O\mathrm{y}, O\mathrm{z}$ is an extreme ray for the octants, which are pointed cones in $\mathbb{R}^3$.

Denote by K_{per} the totality of points of the extreme rays of the cone K.

Theorem 8: *If K is a closed convex cone, $H = \{\mathrm{z}: \langle \mathrm{v}, \mathrm{z} \rangle = 0\}$ is a supporting hyperplane for K, and the ray Λ is such that $\Lambda \subset (K \cap H)_{per}$, then $\Lambda \subset K_{per}$.*

Proof: Let $\langle \mathrm{v}, \mathrm{z}\rangle \leq 0 \ \ \forall \ \ \mathrm{z} \in K$. If $\Lambda \subset (X \cap H)_{per}$ and if we assume that $\Lambda \not\subset K_{per}$, then for some point $\overline{\mathrm{z}} \in \Lambda$ there will exist points z_1 and $\mathrm{z}_2 \in K$, which do not lie simultaneously on the ray Λ, such that $\overline{\mathrm{z}} = \alpha \mathrm{z}_1 + \beta \mathrm{z}_2$, $\alpha > 0$, $\beta > 0$. However, $0 = \langle \mathrm{v}, \overline{\mathrm{z}}\rangle = \alpha\langle \mathrm{v}, \mathrm{z}_1\rangle + \beta\langle \mathrm{v}, \mathrm{z}_2\rangle \leq 0$, therefore $\langle \mathrm{v}, \mathrm{z}_1\rangle = \langle \mathrm{v}, \mathrm{z}_2\rangle = 0$, i.e., $\mathrm{z}_1 \in K \cap H$, $\mathrm{z}_2 \in K \cap H$, which means that $\Lambda \not\subset (K \cap H)_{per}$, a contradiction. Therefore the assumption is wrong, and $\Lambda \subset K_{per}$.

Theorem 9: *A convex closed cone $K \neq \{0\}$ has an extreme ray if and only if K is pointed.*

Proof: *Necessity.* Let $\mathrm{z} \neq 0$ be a point of an extreme ray of this cone K. Assume that K is not a pointed cone. Then K will contain a straight line, for example, $\{\mathrm{y} : \mathrm{y} = \mathrm{z} + t\mathrm{b}, t \in \mathbb{R}\}$.

1) If $\pm b \not\in \{\lambda z : \lambda \geq 0\}$, then $z' \equiv z + b \not\in \{\lambda z : \lambda \geq 0\}$ and $z'' \equiv z - b \not\in \{\lambda z : \lambda \geq 0\}$. However, obviously $z = \frac{1}{2}(z' + z'')$, which means that z is not a point of an extreme ray, a contradiction. Hence the assumption made is wrong, therefore K is a pointed cone.
2) If $\mathrm{b} \in \{\lambda \mathrm{z} : \lambda \geq 0\}$, that is, $\mathrm{z} = t_0\mathrm{b}$, $t_0 > 0$, then $\mathrm{z}'' \equiv \mathrm{z} - t_1 b \not\in \{\lambda z : \lambda \geq 0\}$ with $t_1 > t_0$. Then $\mathrm{z} = \frac{1}{2}(\mathrm{z}' + \mathrm{z}'')$, where $\mathrm{z}' \equiv \mathrm{z} + t_1\mathrm{b}$, a contradiction. The case when $-\mathrm{b} \in \{\lambda \mathrm{z} : \lambda \geq 0\}$ is considered similarly. Therefore the assumption is wrong, hence K is a pointed cone.

Sufficiency. Let a convex closed cone K be pointed. The sufficiency is proved by induction on the dimension of the considered cone K. Concerning a cone with dimension one, the statement of the ``sufficiency" part of Theorem 9 obviously is true because a one-dimensional pointed cone is a ray.

Induction hypothesis. Let statement be true for cones with dimension $p < k$.

