

New analytical tools for investigating $\mathcal{H}$ -stability of time-varying nonlinear dynamics

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Abstract

The present research proposes new requirements for practical $\mathcal{H}$ -stability in non-autonomous nonlinear systems. We introduce new converse Lyapunov theorems specifically for these types of systems. To analyze the $\mathcal{H}$ -stability analysis, we utilize $\mathcal{H}$ -stable practical functions. We investigate the uniform practical $\mathcal{H}$ -stability of non-autonomous systems:

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perturbed ODEs and Interconnected systems. Additionally, several examples are provided for illustration.

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1. Introduction

The second Lyapunov method has been widely employed in the literature to investigate the stability of nonlinear systems, yielding important contributions, see [1], [2], [9], [13], [19], [27]–[29]. Theoretical interest in practical tools for control design and system analysis is the cause of this. The Lyapunov function method is widely recognized as a fundamental approach for simplifying complex systems and has significant applications in control theory (see [16]). The author in [23] proposes a hybrid method that combines the bat algorithm with Lyapunov's artificial small parameter technique to address fractional calculus equations. The application of this method to several examples of fractional order problems demonstrated its reliability and efficiency compared to classical methods. Finding the asymptotic stability conditions has been the only focus of the majority of the intriguing advancements in the search for stability criteria for nonlinear differential systems, see [8], [15]. The theory of $\mathcal{H}$ -stability was introduced in [20]–[22] over the years. Several works, such as [24]–[26], have provided important insights into the behavior of specific dynamical systems, particularly regarding the stability and boundedness of their solutions. Some systems may exhibit instability in practical scenarios. This can happen when the region of attraction or the basin of the desired state is too narrow, or when the target state itself is mathematically unstable. Despite this instability, the system may still operate close enough to the target to achieve satisfactory performance. In these situations, Lyapunov stability may not adequately describe the system's behavior. To address this limitation, the concept of practical stability has been introduced as a more appropriate framework (see [2]–[11], [18]). According to [9], [10], this method yields a novel result: realistic uniform $\mathcal{H}$ -stability. Because it incorporates the traditional idea of practical exponential stability into a single framework, this concept is incredibly adaptable. The work in [9] presents a converse Lyapunov theorem concerning practical $\mathcal{H}$ -stability. Building upon this, the author in [14] proposes a stability result that constructs a Lyapunov function ensuring global asymptotic stability for certain cascaded systems, even when their dynamics exhibit unbounded behavior over time. Studies such as [2] and [12] explore converse stability results in the context of nonlinear time-dependent systems. In [30], nonlinear time-varying systems are analyzed using the theory of stable maps. Moreover, the work in [13] introduces a novel methodology to evaluate the practical stability of time-dependent dynamics, leveraging the idea of making stable functions practically viable. The question then becomes whether the concept of a practical stable function may be applied to the $\mathcal{H}$ -stable practical instance.

As an extension of the practical exponential stability presented in [13], this study tackles the practical $\mathcal{H}$ -stability challenges faced by certain non-autonomous nonlinear dynamical systems. Accordingly, the concept of a practical $\mathcal{H}$ -stability function has been introduced. In other words, if a function is $\mathcal{H}$ -stable practical, then the examined system's practical $\mathcal{H}$ -stability can be assured. We introduce novel Lyapunov theorems for converses for the local and global UP $\mathcal{H}$ -stability. A Lyapunov function is consequently developed to ensure that systems subject to perturbations remain practically $\mathcal{H}$ -stable.

Finally, the paper is structured as follows. Section 2 outlines the model and offers preliminary explanations, including notations and the concept of practical $\mathcal{H}$ -stability. Section 3 presents sufficient stability conditions. Section 4 provides new converse stability theorems. The UP $\mathcal{H}$ S of perturbed and interconnected dynamical systems is established in Section 5. Section 6 presents our conclusion.

2. Fundamental Concepts

We begin by establishing the following notations:

- $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{> 0}$ denote the sets of non-negative and strictly positive real numbers respectively.
- For intervals $A, B \subset \mathbb{R}$, we define:
 - $\mathcal{PC}(A, B)$: piecewise continuous functions.
 - $\mathcal{B}_c(A, B)$: bounded continuous functions.
 - $\mathcal{C}_0(A, B)$: continuous functions.
 - $\mathcal{C}_1(A, B)$: continuously differentiable functions.
- The closed ball in $\mathbb{R}^n$: $\mathcal{D}_r = \{z \in \mathbb{R}^n | PzP \leq r\}$
- $\tilde{L}^{\tilde{p}}(\mathbb{R}_{\geq 0})$ denotes $\tilde{p}$ -integrable functions with $\tilde{p} \geq 1$.
- A function $\bar{\alpha} \in \mathcal{C}_0(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$ is said to belong to the class $\mathcal{B}_\infty$ if it satisfies the following properties: it vanishes at zero, it is strictly increasing over $\mathbb{R}_{\geq 0}$, and $\lim_{r \rightarrow \infty} \bar{\alpha}(r) = +\infty$.

Consider the nonlinear system

$$\dot{\eta}(\wp) = \Psi(\wp, \eta(\wp)), \quad (1)$$

where $\Psi: \mathbb{R}_{> 0} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is:

- Piecewise continuous in $\wp$.
- Locally Lipschitz continuous in η .

Define the function $\bar{\mathcal{F}}$ as the solution flow of equation (1), which satisfies the condition $\bar{\mathcal{F}}(\wp, \wp, \eta) = \eta$.

Definition 2.1: [9] For $\mathcal{H} \in \mathcal{B}_c(\mathbb{R}_{\geq 0}, \mathbb{R}_{> 0})$.

- Dynamic (1) is *Locally Uniformly Practically $\mathcal{H}$ -Stable* (LUP $\mathcal{H}$ S) if $\exists r, \ell > 0$, and $\bar{c} \geq 1$, so that for any η with $P\eta P \leq r$, the following condition holds

$$\|\bar{\mathcal{F}}(j, \wp, \eta)\| \leq \bar{c}\|\eta\|\mathcal{H}(j)\mathcal{H}(\wp)^{-1} + \ell, \quad \forall j \geq \wp. \quad (2)$$

- Dynamic (1) is *Globally Uniformly Practically $\mathcal{H}$ -Stable* (GUP $\mathcal{H}$ S) if $\exists \ell > 0$ and $\bar{c} \geq 1$ so that $\forall \wp$ and $\forall \eta$, the solutions satisfy (2).

Accordingly, $\mathcal{H}(\wp)^{-1}$ is defined as $\frac{1}{\mathcal{H}(\wp)}$.

Lemma 2.1: [17] Set $\bar{\chi}, \bar{\xi} \in \mathcal{P}_c(\mathbb{R}_{\geq 0}, \mathbb{R})$, $\bar{\omega} \in \mathcal{C}_1(\mathbb{R}_{\geq 0}, \mathbb{R})$ be functions that satisfy the differential equation

$$\dot{\bar{\omega}}(\wp) = \bar{\xi}(\wp) + \bar{\chi}(\wp)\bar{\omega}(\wp). \quad (3)$$

Then, $\forall j \geq \wp$, the solution is given by

$$\bar{\omega}(j) = \bar{\omega}(\wp)\Theta(j, \wp) + \int_{\wp}^j \bar{\xi}(\iota)\bar{\Theta}(j, \iota)d\iota,$$

where $\bar{\Theta}(j, \wp) = \exp\left(\int_{\wp}^j \bar{\chi}(\iota)d\iota\right)$.

Definition 2.2: [10] Set functions $\bar{\chi}, \bar{\xi} \in \mathcal{P}_c(\mathbb{R}_{\geq 0}, \mathbb{R})$, and let $\mathcal{H} \in \mathcal{C}_1(\mathbb{R}_{\geq 0}, \mathbb{R}_{>0}) \cap \mathcal{B}_c(\mathbb{R}_{\geq 0}, \mathbb{R}_{>0})$. The function $\bar{\chi}$ is referred to as globally uniformly practically $\mathcal{H}$ -stable relative to $\bar{\xi}$ (GUP $\mathcal{H}$ S with respect to $\bar{\xi}$) if there are constants $\bar{\lambda} \in \mathbb{R}_{\geq 0}$ and $\bar{\rho} \in \mathbb{R}_{>0}$ so that $\forall j \geq \wp$, the following inequalities hold:

$$\int_{\wp}^j \bar{\chi}(\iota)d\iota \leq \int_{\wp}^j \mathcal{H}'(\iota)\mathcal{H}(\iota)^{-1}d\iota + \bar{\lambda},$$

and

$$\int_{\wp}^{\iota} |\bar{\xi}(\iota)|\bar{\Theta}(\iota, \wp)d\iota \leq \bar{\rho},$$

where

$$\bar{\Theta}(j, \wp) = e^{\bar{\lambda}\mathcal{H}(j)\mathcal{H}(\wp)^{-1}}.$$

3. Lyapunov-based analysis of UP $\mathcal{H}$ S

This section examines the UP $\mathcal{H}$ S of the solutions to system (1) through the application of Lyapunov theory.

