Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 29 (2026), No. 6, pp. 1771–1793
DOI : 10.47974/JIM-2271

Comprehensive analysis of product inequalities for Hewitt-Stromberg measures in separable metric spaces

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Abstract

This paper investigates generalized Hewitt-Stromberg (H-S) measures, focusing on how they behave in product sets. These sets are part of a special type of space called separable metric spaces. The goal is to understand inequalities that appear when working with such products. By introducing a pseudo H-S measure derived from pseudo-packing methods, we establish new product inequalities that do not require the doubling condition on component measures or dimension functions. These results extend classical inequalities in Euclidean spaces, providing a versatile framework for analyzing product measures in fractal geometry.

Subject Classification: *28A78, 28A80.*

Keywords: *Generalized Hewitt-Stromberg measures, Product sets, Pseudo-packing.*

1. Introduction

Following Hewitt and Stromberg [16], for each $t \geq 0$ we set H^t to be the lower t-dimensional Hewitt–Stromberg measure (abbreviated H-S) and P^t the upper corresponding measure. Originally presented in [16, Exercise (10.51)], these measures have since drawn considerable attention, with numerous authors exploring their role in local fractal properties and behavior under product operations (see, for instance, [2, 5, 6, 31, 27, 26, 8]). In particular, in [6], the authors were interested in the study of the lower and upper dimensional H-S measures of the product set $E \times F$, where E and F are subsets of real spaces of dimensions n and m, respectively, each of which is at least one. They showed that there are positive constant α_1 and α_2 for which for every $t, s \geq 0$, one has

$$\mathrm{H}^t(E)\,\mathrm{H}^s(F) \leq \alpha_1\,\mathrm{H}^{t+s}(E \times F) \leq \alpha_2\,\mathrm{H}^t(E)\,\mathrm{P}^s(F).$$

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Similarly, there exist $\alpha_3, \alpha_4 > 0$ with

$$\mathrm{P}^t(E)\, \mathrm{H}^s(F) \leq \alpha_3\, \mathrm{P}^{t+s}(E \times F) \leq \alpha_4\, \mathrm{P}^t(E)\, \mathrm{P}^s(F).$$

Consider two separable metric spaces $(\mathbb{X}, d)$ and $(\mathbb{Y}, d')$. Write $\mathcal{P}(\mathbb{X})$ for the family of probability measures on $\mathbb{X}$ equipped with the Borel σ-algebra. For a given measure $\mu \in \mathcal{P}(\mathbb{X})$ and a parameter $a > 1$, the doubling constant associated with μ is defined as:

$$P_a(\mu) = \limsup_{r \downarrow 0} \sup_{x \in \mathrm{supp}(\mu)} \frac{\mu(B(x, ar))}{\mu(B(x, r))},$$

where the limit superior captures the maximal asymptotic growth rate of the measure ratios as the radius r approaches zero. The doubling property forμ is satisfied when one can find $a > 1$ such that the associated doubling constant $P_a(\mu)$ is finite. Equivalently, if $P_a(\mu) < \infty$ for one such a, then the same holds for all $a > 1$. The class of Borel probability measures on $\mathbb{X}$ fulfilling the doubling property is given by:

$$\mathcal{P}_D(\mathbb{X}) = \{\mu \in \mathcal{P}(\mathbb{X}) | \mu \text{ has the doubling property}\}.$$