We will prove that the statement of the "sufficiency" part of Theorem 9 is also true for the $k-$dimensional cone K. If $\mathrm{z} \in K$ and $\mathrm{z} \neq 0$ is a relative boundary point, then $\exists$ a supporting hyperplane H through z with $K \not\subset H$. Then $K \cap H$ is a pointed cone, $\mathrm{d}im\ (K \cap H) < k$, and for $K \cap H$ the statement is true according to the Induction hypothesis, that is, $(K \cap H)_{per} \neq \emptyset$. However, $(K \cap H)_{per} \subset K_{per}$ implies $K_{per} \neq \emptyset$.

Theorem 10: *Each relative interior point of a pointed cone K with $dim\, K > 1$ is presented as the convex combination of two relative boundary points.*

Proof: Let $\mathrm{z} \in \mathrm{ri}\, K$. Since K is a pointed cone, then $\exists$ a hyperplane H through the vertex of K, such that H has no other points in common with the cone K according to Theorem 7. Construct a straight line through z with direction, determined by the vector $\mathrm{b} \in (H \cap \mathrm{a}ff\ K)$, $\mathrm{b} \neq 0$. Then $\pm\mathrm{b} \notin K$. We will show that

$$t_1 \stackrel{\text{def}}{=} \sup\{t: \mathrm{z} + t\mathrm{b} \in K\} < \infty.$$

Assuming the contrary, that $\mathrm{z} + t\mathrm{b} \in K$ for each $t > 0$, then $\frac{1}{t}\mathrm{z} + \mathrm{b} \in K$ for every $t > 0$, therefore $\mathrm{b} \in K$, a contradiction. Hence the assumption made is wrong, therefore $t_1 < \infty$.

Similarly, we can prove that

$$t_2 \stackrel{\text{def}}{=} \sup\{t: \mathrm{z} - t\mathrm{b} \in K\} < \infty.$$

The points $\mathrm{z}_1 \stackrel{\text{def}}{=} \mathrm{z} + t_1\mathrm{b}$ and $\mathrm{z}_2 \stackrel{\text{def}}{=} \mathrm{z} - t_2\mathrm{b}$ are relative boundary points and

$$z \equiv \frac{t_2}{t_1+t_2}\mathrm{z}_1 + \frac{t_1}{t_1+t_2}\mathrm{z}_2 = \frac{t_2}{t_1+t_2}(\mathrm{z} + t_1\mathrm{b}) + \frac{t_1}{t_1+t_2}(\mathrm{z} - t_2\mathrm{b}),$$

where

$$\frac{t_2}{t_1+t_2} > 0, \frac{t_1}{t_1+t_2} > 0, \quad \frac{t_2}{t_1+t_2} + \frac{t_1}{t_1+t_2} = 1,$$

that is, the relative interior point z is represented as a convex combination of two relative boundary points.

The next Theorem 11 is an analogue of the Minkowski-Krein-Milman theorem.

Theorem 11: *Each point of a finite pointed cone K is presented as a convex combination of the points of the extreme rays of K, that is,* $K = co\, K_{per}$.

Proof: (By induction on the dimension of the cone K).

The statement of Theorem 11 is obviously true for the one-dimensional cone, that is, for the ray.

Induction hypothesis. Assume that the statement is true for cones with dimension $p < k$.

Suppose that K is a k −dimensional cone.

a) Let $\mathrm{z} \in r\,\partial K$ be a relative boundary point. Therefore $\exists$ a supporting hyperplane H through this point, such that $H = \{\mathrm{x}: \langle \mathrm{v}, \mathrm{x}\rangle = 0\}$, $\mathrm{z} \in H$, $K \not\subset H$. Then the set $K \cap H$ is a pointed cone with dimension $p < k$, and according to Induction hypothesis, we have

$\mathrm{z} = \sum_{i=1}^{s} \alpha_i \overline{\mathrm{z}}_i, \quad \alpha_i \geq 0, \overline{\mathrm{z}}_i \in (K \cap H)_{per}, i = 1, \dots, s, \quad \sum_{i=1}^{s} \alpha_i = 1.$

However, $(K \cap H)_{per} \subset K_{per}$ according to Theorem 8. Therefore $\overline{z}_i \in K_{per}$, $i = 1, \dots, s$, which proves the required representation in this case.