Theorem 3.1: Set $\mathcal{H} \in \mathcal{C}_1(\mathbb{R}_{\geq 0}, \mathbb{R}_{>0})$ be a decreasing function. Assume there exist a continuously differentiable mapping $\Xi: \mathbb{R}_{\geq 0} \times \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$, a function $\bar{\alpha} \in \mathcal{B}_{\infty}$, constants $\kappa \in \mathbb{R}_{>0}$, $\bar{p} \geq 1$, and $\zeta \in \mathbb{R}_{\geq 0}$, as well as $\bar{\chi}, \bar{\xi} \in \mathcal{P}_c(\mathbb{R}_{\geq 0}, \mathbb{R})$, so that $\forall \wp \in \mathbb{R}_{\geq 0}$ and all $\eta \in \mathbb{R}^n$, the following inequality holds:

$$\kappa\|\eta\|^{\bar{p}} \leq \Xi(\wp, \eta) \leq \zeta + \bar{\alpha}(\|\eta\|), \quad (4)$$

$$\frac{\partial \Xi}{\partial \eta}(\wp, \eta)\Psi(\wp, \eta) + \frac{\partial \Xi}{\partial \wp}(\wp, \eta) \leq \bar{\chi}(\wp)\Xi(\wp, \eta) + \bar{\xi}(\wp). \quad (5)$$

Thus, system (1) is LUP $\mathcal{H}^{\frac{1}{\bar{p}}}$ S if $\bar{\chi}$ is GUP $\mathcal{H}$ S with respect to $\bar{\xi}$.

Proof: Invoking Lemma 2.1, it follows that the following holds $\iota \geq \wp$

$$\Xi(\iota, \bar{\mathcal{F}}(\iota, \wp, \eta)) \leq \Theta(\iota, \wp)\Xi(\wp, \eta) + \int_{\wp}^{\iota} \Theta(\iota, \sigma)\bar{\xi}(\sigma)d\sigma,$$

where $\Theta(\iota, \wp) = \exp\left(\int_{\wp}^{\iota} \bar{\chi}(j)dj\right)$.

Thus, for all $\iota \geq \wp$, we have

$$\kappa \|\bar{\mathcal{F}}(\iota, \wp, \eta)\|^{\tilde{p}} \leq \Xi(\wp, \eta) \Theta(\iota, \wp) + \int_{\wp}^{\iota} \bar{\xi}(j) \Theta(\iota, j) dj.$$

Since $\bar{\chi}$ is GUP $\mathcal{H}$ S with respect to $\bar{\xi}$, for all $\iota \geq \wp$, we obtain

$$\kappa \|\bar{\mathcal{F}}(\iota, \wp, \eta)\|^{\tilde{p}} \leq \bar{\alpha}(\|\eta\|) e^{\lambda \mathcal{H}(\iota) \mathcal{H}(\wp)^{-1}} + \zeta e^{\lambda} + \bar{\rho}.$$

Then, for any $\iota \geq \wp$, it holds that

$$\|\bar{\mathcal{F}}(\iota, \wp, \eta)\| \leq \kappa^{-\frac{1}{\tilde{p}}} \bar{\alpha}^{\frac{1}{\tilde{p}}}(\|\eta\|) e^{\frac{\lambda}{\tilde{p}} \mathcal{H}(\iota) \mathcal{H}(\wp)^{-1}} + \kappa^{-\frac{1}{\tilde{p}}} (\zeta e^{\lambda} + \bar{\rho})^{\frac{1}{\tilde{p}}}.$$

For any $\varepsilon \in \mathbb{R}_{>0}$, we select $\bar{c} \geq 1$, so that

$$\bar{\delta} = \bar{\alpha}^{-1}(\kappa \varepsilon^{\tilde{p}} e^{-\lambda}) \geq \frac{\varepsilon}{\bar{c}}.$$

If we choose an initial state η , such that

$$\bar{\delta} \geq \|\eta\| \geq \frac{\varepsilon}{\bar{c}},$$

then $\forall \iota \geq \wp$, we have

$$\|\bar{\mathcal{F}}(\iota, \wp, \eta)\| \leq \varepsilon \mathcal{H}^{\frac{1}{\tilde{p}}}(\iota) \mathcal{H}(\wp)^{-\frac{1}{\tilde{p}}} + \kappa^{-\frac{1}{\tilde{p}}} (\zeta e^{\lambda} + \bar{\rho})^{\frac{1}{\tilde{p}}}.$$

Thus, we can conclude that

$$\|\bar{\mathcal{F}}(\iota, \wp, \eta)\| \leq \bar{c} \|\eta\| \mathcal{H}^{\frac{1}{\tilde{p}}}(\iota) \mathcal{H}(\wp)^{-\frac{1}{\tilde{p}}} + \kappa^{-\frac{1}{\tilde{p}}} (\zeta e^{\lambda} + \bar{\rho})^{\frac{1}{\tilde{p}}}.$$

Hence, the system (1) is LUP $\mathcal{H}^{\frac{1}{\tilde{p}}}$ S.

Theorem 3.2: Set $\mathcal{H} \in \mathcal{C}_1(\mathbb{R}_{\geq 0}, \mathbb{R}_{>0})$ be a monotonically decreasing map. Suppose there exists a $\mathcal{C}_1$ function $\Xi: \mathbb{R} + \times \mathbb{R}^n \rightarrow \mathbb{R}$, positive constants b_1, b_2, ζ with $b_1 > 0$, $\tilde{p} \geq 1$ and functions $\bar{\chi}, \bar{\xi} \in \mathcal{P}_c(\mathbb{R}_{\geq 0}, \mathbb{R})$. The following statements hold for all $\wp$ and all η :

$$b_1 \|\eta\|^{\tilde{p}} \leq \Xi(\wp, \eta) \leq b_2 \|\eta\|^{\tilde{p}} + \zeta, \quad (6)$$

$$\frac{\partial \Xi}{\partial \wp}(\wp, \eta) + \frac{\partial \Xi}{\partial \eta}(\wp, \eta) \Psi(\wp, \eta) \leq \bar{\chi}(\wp) \Xi(\wp, \eta) + \bar{\xi}(\wp). \quad (7)$$

Thus, system (1) is GUP $\mathcal{H}^{\frac{1}{\tilde{p}}}$ S if $\bar{\chi}$ is GUP $\mathcal{H}$ S with respect to $\bar{\xi}$.

Proof: Using similar reasoning as stated in Theorem 3.1, for all $\bar{s} \geq \wp$, we derive:

$$\|\bar{\mathcal{F}}(j, \wp, \eta)\|^{\tilde{p}} \leq \frac{b_2}{b_1} e^{\lambda} \|\eta\|^{\tilde{p}} \mathcal{H}(j) \mathcal{H}(\wp)^{-1} + \frac{\zeta e^{\lambda}}{b_1} + \frac{\bar{\rho}}{b_1}.$$

Then,

$$\|\bar{\mathcal{F}}(j, \wp, \eta)\| \leq \left(\frac{b_2 e^{\lambda}}{b_1} \right)^{\frac{1}{\tilde{p}}} \|\eta\| \mathcal{H}^{\frac{1}{\tilde{p}}}(j) \mathcal{H}(\wp)^{-\frac{1}{\tilde{p}}} + \left(\frac{\zeta e^{\lambda}}{b_1} + \frac{\bar{\rho}}{b_1} \right)^{\frac{1}{\tilde{p}}}.$$

Thus, system (1) is GUP $\mathcal{H}^{\frac{1}{\tilde{p}}}$ S with the constants $\bar{c} = \left(\frac{b_2 e^{\lambda}}{b_1} \right)^{\frac{1}{\tilde{p}}}$ and $\ell =$

$$\left(\frac{\zeta e^{\lambda}}{b_1} + \frac{\bar{\rho}}{b_1} \right)^{\frac{1}{\tilde{p}}}.$$

Remark 3.1: It can be observed that Theorem 3.2 extends the results presented in [13].

Example 3.1: Consider:

$$\dot{\eta}(\wp) = \left(-\bar{\gamma}(\wp) + \frac{1}{1+\wp^2} \right) \eta + \frac{2\wp\gamma(\wp)|\cos(\wp)|\bar{u}(\wp,\eta)}{1+\eta^2}, \quad (8)$$

where $\eta \in \mathbb{R}$, $\bar{u} \in \mathbb{R}$ and $\bar{\gamma}(\wp) \in \mathbb{R}_{\geq 0}$.

Let $\Xi = \frac{1}{2}\eta^2$, which satisfies the inequalities given in (6), with $b_1 = b_2 = \frac{1}{2}$ and $\zeta = 0$, for all $\wp \in \mathbb{R}_{\geq 0}$. Then,

$$\dot{\Xi}(\wp, \eta) \leq 2 \left(-\bar{\gamma}(\wp) + \frac{1}{1+\wp^2} \right) \Xi(\wp, \eta) + \wp \bar{\gamma}(\wp) |\bar{u}(\wp, \eta)|.$$

If $\bar{u} = 0$, for all $\wp \in \mathbb{R}_{\geq 0}$ and all $\eta \in \mathbb{R}$, then the system (8) is GUPHS with $\mathcal{H}(\wp) = \exp(-2 \int_0^\wp \bar{\gamma}(\sigma) d\sigma)$.