In [22], a multi-fractal formalism was developed to analyze the geometric and statistical properties of level sets E generated by measures μ defined on $\mathbb{R}^m$. This formalism emphasizes the necessity of integrating the measure μ into the definitions of Hausdorff and packing measures. By doing so, Olsen constructed multi-fractal generalizations, denoted $\mathcal{H}_\mu^{q,t}$ and $\mathcal{P}_\mu^{q,t}$ (for $q, t \in \mathbb{R}$), which refine classical geometric measures to capture the multi-fractal spectrum of μ. Since them, many researchers study some properties of these measures [15, 4]. The study of measures on metric spaces often requires flexible tools to capture subtle geometric or analytic features. While classical frameworks such as Hausdorff and packing measures have been widely used, recent years have witnessed the emergence of more general metric structures to address convergence, regularity, and fixed point phenomena. In particular, partial soft metric spaces [7] and B-multiplicative metric spaces [9] have been introduced to deal with such issues, revealing an increasing interest in adaptable distance-like frameworks. We focus here on the generalized versions of the lower and upper H-S measures $\mathrm{H}_\mu^{q,t}$ and $\mathrm{P}_\nu^{q,s}$ defined on $\mathbb{X}$ (see Section 2 for the definition) which are the counterparts of the multi-fractal Hausdorff and packing measures presented above. These measure interpolate between $\mathcal{H}_\mu^{q,t}$ and $\mathcal{P}_\mu^{q,t}$ which makes them particularly important, when multi-fractal formalism is not valid; that is,

$$\dim_\mu^q \mathrm{E} \neq Dim_\mu^q \mathrm{E},$$

where the symbols $\dim_\mu^q$ and Dim_μ^q stand for the multi-fractal Hausdorff and packing dimensions, as defined in [1, 28]. When $\mathbb{X}$ and $\mathbb{Y}$ are the Euclidean spaces, using the density approach the authors in [3], proved that the inequalities mentioned above remain true for the measures $\mathrm{H}_\mu^{q,t}$ and $\mathrm{P}_\nu^{q,s}$, when $\mu \in \mathcal{P}_D(\mathbb{X})$ and $\nu \in \mathcal{P}_D(\mathbb{Y})$. This approach cannot be applied here because the density method does not hold true in certain important discussions (see Remark 2).

Furthermore, the technique used in [6] can not be applied here also. Indeed, In [6], the main strategy involves defining an auxiliary outer measure H^{*t} that is equivalent to the lower H-S measure H^t but more tractable for analysis. This construction is done using the collection of all semi-dyadic cubes that are half-open. Semi-dyadic cubes are chosen here due to their reduced sensitivity compared to the usual dyadic cubes (see [19, 23] for more discussion on this construction).

Let $\mathcal{F}$ represent the collection of all Hausdorff functions. By definition, these are continuous, monotonically increasing function h that map $[0,\infty)$ into itself, vanish at the origin, and strictly positive values on $(0,\infty)$. A Hausdorff function h qualifies as blanketed whenever it satisfies the inequality

$$\limsup_{r\to 0^+} \frac{h(2r)}{h(r)} \leq k,$$

for some constant $k \geq 1$. We write $\mathcal{F}_0$ for the collection of Hausdorff functions which are blanketed. For $h, g \in \mathcal{F}$, we introduce and investigate the lower and upper H-S h-measures, denoted respectively by $\mathrm{H}_\mu^{q,h}$ and $\mathrm{P}_\mu^{q,h}$. In Section 2, we show that the previous inequalities remain valid in a metric space using our density approach introduced in that section. To be precise, we present these two results.

Theorem 1: *Let $\mu_1 \in \mathcal{P}_D(\mathbb{X})$, $\mu_2 \in \mathcal{P}_D(\mathbb{Y})$, $h, g \in \mathcal{F}_0$ and $q \in \mathbb{R}$. Assume that $\mathcal{P}_{\mu_1}^{q,h}(E) < \infty$ and $\mathcal{P}_{\mu_2}^{q,g}(F) < \infty$. Then, we have*

$$\mathrm{H}_{\mu_1}^{q,h}(E)\, \mathrm{H}_{\mu_2}^{q,g}(F) \leq \mathrm{H}_{\mu_1\times\mu_2}^{q,hg}(E \times F) \tag{1.1}$$

and

$$\mathrm{H}_{\mu_1}^{q,h}(E)\, \mathrm{P}_{\mu_2}^{q,g}(F) \leq \mathrm{P}_{\mu_1\times\mu_2}^{q,hg}(E \times F). \tag{1.2}$$