b) Let $\mathrm{z} \in \mathrm{ri}\, K$. From Theorem 10 we have

$$\mathrm{z} = \lambda \mathrm{z}' + (1 - \lambda)\mathrm{z}'', \quad \lambda \in [0,1], \quad \mathrm{z}', \mathrm{z}'' \in r\,\partial K.$$

However,

$$z' = \sum_{i=1}^{s_1} \alpha_i \overline{\mathrm{y}}_1, \quad \mathrm{z}'' = \sum_{i=1}^{s_2} \beta_j \overline{\mathrm{z}}_j, \quad \overline{\mathrm{y}}_i \in K_{per}, i = 1, \dots, s_1,$$

$$\bar{z}_j \in K_{per}, \quad j = 1, \dots, s_2,$$

$$\alpha_i \geq 0, i = 1, \dots, s_1, \quad \beta_j \geq 0, j = 1, \dots, s_2, \quad \sum_{i=1}^{s_1} \alpha_i = 1, \quad \sum_{j=1}^{s_2} \beta_j = 1$$

in accordance with case a), considered above, and applied to $z', z'' \in r\,\partial K$.

Therefore

$$z = \lambda z' + (1-\lambda) z'' = \sum_{i=1}^{s_1} \lambda \alpha_i \bar{y}_i + \sum_{j=1}^{s_2} (1-\lambda)\beta_j \bar{z}_j,$$

where

$$\bar{y}_i \in K_{per}, i = 1, \dots, s_1, \quad \bar{z}_j \in K_{per}, \quad j = 1, \dots, s_2,$$

$$\lambda\alpha_i \geq 0, i = 1, \dots, s_1, \quad (1-\lambda)\beta_j \geq 0, \quad j = 1, \dots, s_2,$$

$$\sum_{i=1}^{s_1} \lambda\alpha_i + \sum_{j=1}^{s_2} (1-\lambda)\beta_j = \lambda \sum_{i=1}^{s_1} \alpha_i + (1-\lambda) \sum_{j=1}^{s_2} \beta_j = \lambda.1 + (1-\lambda).1 = 1,$$

that is, the convex combination z of convex combinations z' and z'' is also a convex combination of the points of the extreme rays of cone K.

We have shown that in both possible cases a) and b), the point z is represented as a convex combination of points of extreme rays of cone K, that is, $K \subseteq \mathrm{co}\, K_{per}$.

Obviously $\mathrm{co} K_{per} \subseteq K$ by definition of K_{per}.

From the two opposite inclusions it follows that $K = \mathrm{co}\, K_{per}$.

Theorem 12: *Let K be a pointed cone. Then each supporting hyperplane for K passes through its vertex* $\mathbf{0}$.

Proof: Let $x_0 \in K$, $H = \{x : \langle v, x\rangle = \alpha\}$ be a supporting hyperplane for K with $\alpha = \langle v, x_0\rangle$, hence

$$\langle v, x\rangle \leq \langle v, x_0\rangle \quad \forall x \in K. \tag{4}$$

For $x = 2x_0$ ($\in K$ according to Definition 1 with $t = 2 > 0$) we have

$$2\langle v, x_0\rangle \leq \langle v, x_0\rangle,$$

hence $\langle v, x_0\rangle \leq 0$.

However, for $x = 0 \in K$ from (4) we obtain $0 \equiv \langle v, 0\rangle \leq \langle v, x_0\rangle$, hence $\langle v, x_0\rangle \geq 0$.

From the two opposite inequalities, obtained above, it follows that $\langle v, x_0\rangle = 0$ and $\alpha \overset{\text{def}}{=} \langle v, x_0\rangle = 0$.

Therefore $0 \in H$ because $\mathbf{0}$ satisfies the equation, defining H, with $\alpha = 0$.

Definition 7: The set

$$\mathrm{ray}\, X = \{z \in \mathbb{R}^n : z = \lambda x, x \in X, \lambda \geq 0\}$$

is said to be the *conic hull of the set* $X \subseteq \mathbb{R}^n$.