Now, assuming there exists $\bar{\xi}$, so that $\forall \wp$ and $\forall \eta$, we have $|\wp u(\wp, \eta)| \leq \kappa$, then we obtain the inequality:

$$\dot{\Xi}(\wp, \eta) \leq \left(-2\bar{\gamma}(\wp) + \frac{2}{1+\wp^2} \right) \Xi(\wp, \eta) + \kappa \bar{\gamma}(\wp).$$

Let us define

$$\bar{\chi}(\wp) = 2 \left(\frac{1}{1+\wp^2} - \bar{\gamma}(\wp) \right), \quad \bar{\xi}(\wp) = \kappa \bar{\gamma}(\wp).$$

A constant $\bar{\rho} \in \mathbb{R}_{>0}$ can be found such that the inequality below is satisfied for all $\bar{s} \geq \wp$:

$$\int_\wp^j \bar{\chi}(\sigma) d\sigma \leq \int_\wp^j -2\bar{\gamma}(\sigma) d\sigma + \pi$$

and

$$\int_\wp^j \bar{\xi}(\tau) \Theta(j, \sigma) d\sigma \leq \bar{\rho},$$

where $\Theta(j, \wp) = \exp \left(\int_\wp^j \bar{\chi}(\sigma) d\sigma \right)$ and $\bar{\rho} = \frac{\kappa}{2} e^\pi$. Therefore, the system $\bar{\chi}$ satisfies the property of GUPHS with respect to $\bar{\xi}$, where the function $\mathcal{H}(\wp)$ is defined by

$$\mathcal{H}(\wp) = \exp(-2 \int_0^\wp \bar{\gamma}(\sigma) d\sigma).$$

Consequently, all the conditions required by Theorem 3.2 are fulfilled, which implies that the system given by equation (8) is GUPHS.

For the simulation, we choose $\bar{\gamma}(\wp) = \frac{1}{1+\wp}$, $u(\wp, \eta) = \frac{2}{(1+\wp^2)\sqrt{1+\eta^2}}$, and $\eta(0) = 0.5$. The evolution of the corresponding solution $\eta(\wp)$ is illustrated in Figure 1.

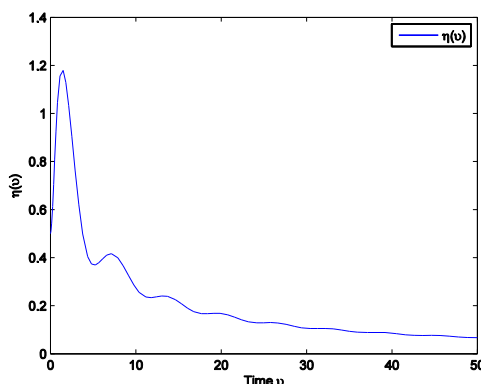


Figure 1
Evolution over time of the solution $\eta(\varphi)$ corresponding to system (8).

4. Inverse results for $\mathcal{H}$ -stability theorems

We introduce novel converse Lyapunov theorems for the system (1) in this section, considering both local and global uniform practical $\mathcal{H}$ -stability. This extends the converse theorem presented in [13], which addresses a specific case where $\mathcal{H}(\varphi) = e^{-b\varphi}$ for $b \in \mathbb{R}_{\geq 0}$. Our results broaden the scope of the converse theorems discussed in [2], [12], [13].

To derive the necessary properties of $\Psi(\cdot, \cdot)$, we first introduce certain subclasses of $\mathcal{C}_0(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$, following the framework proposed in [12].

Definition 4.1: A continuous function $\mathcal{L}$ belongs to class $\mathcal{B}_N$ if $\exists M \in \mathcal{C}_0(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$ satisfying $\forall \varphi, \theta \in \mathbb{R}_{\geq 0}$:

$$\int_{\varphi}^{\varphi+\theta} \mathcal{L}(\sigma) d\sigma \leq \bar{M}(\theta).$$

Definition 4.2: A continuous function $\mathcal{L}$ belongs to $\mathcal{B}_N^2$ if $\exists N \in \mathcal{C}_0(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$, satisfying $\forall \varphi, \theta \in \mathbb{R}_{\geq 0}$:

$$\int_{\varphi}^{\varphi+\theta} \mathcal{L}^2(\sigma) d\sigma \leq \bar{N}(\theta).$$

($\mathcal{A}_1$) One can find functions $\bar{R}(\cdot) \in \mathcal{B}_N$ and $\mathcal{K}(\cdot) \in \mathcal{B}_N^2$ verifying the following condition:

$$\|\Psi(\varphi, \eta)\| \leq \mathcal{L}(\varphi)\|\eta\| + \mathcal{K}(\varphi), \mathcal{L}(\varphi) = \mathcal{L} + \bar{R}(\varphi) > 0.$$

($\mathcal{A}_2$) System (1) is termed LUPHS, that is $\exists r, \ell \in \mathbb{R}_{\geq 0}$, and $\bar{c} \geq 1$, so that $\forall \varphi \in \mathbb{R}_{\geq 0}$ and $\forall \eta$, $P\eta P \leq r$, system (1) satisfies (2).

Theorem 4.1: Under the assumptions ($\mathcal{A}_1$) and ($\mathcal{A}_2$), suppose there exists a constant $\rho > 0$ such that:

- (i) The function $\mathcal{H}: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{>0}$ is strictly decreasing.
- (ii) The following inequality holds:

$\mathcal{H}$ is decreasing and $\mathcal{H}(v)\mathcal{H}(v + \theta)^{-1} \leq \rho$, for all $v \in \mathbb{R}_{\geq 0}$, $\theta \in \mathbb{R}_{>0}$. (9)

Then, there exists a continuously differentiable function $\Xi: \mathbb{R}_{\geq 0} \times \mathbb{R}^n \rightarrow \mathbb{R}$, $b_1 > 0$, $b_2 \geq 0$, $\zeta, r > 0$, $\tilde{p} \geq 2$, and $\bar{\chi}, \bar{\xi} \in \mathcal{P}_c(\mathbb{R}_{\geq 0}, \mathbb{R})$, such that the following inequality holds:

$$b_1 \|\eta\|^{\tilde{p}} \leq \Xi(\wp, \eta) \leq b_2 \|\eta\|^{\tilde{p}} + \zeta,$$

$$\frac{\partial \Xi}{\partial \eta}(\wp, \eta) \Psi(\wp, \eta) + \frac{\partial \Xi}{\partial \wp}(\wp, \eta) \leq \bar{\chi}(\wp) \Xi(\wp, \eta) + \bar{\xi}(\wp),$$

for all $(\wp, \eta) \in \mathbb{R}_{\geq 0} \times \mathcal{D}_r$, where $\bar{\chi}$ is a GUP $\mathcal{H}S$ with respect to $\bar{\xi}$.

Proof: Consider:

$$\Xi(\wp, \eta) = \mathcal{H}(\wp) \int_{\wp}^{\wp+\theta} \omega(j) (\mathcal{P}\bar{\mathcal{F}}(j, \wp, \eta) \mathcal{P}^{\tilde{p}} + [\frac{e^{2\bar{M}(\theta)}}{2\mathcal{L}} \int_{\wp}^s \mathcal{K}^2(\sigma) d\sigma]^{\tilde{p}^2}) \mathcal{H}^{-1}(j) j,$$

where $\omega \in \mathcal{C}_0(\mathbb{R}_{\geq 0}, \mathbb{R}_{>0}) \cap \tilde{L}^1(\mathbb{R}_{\geq 0})$ denotes a non-increasing function with the property that $\inf_{\wp \geq 0} \omega(\wp) > 0$.

We have

$$\begin{aligned} \Xi(\wp, \eta) &= \mathcal{H}(\wp) \int_{\wp}^{\wp+\theta} \omega(j) (\|\bar{\mathcal{F}}(j, \wp, \eta)\|^{\tilde{p}} + [e^{2\bar{M}(\theta)} 2\mathcal{L} \int_{\wp}^s \mathcal{K}^2(\sigma) d\sigma]^{\tilde{p}^2}) \mathcal{H}(j)^{-1} dj \\ &\leq \mathcal{H}(\wp) \int_{\wp}^{\wp+\theta} \omega(j) (\bar{c}\mathcal{H}(j) \mathcal{H}(\wp)^{-1} \|\eta\| + \ell)^{\tilde{p}} + [e^{2\bar{M}(\theta)} 2\mathcal{L} \int_{\wp}^j \mathcal{K}^2(\sigma) d\sigma]^{\tilde{p}^2}) \mathcal{H}(j)^{-1} dj. \end{aligned}$$

By applying the properties of $\mathcal{H}$, we obtain the bound

$$\Xi(\wp, \eta) \leq 2^{\tilde{p}-1} \bar{c}^{\tilde{p}} \theta \omega(0) \|\eta\|^{\tilde{p}} + (2^{\tilde{p}-1} \ell^{\tilde{p}} + [e^{2\bar{M}(\theta)} 2\mathcal{L} \bar{N}(\theta)]^{\tilde{p}^2}) \rho \theta \omega(0).$$

On the other hand, we get

$$\begin{aligned} \left| \frac{d}{dj} \bar{\mathcal{F}}^T(j, \wp, \eta) \bar{\mathcal{F}}(j, \wp, \eta) \right| &\leq 2 \|\bar{\mathcal{F}}(j, \wp, \eta)\| \|\Psi(j, \eta(j, \wp, \eta))\| \\ &\leq 2\mathcal{L}(j) \|\bar{\mathcal{F}}(j, \wp, \eta)\|^2 + 2\mathcal{K}(j) \|\bar{\mathcal{F}}(j, \wp, \eta)\|. \end{aligned}$$

Then,

$$\frac{d}{dj} \bar{\mathcal{F}}^T(j, \wp, \eta) \bar{\mathcal{F}}(j, \wp, \eta) \geq -2\mathcal{L}(j) \|\bar{\mathcal{F}}(j, \wp, \eta)\|^2 - 2\mathcal{K}(j) \|\bar{\mathcal{F}}(j, \wp, \eta)\|. \quad (10)$$

Let $\iota(j) = -\|\bar{\mathcal{F}}(j, \wp, \eta)\|$, and by utilizing (10), we derive, similar to the approach in [16], that

$$D^+ \iota(j) \leq \mathcal{K}(j) - \mathcal{L}(j) \iota(j),$$

such that

$$D^+ \iota(\wp) = \limsup_{\mathcal{T} \rightarrow 0^+} \frac{1}{\mathcal{T}} (\iota(\wp + \mathcal{T}) - \iota(\wp)).$$

Let

$$\bar{u}(j) = e^{\int_{\wp}^j \mathcal{L}(\sigma) d\sigma} \iota(j),$$

we obtain

$$D^+ \bar{u}(j) \leq e^{\int_{\wp}^j \mathcal{L}(\sigma) d\sigma} \mathcal{K}(j).$$

Integrating over $[\wp, j]$, we derive

$$\iota(j) \leq \iota(\wp) e^{-\int_{\wp}^j \mathcal{L}(\sigma) d\sigma} + \left(\int_{\wp}^j \mathcal{K}(\sigma) e^{\int_{\wp}^{\sigma} \mathcal{L}(\varsigma) d\varsigma} d\sigma \right) e^{-\int_{\wp}^j \mathcal{L}(\sigma) d\sigma}.$$

On the one side, we have

$$\begin{aligned} \int_{\wp}^j \mathcal{L}(\sigma) d\sigma &= \mathcal{L}(j - \wp) + \int_{\wp}^j \bar{K}(\sigma) d\sigma, \\ &\leq \mathcal{L}(j - \wp) + \bar{M}(\theta), \text{ for all } j \in [\wp, \wp + \theta], \text{ all } \wp \in \mathbb{R}_{\geq 0}, \end{aligned}$$

As a result, we establish the bound

$$e^{\int_{\wp}^j \mathcal{L}(\sigma) d\sigma} \leq e^{\mathcal{L}(j - \wp)} e^{\bar{M}(\theta)}, \text{ for all } j \in [\wp, \wp + \theta], \text{ all } \wp \in \mathbb{R}_{\geq 0}.$$

On the other side, we have

$$\begin{aligned} &\left[\int_{\wp}^j \mathcal{K}(\sigma) e^{\int_{\wp}^{\sigma} \mathcal{L}(\varsigma) d\varsigma} d\sigma \right] e^{-\int_{\wp}^j \mathcal{L}(\sigma) d\sigma} \leq e^{\bar{M}(\theta)} \left(\int_{\wp}^j \bar{K}(\sigma) e^{\mathcal{L}(\sigma - \wp)} d\sigma \right) e^{-\int_{\wp}^j \mathcal{L}(\sigma) d\sigma} \\ &\leq e^{\bar{M}(\theta)} \left(\int_{\wp}^j \mathcal{K}^2(\sigma) d\sigma \right)^{12} \left[\int_{\wp}^s e^{2\mathcal{L}(\sigma - \wp)} d\sigma \right]^{12} e^{-\mathcal{L}(j - \wp)} \\ &\leq \left(\frac{e^{2\bar{M}(\theta)}}{2\mathcal{L}} \int_{\wp}^j \mathcal{K}^2(\sigma) d\sigma \right)^{12}. \end{aligned}$$

Thus, for all $j \in [\wp, \wp + \theta]$, and all $\wp \in \mathbb{R}_{\geq 0}$,

$$\|\bar{\mathcal{F}}(j, \wp, \eta)\| + \left(\frac{e^{2\bar{M}(\theta)}}{2\mathcal{L}} \int_{\wp}^j \mathcal{K}^2(\sigma) d\sigma \right)^{12} \geq \|\eta\| e^{-\mathcal{L}(j - \wp)} e^{-\bar{M}(\theta)}.$$

Observing that

$$\begin{aligned} &\left[\|\bar{\mathcal{F}}(j, \wp, \eta)\| + \left(\frac{e^{2\bar{M}(\theta)}}{2\mathcal{L}} \int_{\wp}^j \mathcal{K}^2(\sigma) d\sigma \right)^{12} \right]^p \leq 2^{\tilde{p}-1} \|\bar{\mathcal{F}}(j, \wp, \eta)\|^{\tilde{p}} + \\ &2^{\tilde{p}-1} \left[\left(\frac{e^{2\bar{M}(\theta)}}{2\mathcal{L}} \int_{\wp}^j \mathcal{K}^2(\sigma) d\sigma \right)^{12} \right]^{\tilde{p}}, \end{aligned}$$

we obtain for all $j \in [\wp, \wp + \theta]$ and all $\wp \in \mathbb{R}_{\geq 0}$,

$$\|\bar{\mathcal{F}}(j, \wp, \eta)\|^{\tilde{p}} + \left[\frac{e^{2\bar{M}(\theta)}}{2\mathcal{L}} \int_{\wp}^j \mathcal{K}^2(\sigma) d\sigma \right]^{\tilde{p}2} \geq 12^{\tilde{p}-1} \mathbf{P} \eta \mathbf{P}^{\tilde{p}} e^{-\tilde{p}\mathcal{L}(j - \wp)} e^{-\tilde{p}\bar{M}(\theta)}. \quad (11)$$

Consequently,

$$\begin{aligned} \Xi(\wp, \eta) &\geq \mathcal{H}(\wp) \int_{\wp}^{\wp + \theta} \frac{\omega(j)}{2^{\tilde{p}-1}} \|\eta\|^{\tilde{p}} e^{-\tilde{p}(\theta)} e^{-\tilde{p}(j - \wp)} \mathcal{H}(j)^{-1} ds \\ &\geq \frac{e^{-\tilde{p}(\theta)} (1 - e^{-\tilde{p}\theta})}{2^{\tilde{p}-1} \tilde{p}\mathcal{L}} \inf_{\wp \geq 0} (\omega(\wp)). \end{aligned}$$

It follows that $\Xi(\wp, \eta)$ fulfills the inequality stated in Theorem 4.1 with

$$b_1 = \frac{e^{-\bar{M}(\theta)} \tilde{p} (1 - e^{-\tilde{p}\theta})}{2^{\tilde{p}-1} \tilde{p}\mathcal{L}} \inf_{\wp \geq 0} (\omega(\wp)),$$

$$b_2 = 2^{\tilde{p}-1} \bar{c}^{\tilde{p}} \theta \omega(0),$$

and

$$\zeta = (2^{\tilde{p}-1} \rho^{\tilde{p}} + \left[\frac{e^{2\bar{M}(\theta)}}{2\mathcal{L}} \bar{N}(\theta) \right]^{\tilde{p}2}) \rho \theta \omega(0).$$

Set,

$$\begin{aligned}\Xi(j, \bar{\mathcal{F}}(j, \wp, \eta)) &= \mathcal{H}(j) \int_j^{j+\theta} \omega(\bar{u}) (\|\bar{\mathcal{F}}(\bar{u}, j, \bar{\mathcal{F}}(j, \wp, \eta))\|^{\tilde{p}} + \\ &\quad [\frac{e^{2\bar{M}(\theta)}}{2\mathcal{L}} \int_j^{\bar{u}} \mathcal{K}^2(\sigma) d\sigma]^{\tilde{p}^2}) \mathcal{H}(\bar{u})^{-1} d\bar{u}.\end{aligned}$$

Therefore,

$$\begin{aligned}\Xi(j, \bar{\mathcal{F}}(j, \wp, \eta)) &= \mathcal{H}(j) \int_j^{j+\theta} \omega(\bar{u}) (\|\bar{\mathcal{F}}(\bar{u}, \wp, \eta)\|^{\tilde{p}} + \\ &\quad [\frac{e^{2\bar{M}(\theta)}}{2\mathcal{L}} \int_j^{\bar{u}} \mathcal{K}^2(\sigma) d\sigma]^{\frac{\tilde{p}}{2}}) \mathcal{H}(\bar{u})^{-1} d\bar{u}.\end{aligned}$$