Definition 8: The intersection of all convex cones, which contain the set X, is said to be the *convex conic hull of the set* $X \subseteq \mathbb{R}^n$ and it is denoted by $\mathrm{con}\, X$.

The following property holds true.

Property: $con\ X = co(ray\ X) = ray(co\ X)$, that is,

$$conX = \{z : z = \sum_{i=1}^{n+1} \alpha_i z_i, \quad \alpha_i \geq 0, z_i \in X, i = 1, \dots, n+1\}.$$

In this representation, $\sum_{i=1}^{n+1} \alpha_i = 1$ is *not* required, and the number of addends of $z = \sum_{i=1}^{n+1} \alpha_i z_i$ is finite.

Remark 4: The set $ray\ X$ is always a cone, regardless of whether X is a cone or not. In the case when K is a cone, then $K = ray\ K$.

If K is a closed convex cone and $\text{ray}\ X = K_{per}$, then $K = con\ X$.

Indeed,

$$K = \text{co}\ K_{per} = co(ray\ X) = con\ X,$$

where for the first equality we have used Theorem 11; for the second equality, the assumption that $ray\ X = K_{per}$ is used; and for the third equality, the above Property is applied.

5. Conclusion

In this paper, the concepts of cones, polar cones, double polar cones, and pointed cones are introduced. Some properties of these cones are proved, and the topic of representation of convex cones is discussed.

An open problem for research is to study and establish other properties of the cones.

References

[1] C. D. Aliprantis and K. C. Border, Infinite Dimensional Analysis: A Hitchhiker's Guide, 3rd ed., Springer, Berlin-Heidelberg-New York (2006).

[2] V. G. Boltyanski, H. Martini, and P. Soltan, Excursions into Combinatorial Geometry, Springer, Berlin-Heidelberg-New York (1997).

[3] S. Boyd, L. Vandenberghe, Convex Optimization, Cambridge University Press, Cambridge (2004).

[4] C. J. Goh, X. Q. Yang, Duality in Optimization and Variational Inequalities, Taylor & Francis, London-New York (2002).

[5] B. Grünbaum, Convex Polytopes, 2nd ed., Springer, New York-Berlin-Heidelberg (2003).

[6] B. Iochum, Cônes autopolaires et algèbres de Jordan, Springer, Berlin-Heidelberg-New York-Tokio (1984).

[7] L. Narici and E. Beckenstein, Topological Vector Spaces, 2nd ed., Chapman and Hall / CRC Press, Boca Raton (2010).

[8] A. G. Ramm, P. N. Shivakumar, and A. V. Strauss (eds.), Operator Theory and Its Applications, American Mathematical Society, Providence (2000).

[9] R. T. Rochafellar, Convex Analysis, Princeton University Press, Princeton (1970).

[10] H. H. Schaefer and M. P. Wolff, Topological Vector Spaces, 2nd ed., Springer, New York (1999).

[11] S. M. Stefanov, "Valid inequalities and cutting planes for some polytopes", Mathematical Inequalities and Applications, vol. 1, no. 2, pp. 285-294 (1998).

[12] S. M. Stefanov, Convex Separable Programming, Kluwer Academic Publishers, Dordrecht (2000).

[13] S. M. Stefanov, Separable Programming: Theory and Methods, Kluwer Academic Publishers, Dordrech-Boston-London (2001).

[14] S. M. Stefanov, "Valid inequalities, cutting planes and integrality of the knapsack polytope", *Journal of Interdisciplinary Mathematics*, vol. 14, no. 4, pp. 389-406 (2011).

[15] S. M. Stefanov, "On the solution of quadratic programming problem with a feasible region defined as a Minkowski sum of a compact set and finitely generated convex closed cone", *Journal of Information and Optimization Sciences*, vol. 39, no. 6, pp. 1223-1230 (2018).

[16] S. M. Stefanov, Separable Optimization: Theory and Methods, Springer, Cham (2021).

[17] S. M. Stefanov, "Continuous linear knapsack problems revisited", *Journal of Information and Optimization Sciences*, vol. 44, no. 5, pp. 909-922 (2023).

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