Thus,

$$\begin{aligned}\dot{\Xi}(\wp, \eta) &= \frac{d}{dj} (\Xi(j, \bar{\mathcal{F}}(j, \wp, \eta)))_{/j=\wp} \\ &= \mathcal{H}'(\wp) \mathcal{H}(\wp)^{-1} \Xi(\wp, \eta) - \omega(\wp) \|\eta\|^{\tilde{p}} \\ &\quad + \omega(\wp + \theta) \mathcal{H}(\wp) \mathcal{H}(\wp + \theta)^{-1} \|\bar{\mathcal{F}}(\wp + \theta, \wp, \eta)\|^{\tilde{p}} \\ &\quad + \mathcal{H}(\wp) \mathcal{H}(\wp + \theta)^{-1} \omega(\wp + \theta) [\frac{e^{2\bar{M}(\theta)}}{2\mathcal{L}} \int_{\wp}^{\wp+\theta} \mathcal{K}^2(\sigma) d\sigma]^{\frac{\tilde{p}}{2}} \\ &\quad - \frac{\tilde{p}\theta e^{2\bar{M}(\theta)}}{4\mathcal{L}} \mathcal{K}^2(\wp) \int_{\wp}^{\wp+\theta} \omega(\bar{u}) [\frac{e^{2\bar{M}(\theta)}}{2\mathcal{L}} \int_{\wp}^{\bar{u}} \mathcal{K}^2(\sigma) d\sigma]^{\tilde{p}^2-1} d\bar{u} \\ &\quad + \omega(\wp + \theta) \mathcal{H}(\wp) \mathcal{H}(\wp + \theta)^{-1} [\frac{e^{2\bar{M}(\theta)}}{2\mathcal{L}} \int_{\wp}^{\wp+\theta} \mathcal{K}^2(\sigma) d\sigma]^{\frac{\tilde{p}}{2}} \\ &\leq \mathcal{H}'(\wp) \mathcal{H}(\wp)^{-1} \Xi(\wp, \eta) + \omega(\wp) (2^{\tilde{p}-1} \bar{c}^{\tilde{p}} - 1) r^{\tilde{p}} \\ &\quad + \omega(\wp) \rho (2^{\tilde{p}-1} \ell^{\tilde{p}} + [\frac{e^{2\bar{M}(\theta)}}{2\mathcal{L}} \bar{N}(\theta)]^{\frac{\tilde{p}}{2}}).\end{aligned}$$

Consequently, the second inequality in Theorem 4.1 holds with

$$\bar{\chi}(\wp) = \mathcal{H}'(\wp) \mathcal{H}(\wp)^{-1},$$

and

$$\bar{\xi}(\wp) = \omega(\wp) [(2^{\tilde{p}-1} \bar{c}^{\tilde{p}} - 1) r^{\tilde{p}} + \rho (2^{\tilde{p}-1} \ell^{\tilde{p}} + [\frac{e^{2\bar{M}(\theta)}}{2\mathcal{L}} \bar{N}(\theta)]^{\frac{\tilde{p}}{2}})].$$

where $\bar{\chi}$ is GUPHS with respect to $\bar{\xi}$.

Now, the question that is raised here: is there a Lyapunov function that fulfills the conditions specified in Theorem 3.2 when the system (1) is GUPHS? The response to this question is as follows.

($\mathcal{A}_3$) The system (1) is GUPHS, that is One can find constants $\ell \in \mathbb{R}_{>0}$ and $\bar{c} \geq 1$ for which, for all $\wp \in \mathbb{R}_{\geq 0}$ and any initial condition η , the system (1) satisfies (2)

Theorem 4.2: Under assumptions ($\mathcal{A}_1$), ($\mathcal{A}_3$) and that the function $\mathcal{H}$ satisfies condition (9). Then, one can construct a continuously differentiable function $\Xi: \mathbb{R}_{\geq 0} \times \mathbb{R}^n \rightarrow \mathbb{R}$, along with constants $b_1 \in \mathbb{R}_{>0}$, $b_2 \in \mathbb{R}_{\geq 0}$, $\zeta, b \in \mathbb{R}_{>0}$, and $\tilde{p} \geq 2$ and functions $\bar{\chi}, \bar{\xi} \in \mathcal{P}_c(\mathbb{R}_{\geq 0}, \mathbb{R})$, so that for all $(\wp, \eta) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$:

$$b_1 \|\eta\|^{\tilde{p}} \leq \Xi(\wp, \eta) \leq b_2 \|\eta\|^{\tilde{p}} + \zeta,$$

$$\frac{\partial \Xi}{\partial \wp}(\wp, \eta) + \frac{\partial \Xi}{\partial \eta}(\wp, \eta) \Psi(\wp, \eta) \leq \bar{\xi}(\wp) + \bar{\chi}(\wp) \Xi(\wp, \eta),$$

such that $\bar{\chi}(\wp)$ is GUPHS with respect to $\bar{\xi}$ where $\tilde{\mathcal{H}}(\wp) = \mathcal{H}(\wp)e^{b\wp}$.

Proof: Let

$$\begin{aligned} \Xi(\wp, \eta) &= \mathcal{H}(\wp) \int_{\wp}^{\wp+\theta} \omega(j) (\|\bar{\mathcal{F}}(j, \wp, \eta)\|^{\tilde{p}} + \\ &\quad [\frac{e^{2\bar{M}(\theta)}}{2\mathcal{L}} \int_{\wp}^j \mathcal{K}^2(\sigma) d\sigma]^{\tilde{p}^2}) \mathcal{H}^{-1}(j) dj, \end{aligned}$$

where $\omega \in \mathcal{C}_0(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$ is a decreasing map, satisfies $\omega(\wp)\omega(\wp+\theta) \leq \nu$, and $\omega\tilde{\mathcal{H}}^{-1} \in \tilde{L}^{\tilde{q}}(\mathbb{R}_{\geq 0})$, with $\tilde{q} > 1$.

Using the same arguments as in the theorem 4.1 with the same $c_1(\wp)$, $c_2(\wp)$ and $a(\wp)$.

Moreover, we have

$$\begin{aligned} \dot{\Xi}(\wp, \eta) &= \frac{d}{dj} (\Xi(j, \bar{\mathcal{F}}(j, \wp, \eta)))_{/j=\wp} \\ &= \mathcal{H}'(\wp) \mathcal{H}(\wp)^{-1} \Xi(\wp, \eta) \\ &\quad + \omega(\wp + \theta) \mathcal{H}(\wp) \mathcal{H}(\wp + \theta)^{-1} \|\bar{\mathcal{F}}(\wp + \theta, \wp, \eta)\|^{\tilde{q}} - \omega(\wp) \|\eta\|^{\tilde{p}} \\ &\quad + \omega(\wp + \theta) \mathcal{H}(\wp) \mathcal{H}(\wp + \theta)^{-1} \left[\frac{e^{2\bar{M}(\theta)}}{2\mathcal{L}} \int_{\wp}^{\wp+\theta} \mathcal{K}^2(\sigma) d\sigma \right]^{\frac{\tilde{p}}{2}} \\ &\quad - \frac{\tilde{p}\theta e^{2\bar{M}(\theta)}}{4\mathcal{L}} \mathcal{K}^2(\wp) \int_{\wp}^{\wp+\theta} \omega(\bar{u}) \left[\frac{e^{2\bar{M}(\theta)}}{2\mathcal{L}} \int_{\wp}^{\bar{u}} \mathcal{K}^2(\sigma) d\sigma \right]^{\frac{\tilde{p}}{2}-1} d\bar{u} \\ &\leq \mathcal{H}'(\wp) \mathcal{H}(\wp)^{-1} \Xi(\wp, \eta) \\ &\quad + \omega(\wp + \theta) \mathcal{H}(\wp) \mathcal{H}(\wp + \theta)^{-1} [\bar{c} \|\eta\| \mathcal{H}(\wp + \theta) \mathcal{H}(\wp)^{-1} + \ell]^{\tilde{p}} \\ &\quad - \omega(\wp) \|\eta\|^{\tilde{p}} + \omega(\wp + \theta) \mathcal{H}(\wp) \mathcal{H}(\wp + \theta)^{-1} \left[\frac{e^{2\bar{M}(\theta)}}{2\mathcal{L}} \int_{\wp}^{\wp+\theta} \mathcal{K}^2(\sigma) d\sigma \right]^{\frac{\tilde{p}}{2}} \\ &\leq (\mathcal{H}'(\wp) \mathcal{H}(\wp)^{-1} + b) \Xi(\wp, \eta) + \omega(\wp) \left[\rho \left(2^{\tilde{p}-1} \ell^{\tilde{p}} + \left[\frac{e^{2\bar{M}(\theta)}}{2\mathcal{L}} \bar{N}(\theta) \right]^{\frac{\tilde{p}}{2}} \right) \right], \end{aligned}$$

where

$$b = \frac{2^{\tilde{p}-1} \tilde{p} \mathcal{L} (2^{\tilde{p}-1} \bar{c}^{\tilde{p}-1} \nu)}{e^{-\tilde{p}\bar{M}(\theta)} (1 - e^{-\tilde{p}\mathcal{L}\theta})}.$$

Therefore, the second inequality in Theorem 4.2 holds with

$$\bar{\chi}(\wp) = \mathcal{H}'(\wp) \mathcal{H}(\wp)^{-1} + b,$$

and

$$\bar{\xi}(\wp) = \omega(\wp) [\rho (2^{\tilde{p}-1} \ell^{\tilde{p}} + [e^{2\bar{M}(\theta)} 2\mathcal{L} \bar{N}(\theta)]^{\frac{\tilde{p}}{2}})].$$

We have

$$\begin{aligned} \int_{\wp}^j \bar{\chi}(\sigma) d\sigma &\leq \int_{\wp}^j \mathcal{H}'(\sigma) \mathcal{H}^{-1}(\sigma) d\sigma + b(j - \wp) \\ &\leq \int_{\wp}^j \tilde{\mathcal{H}}'(\sigma) \tilde{\mathcal{H}}^{-1}(\sigma) d\sigma. \end{aligned}$$

Now, let's

$$\Theta(s, \wp) = \exp \left(\int_{\wp}^s \bar{\chi}(\sigma) d\sigma \right)$$

and $\mathcal{H}$ such that $\tilde{\mathcal{H}}(\wp) = \mathcal{H}(\wp)e^{b\wp} \in \mathcal{B}_c(\mathbb{R}_{\geq 0}, \mathbb{R}_{>0})$. Thus,

$$\begin{aligned} \int_{\wp}^s |\bar{\xi}(\sigma)| \Theta(s, \sigma) d\sigma &= \\ \rho(2^{\bar{p}-1} \ell^{\bar{p}} + [\frac{e^{2M(\theta)}}{2L} N(\theta)]^{\frac{\bar{p}}{2}}) \int_{\wp}^s \omega(\sigma) \mathcal{H}(s) e^{bs} \tilde{\mathcal{H}}^{-1}(\sigma) e^{-b\sigma} d\sigma. \\ &\leq \rho(2^{\bar{p}-1} \ell^{\bar{p}} + [\frac{e^{2M(\theta)}}{2L} N(\theta)]^{\frac{\bar{p}}{2}}) \|\tilde{\mathcal{H}}\|_{\infty} \int_{\wp}^s \omega(\sigma) \tilde{\mathcal{H}}^{-1}(\sigma) e^{-b\sigma} d\sigma. \end{aligned}$$

Given that $\omega \tilde{\mathcal{H}}^{-1} \in \tilde{L}^{\bar{q}}(\mathbb{R}_{\geq 0})$ and applying Hölder's inequality, one obtains

$$\int_{\wp}^s |\bar{\xi}(\sigma)| \Theta(s, \sigma) d\sigma \leq (2^{\bar{p}-1} \ell^{\bar{p}} + [\frac{e^{2M(\theta)}}{2L} N(\theta)]^{\frac{\bar{p}}{2}}) \frac{\rho \|\mathcal{H}\|_{\infty}^{(\bar{q}-1)}}{b\bar{q}} \|\omega \tilde{\mathcal{H}}^{-1}\|_{\bar{q}}.$$

Thus, $\bar{\chi}$ is GUPHS with respect to $\bar{\xi}$.

5. Dynamical systems with perturbations

Consider the following perturbed ODEs:

$$\dot{\eta}(\wp) = \bar{\Delta}(\wp, \eta) + \vartheta(\wp, \eta), \quad (12)$$

where where $\bar{\Delta}, \vartheta: \mathbb{R}_{\geq 0} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ are:

- Piecewise continuous in $\wp$
- Locally Lipschitz continuous in η .

Earlier research on the stability of equation (12) generally starts by analyzing the stability characteristics of the related system.:

$$\dot{\eta}(\wp) = \bar{\Delta}(\wp, \eta(\wp)). \quad (13)$$

Typically, a function $\Xi(\wp, \eta)$ for (13) is also considered for the perturbed ODEs, provided that the perturbation magnitude is well characterized (see [1], [16]). However, using $\Xi(\wp, \eta)$ directly may not always allow us to infer the performance of solutions to the system (12).

Example 5.1: Consider:

$$\dot{\eta}(\wp) = -\eta(\wp)\gamma(\wp) + \frac{e^{-\wp}}{1+\eta^2} + \Gamma(\wp)\eta(\wp), \quad (14)$$

where $\Gamma(\wp) = \frac{1}{1+\wp} \in \mathcal{B}_N$ and γ is continuous such that $\wp \in \mathbb{R}_{\geq 0} \mapsto \exp(\wp - \int_0^{\wp} 2\gamma(s)ds)$ is bounded, with $\bar{\Delta}(\wp, \eta) = -\gamma(\wp)\eta + \frac{e^{-\wp}}{1+\eta^2}$ and $\vartheta(\wp, \eta) = \Gamma(\wp)\eta$.

One can see that the system $\dot{\eta}(\wp) = \frac{e^{-\wp}}{1+\eta^2} - \gamma(\wp)\eta$ is GUPHS with $\Xi(\wp, \eta) = \frac{\eta^2}{2}$, $\bar{\chi}(\wp) = -2\gamma(\wp)$, $\bar{\xi}(\wp) = e^{-\wp}$ and $\mathcal{H}(\wp) = \exp(-2 \int_0^{\wp} \gamma(s)ds)$. If we differentiate Ξ along (14), we get

$$\dot{\Xi}(\wp, \eta) \leq (-2\gamma(\wp) + 2\Gamma(\wp))\Xi(\wp, \eta) + e^{-\wp}.$$

Here, $\tilde{\chi}(\wp) = -2\gamma(\wp) + 2\Gamma(\wp)$ is not GUP $\mathcal{H}$ S with respect to $\tilde{\xi}$ although the system (14) is GUP $\mathcal{H}$ S.

The objective consists in demonstrating that a system (12) preserves the same stability properties as a system (13). Subsequently, we strive to design a continuously differentiable function that guarantees the GUP $\mathcal{H}$ S of (12).

($\mathcal{A}_4$) One can find a continuously differentiable function Ξ , $\tilde{b}_1, \tilde{b}_2 > 0$, $\tilde{\zeta} \geq 0$, $\tilde{p} \geq 1$ and $\tilde{\chi}, \tilde{\xi} \in \mathcal{P}_c(\mathbb{R}_{\geq 0}, \mathbb{R})$, so that $\forall \wp$ and $\forall \eta$,

$$\tilde{b}_1 \|\eta\|^{\tilde{p}} \leq \Xi(\wp, \eta) \leq \tilde{\zeta} + \tilde{b}_2 \|\eta\|^{\tilde{p}}, \quad (15)$$

$$\frac{\partial \Xi}{\partial \wp}(\wp, \eta) + \frac{\partial \Xi}{\partial \eta}(\wp, \eta) \Psi(\wp, \eta) \leq \tilde{\xi}(\wp) + \tilde{\chi}(\wp) \Xi(\wp, \eta). \quad (16)$$

($\mathcal{A}_5$) One can find $\Gamma \in \mathcal{B}_N$, such that, $\forall \wp$ and $\forall \eta$,

$$\left| \frac{\partial \Xi}{\partial \eta}(\wp, \eta) \vartheta(\wp, \eta) \right| \leq \Gamma(\wp) \Xi(\wp, \eta). \quad (17)$$

Theorem 5.1: Consider $\tilde{\mathcal{H}} \in \mathcal{C}_1(\mathbb{R}_{\geq 0}, \mathbb{R}_{>0})$, and define $\mathcal{H}(\wp) = \tilde{\mathcal{H}}(\wp) e^{b\wp} > 0$, where $b > 0$ and $\tilde{\mathcal{H}}^{-1}|\tilde{\xi}| \in \tilde{L}^{\tilde{q}}(\mathbb{R}_{\geq 0})$, $\tilde{q} > 1$. If assumptions ($\mathcal{A}_4$) and ($\mathcal{A}_5$) hold, then the system (12) is GUP $\mathcal{H}^{\frac{1}{\tilde{p}}}$ S if $\tilde{\chi}$ is GUP $\mathcal{H}$ S with respect to $\tilde{\xi}$.

Proof: Given $\aleph \in \mathbb{R}_{>0}$, we define the following function:

$$Z_{\theta}(\wp, \eta) := \Xi(\wp, \eta) e^{\Lambda_{\theta}(\wp, \eta)}, \quad (18)$$

where

$$\Lambda_{\aleph}(\wp, \eta) = \int_{\wp}^{\wp + \aleph} \int_{\wp}^s \frac{1}{\aleph \Xi(j, \bar{\mathcal{F}}(j, \wp, \eta))} \frac{\partial \Xi}{\partial \eta}(j, \bar{\mathcal{F}}(j, v, \eta)) \vartheta(j, \bar{\mathcal{F}}(j, \wp, \eta)) dj ds.$$

Thus, by appropriately selecting $\aleph$, it is necessary to demonstrate that $Z_{\aleph}$ meets the requirements of Theorem 3.1, ensuring that (12) is GUP $\mathcal{H}^{\frac{1}{\tilde{p}}}$ S.

Since

$$\left| \frac{\partial \Xi}{\partial \eta}(\wp, \eta) \vartheta(\wp, \eta) \right| \leq \Gamma(\wp) \Xi(\wp, \eta),$$

then,

$$|\Lambda_{\aleph}(\wp, \eta)| \leq M(\aleph).$$

Therefore, $Z_{\aleph}$ satisfies the inequality (6) with

$$b_1 = \tilde{b}_1 e^{M(\aleph)}, b_2 = \tilde{b}_2 e^{M(\aleph)} \text{ and } \zeta = \tilde{\zeta} e^{M(\aleph)}.$$

The derivative of $Z_{\aleph}(\wp, \eta)$ along the trajectories of system (12) satisfies

$$\begin{aligned} \dot{Z}_{\aleph}(\wp, \eta) &= \dot{\Xi}(\wp, \eta) \exp(\Lambda_{\aleph}(\wp, \eta)) + \Lambda_{\aleph}(\wp, \eta) \Xi(\wp, \eta) \exp(\Lambda_{\aleph}(\wp, \eta)) \\ &\leq \tilde{\chi}(\wp) Z_{\aleph}(\wp, \eta) + \tilde{\xi}(\wp) e^{M(\aleph)} + \frac{M(\aleph)}{\aleph} Z_{\aleph}(\wp, \eta) \\ &\leq (\tilde{\chi}(\wp) + \frac{M(\aleph)}{\aleph}) Z_{\aleph}(\wp, \eta) + \tilde{\xi}(\wp) e^{M(\aleph)}, \end{aligned}$$

Accordingly, if $\aleph$ is set such that

$$M(\mathfrak{K}) < \mathfrak{K}b.$$

Thus, $Z_{\mathfrak{K}}$ fulfills (7) having

$$\bar{\chi}(\wp) = b + \tilde{\chi}(\wp), \bar{\xi}(\wp) = \tilde{\xi}(\wp)e^{M(\mathfrak{N})},$$

Because $\tilde{\chi}$ is GUPHS with respect to $\tilde{\xi}$, it follows that

$$\begin{aligned} \int_{\wp}^s \bar{\chi}(\sigma) d\sigma &\leq \int_{\wp}^s \tilde{\mathcal{H}}'(\sigma) \tilde{\mathcal{H}}^{-1}(\sigma) d\sigma + b(s - \wp) + \lambda \\ &\leq \int_{\wp}^s \mathcal{H}'(\sigma) \mathcal{H}^{-1}(\sigma) d\sigma + \lambda. \end{aligned}$$

Now, let's

$$\Theta(s, \wp) = e^{\lambda \mathcal{H}(s)} \mathcal{H}(\wp)^{-1}.$$

We have,

$$\int_{\wp}^s |\bar{\xi}(\sigma)| \Theta(s, \sigma) d\sigma = e^{(M(\mathfrak{N})+\lambda)} \int_{\wp}^s |\tilde{\xi}(\sigma)| \tilde{\mathcal{H}}(s) e^{bs} \tilde{\mathcal{H}}^{-1}(\sigma) e^{-b\sigma} d\sigma.$$

Since, $\mathcal{H} \in \mathcal{B}_c(\mathbb{R}_{\geq 0}, \mathbb{R}_{>0})$, then

$$\int_{\wp}^s |\bar{\xi}(\sigma)| \Theta(s, \sigma) d\sigma \leq e^{(M(\mathfrak{N})+\lambda)} \|\mathcal{H}\|_{\infty} \int_{\wp}^s |\tilde{\xi}(\sigma)| \tilde{\mathcal{H}}^{-1}(\sigma) e^{-b\sigma} d\sigma,$$

Thus, by utilizing the property that $|\tilde{\xi}| \tilde{\mathcal{H}}^{-1} \in \tilde{L}^{\tilde{q}}(\mathbb{R}_{\geq 0})$, one can deduce that

$$\int_{\wp}^s |\bar{\xi}(\sigma)| \Theta(s, \sigma) d\sigma \leq \left\| |\tilde{\xi}| \tilde{\mathcal{H}}^{-1} \right\|_{\tilde{q}} \frac{e^{(M(\mathfrak{N})+\lambda)} \|\mathcal{H}\|_{\infty} (\tilde{q}-1)}{b\tilde{q}}.$$

So, $\bar{\chi}$ is GUPHS with respect to $\bar{\xi}$.

According to the Theorem 3.2 the system (12) is GUPH $\frac{1}{p}$ S.

5.1 Interconnected dynamical systems

Consider:

$$\begin{cases} \dot{\eta}_1(\wp) = f_1(\wp, \eta_1(\wp)) + \vartheta(\wp, \eta) \eta_2(\wp), \\ \dot{\eta}_2(\wp) = f_2(\wp, \eta_2(\wp)), \end{cases} \quad (19)$$

where $\wp \in \mathbb{R}_{\geq 0}$, $\eta_1 \in \mathbb{R}^n$, $\eta_2 \in \mathbb{R}^m$, $\eta := \text{col}(\eta_1, \eta_2)$, and $f_1(\wp, \eta_1)$, $f_2(\wp, \eta_2)$ and $\vartheta(\wp, \eta)$ are piecewise continuous with respect to time and local Lipschitz in the state. This system can be viewed as a non-autonomous perturbed model.

($\mathcal{A}_6$) One can find $\Xi_1: \mathbb{R}_{\geq 0} \times \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ a continuously differentiable $b_1 \in \mathbb{R}_{>0}$, b_2 , $\zeta_1 \in \mathbb{R}_{>0}$, $\tilde{p} \geq 1$ and $\tilde{\chi}_1$, $\tilde{\xi}_1 \in \mathcal{P}_c(\mathbb{R}_{\geq 0}, \mathbb{R})$ so that $\forall \wp \in \mathbb{R}_{>0}$ and $\forall \eta_1 \in \mathbb{R}^n$,

$$b_1 \|\eta_1\|^{\tilde{p}} \leq \Xi_1(\wp, \eta_1) \leq b_2 \|\eta_1\|^{\tilde{p}} + \zeta_1, \quad (20)$$

$$\frac{\partial \Xi_1}{\partial \wp}(\wp, \eta_1) + \frac{\partial \Xi_1}{\partial \eta_1}(\wp, \eta_1) f_1(\wp, \eta_1) \leq \tilde{\chi}(\wp) \Xi_1(\wp, \eta_1) + \tilde{\xi}_1(\wp). \quad (21)$$

($\mathcal{A}_7$) There exist $\Xi_2: \mathbb{R}_{\geq 0} \times \mathbb{R}^m \rightarrow \mathbb{R}_{\geq 0}$ a continuously differentiable $\tilde{b}_1 \in \mathbb{R}_{>0}$, $\tilde{b}_2 \in \mathbb{R}_{\geq 0}$, $\zeta_2 \in \mathbb{R}_{\geq 0}$, $\tilde{p} \geq 1$ and $\tilde{\chi}_2$, $\tilde{\xi}_2 \in \mathcal{P}_c(\mathbb{R}_{\geq 0}, \mathbb{R})$ so that $\forall \wp \in \mathbb{R}_{>0}$ and $\forall \eta_2 \in \mathbb{R}^m$,

$$\tilde{b}_1 \|\eta_2\|^{\tilde{p}} \leq \Xi_2(\wp, \eta_2) \leq \tilde{b}_2 \|\eta_2\|^{\tilde{p}} + \zeta_2, \quad (22)$$

$$\frac{\partial \Xi_2}{\partial \wp}(\wp, \eta_2) + \frac{\partial \Xi_2}{\partial \eta_2}(\wp, \eta_2)f_2(\wp, \eta_2) \leq \tilde{\chi}_2(\wp)\Xi_2(\wp, \eta_2) + \tilde{\xi}_2(\wp). \quad (23)$$

($\mathcal{A}_8$) One can find $\Gamma \in \mathcal{B}_N$, such that $\forall \wp \in \mathbb{R}_{\geq 0}$ and $\forall \eta = (\eta_1, \eta_2) \in \mathbb{R}^n \times \mathbb{R}^m$,

$$\left| \frac{\partial \Xi_1}{\partial \eta_1}(\wp, \eta_1)\vartheta(\wp, \eta)\eta_2 \right| \leq \Gamma(\wp)\Xi_1(\wp, \eta_1). \quad (24)$$

Corollary 5.1: Consider $\tilde{\mathcal{H}} \in \mathcal{C}_1(\mathbb{R}_{\geq 0}, \mathbb{R}_{>0})$, and define $\mathcal{H}(\wp) = \tilde{\mathcal{H}}(\wp)e^{b\wp}$, where $b > 0$ and $|\tilde{\xi}_1|\tilde{\mathcal{H}}^{-1}$ lies in the space $\tilde{L}^{\tilde{q}}(\mathbb{R}_{\geq 0})$, for some $\tilde{q} > 1$. Under the hypotheses ($\mathcal{A}_6$), ($\mathcal{A}_7$) and ($\mathcal{A}_8$), the system (19) is GUP $\mathcal{H}^{\frac{1}{\tilde{p}}}$ provided that $\tilde{\chi}_1$ is GUPHS with respect to $\tilde{\xi}_1$ and $\tilde{\chi}_2$ is GUPHS with respect to $\tilde{\xi}_2$.

Proof: It is enough to observe that

$$\tilde{Z}_{\kappa}(\wp, \eta) = Z_{\kappa}(\wp, \eta_1) + \Xi_2(\wp, \eta_2),$$

with

$$\begin{aligned} Z_{\kappa}(\wp, \eta) &= \Xi_1(\wp, \eta_1) \exp\left(\int_{\wp}^{\wp+\kappa} \int_{\wp}^s \frac{1}{\kappa \Xi_1(j, \bar{\mathcal{F}}_1(j, \wp, \eta))} \frac{\partial \Xi_1}{\partial \eta_1}(j, \bar{\mathcal{F}}_1(j, \wp, \eta)) \right. \\ &\quad \left. \vartheta(j, \bar{\mathcal{F}}(j, \wp, \eta)) \bar{\mathcal{F}}_2(j, \wp, \eta_2) dj ds\right), \end{aligned}$$

Here, $\bar{\mathcal{F}}(\eta, \wp, \eta) = (\bar{\mathcal{F}}_1(\cdot, \wp, \eta), \bar{\mathcal{F}}_2(\cdot, \wp, \eta_2))$ represents the solution of dynamic (19), satisfying $\bar{\mathcal{F}}(\wp, \wp, \eta) = (\eta_1, \eta_2)$. This function is continuously differentiable and fulfills the conditions of Theorem 3.1. Therefore, the dynamic (19) is GUP $\mathcal{H}^{\frac{1}{\tilde{p}}}$ -stable.

Example 5.2: Consider the system described by equation (19), which is expressed as

$$\begin{aligned} f_1(\wp, \eta_1) &:= \begin{cases} \tilde{\chi}(\wp)\eta_1 + \frac{\tilde{\xi}_1(\wp)\sin(\eta_1)}{\eta_1}, & \text{if } \eta_1 \neq 0, \\ \tilde{\xi}_1(\wp), & \text{if } \eta_1 = 0, \end{cases} \\ f_1(\wp, \eta_2) &:= \begin{cases} \tilde{\chi}(\wp)\eta_2 + \frac{\tilde{\xi}_2(\wp)\sin(\eta_2)}{\eta_2}, & \text{if } \eta_2 \neq 0, \\ \tilde{\xi}_2(\wp), & \text{if } \eta_2 = 0, \end{cases} \end{aligned}$$

and

$$\vartheta(\wp, \eta) = \Gamma(\wp)\left(\frac{\eta_1}{1} + \eta_2^2 + \frac{\omega(\wp)}{1 + \|\eta\|^2}\right), \quad (25)$$

for which $\tilde{\chi}, \tilde{\xi}_1, \tilde{\xi}_2 \in \mathcal{P}_c(\mathbb{R}_{\geq 0}, \mathbb{R})$, $\Gamma \in \mathcal{B}_N$, and $\omega(\wp) = \exp\left(2 \int_0^{\wp} \tilde{\chi}(s) ds\right)$.

Set $\tilde{\mathcal{H}} \in \mathcal{C}_1(\mathbb{R}_{\geq 0}, \mathbb{R}_{>0})$ such that

$$\mathcal{H}(\wp) = e^{b\wp}\tilde{\mathcal{H}}(\wp),$$

where $b \in \mathbb{R}_{>0}$. Suppose that $|\tilde{\xi}_1|\tilde{\mathcal{H}}^{-1} \in \tilde{L}^{\tilde{q}}(\mathbb{R}_{\geq 0})$, with $\tilde{q} > 1$. Let $\Xi_1(\wp, \eta_1) = \eta_1^2 + \omega(\wp)$ and $\Xi_2(\wp, \eta_2) = \eta_2^2 + \omega(\wp)$. Then, we have

$$\left| \frac{\partial \Xi_1}{\partial \eta_1}(\wp, \eta)g(\wp, \eta)\eta_2 \right| \leq \Gamma(t)\Xi_1(\wp, \eta_1).$$

If $\tilde{\chi}_1$ is GUPHS with respect to $\tilde{\xi}_1$ and $\tilde{\chi}_2$ is GUPHS with respect to $\tilde{\xi}_2$, it follows that the system (19) is GUP $\mathcal{H}^{\frac{1}{2}}$ S.

5.2 Example for Illustration

Consider the following time-dependent differential equation

$$\begin{cases} \dot{\eta}_1 = \tilde{\chi}(\wp)\eta_1 + \frac{\tilde{\xi}(\wp)}{1+\eta_1^2} + \Gamma(\wp)\left(\eta_1 + \frac{\tilde{\psi}(\wp)}{1+\|\eta\|^2}\right), \\ \dot{\eta}_2 = \tilde{\chi}(\wp)\eta_2 + \frac{\tilde{\xi}(\wp)}{1+\eta_2^2} + \Gamma(\wp)\left(\eta_2 + \frac{\tilde{\psi}(\wp)}{1+\|\eta\|^2}\right), \end{cases} \quad (26)$$

where $\Gamma \in \mathcal{B}_{\mathcal{N}}$, $\tilde{\chi}, \tilde{\xi} \in \mathcal{P}_c(\mathbb{R}_{\geq 0}, \mathbb{R})$ and $\tilde{\psi}(\wp) = e^{2 \int_0^\wp \tilde{\chi}(s) ds}$. Let $\tilde{\mathcal{H}} \in \mathcal{C}_1(\mathbb{R}_{\geq 0}, \mathbb{R}_{>0})$, such that $\mathcal{H}(\wp) = \tilde{\mathcal{H}}(\wp)e^{b\wp} \in \mathcal{B}_c(\mathbb{R}_{\geq 0}, \mathbb{R}_{>0})$, with $b > 0$. Given that $|\tilde{\xi}_1|\tilde{\mathcal{H}}^{-1} \in \tilde{L}^{\tilde{q}}(\mathbb{R}_{\geq 0})$, where $\tilde{q} > 1$.

The system under consideration can be modeled using equation (12) as:

$$\bar{\Delta} = \begin{pmatrix} \tilde{\chi}(\wp)\eta_1 + \frac{\tilde{\xi}(\wp)}{1+\eta_1^2} \\ \tilde{\chi}(\wp)\eta_2 + \frac{\tilde{\xi}(\wp)}{1+\eta_2^2} \end{pmatrix}$$

and

$$\vartheta(\wp, \eta) = \begin{pmatrix} \Gamma(\wp)\left(\eta_1 + \frac{\tilde{\psi}(\wp)}{1+\|\eta\|^2}\right) \\ \Gamma(\wp)\left(\eta_2 + \frac{\tilde{\psi}(\wp)}{1+\|\eta\|^2}\right) \end{pmatrix}.$$

Let

$$\Xi(\wp, \eta) = \frac{1}{2}(\eta_1^2 + \eta_2^2) + \tilde{\psi}(\wp).$$

The derivative of Ξ evaluated along the solutions of the dynamic

$$\dot{\eta} = \bar{\Delta}(\wp, \eta), \quad (27)$$

satisfies the inequality:

$$\dot{\Xi}(\wp, \eta) \leq 2\tilde{\chi}(\wp)\Xi(\wp, \eta) + \tilde{\xi}(\wp).$$

It follows that $\tilde{\chi}$ is GUPHS with respect to $\tilde{\xi}$. Therefore, the system (27) is GUP $\tilde{\mathcal{H}}^{\frac{1}{2}}$ S.

Also, we derive

$$\left| \frac{\partial \Xi}{\partial \eta}(\wp, \eta) \vartheta(\wp, \eta) \right| \leq \Gamma(\wp)\Xi(\wp, \eta),$$

which ensures that all hypotheses of Theorem 5.1 are achieved. As a result, the system in equation (26) is GUP $\mathcal{H}^{\frac{1}{2}}$ S.

For the specific choice

$$\tilde{\chi}(\wp) = -\frac{(1+\wp)^2}{1+\wp^2} + e^{-\wp}, \tilde{\xi}(\wp) = \frac{e^{-2\wp}}{1+\wp^2}, \Gamma(\wp) = e^{-2\wp}, \tilde{\mathcal{H}}(\wp) = \frac{e^{-\wp}}{1+\wp^2},$$

it can be observed from Figure 2 that system (26) is GUP $\mathcal{H}^{1/2}$ S, with $\mathcal{H}(\wp) = \frac{1}{1+\wp^2}$.

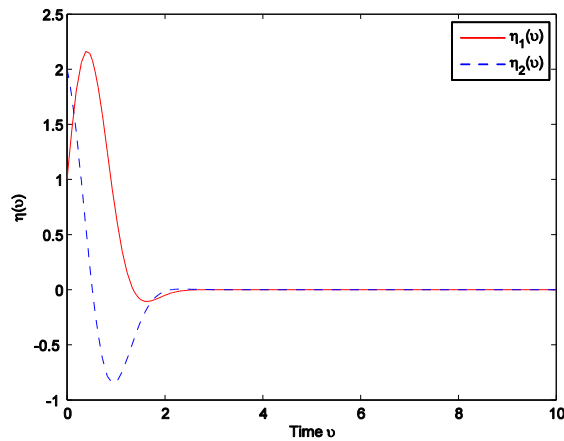


Figure 2
Temporal behavior of the solution $(\eta_1(t), \eta_2(t))$ of system (26) with $\eta_1(0) = 1$ and $\eta_2(0) = 2$.

6. Conclusion

In this study, we utilize the concept of a $\mathcal{H}$ -stable practical function to establish new and adequate conditions for both local and global UP $\mathcal{H}$ -stability in non-autonomous nonlinear dynamical systems. Consequently, we present two contrasting theorems that address cases where the system is practically $\mathcal{H}$ -stable at both local and global levels. Additionally, we introduce new Lyapunov functions, which are applied innovatively to a non-autonomous perturbed systems. To illustrate the effectiveness of the findings, several examples are provided.

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