Theorem 2: *Let $\mu_1 \in \mathcal{P}_D(\mathbb{X})$, $\mu_2 \in \mathcal{P}_D(\mathbb{Y})$, and $q \in \mathbb{R}$. Assume that $\mathcal{P}_{\mu_1}^{q,h}(E) < \infty$ and $\mathcal{P}_{\mu_2}^{q,g}(F) < \infty$. Then, we have*

$$\mathrm{H}_{\mu_1\times\mu_2}^{q,hg}(E \times F) \leq \mathrm{H}_{\mu_1}^{q,h}(E)\, \mathrm{P}_{\mu_2}^{q,g}(F) \tag{1.3}$$

and

$$\mathrm{P}_{\mu_1\times\mu_2}^{q,hg}(E \times F) \leq \mathrm{P}_{\mu_1}^{q,h}(E)\, \mathrm{P}_{\mu_2}^{q,g}(F), \tag{1.4}$$

assuming validity even in boundary cases involving zero-valued factors on the right.

In section 3, we set out our fundamental contributions, aiming to refine the inequalities established in [6] and [3]. Specifically, we will examine similar inequalities within a separable metric space, without assuming the doubling condition. To this end, we introduce a new fractal measure of packing type, which is greater than the standard packing measure. This new measure, called the pseudo generalized H-S h-measure and denoted by $\mathrm{R}_\mu^{q,h}$, is constructed through the pseudo-packing of balls of equal radius, analogous to the pseudo-packing

h-measure $\mathcal{R}_{\mu}^{q,h}$ (see definition in Section 4.1). More precisely, our mains results are the following.

Theorem 3: *Suppose μ is a member of $\mathcal{P}(\mathbb{X})$ and ν belongs to $\mathcal{P}(\mathbb{Y})$, two Hausdorff function h and g, and a real number $q \in \mathbb{R}$. Then, we have*

(1) $H_{\mu\times\nu}^{q,hg}(E \times F) \leq H_{\mu}^{q,h}(E)\, R_{\nu}^{q,g}(F)$, *on the condition that the product does not yield an indeterminate form (specifically $0 \times \infty$ in either order).*

(2) $\mathrm{R}_{\mu}^{q,h}(E)\, \mathrm{H}_{\nu}^{q,g}(F) \leq \mathrm{R}_{\mu\times\nu}^{q,hg}(E \times F)$.

(3) $P_{\mu\times\nu}^{q,hg}(E \times F) \leq R_{\mu}^{q,h}(E)\, P_{\nu}^{q,g}(F)$, *provided the product does not yield an indeterminate form (specifically $0 \times \infty$ in either order).*

Note that inequality (1.1) remains true without any assumption on μ, ν, h and g but using the generalized weighted H-S h-measure, defined using weighted cover of balls with the same radius, instead of the lower generalized H-S h-measure $\mathrm{H}_{\mu}^{q,h}$. The reader can be referred to [4, 18, 21, 11] for the definition of weighted cover and more details on the weighted Hausdorff measures.

The paper unfolds as follows. Section 2 reviews definitions and notation, and establishes the key preliminary results, including an introductory overview to generalized Hewitt-Stromberg (H-S) h-measures and density theorems related to our study. Section 3 presents our main results, where we examine and refine product inequalities for generalized H-S h-measures within separable metric spaces. This includes detailed proofs for Theorems 3, demonstrating the relationships between product measures without assuming the doubling condition. Finally, the Appendix provides essential background on generalized Hausdorff and packing measures, along with methods related to maximal pseudo-packing, which play a crucial role in our analyses.

2. Definitions, notation and preliminary results

2.1 *Generalized H-S h-measures*

We proceed by constructing the upper and lower generalized H–S h-measures, $\mathrm{P}_{\mu}^{q,h}$ and $\mathrm{H}_{\mu}^{q,h}$, associated with a parameter $q \in \mathbb{R}$, a Hausdorff function $h \in \mathcal{F}$, and a probability measure $\mu \in \mathcal{P}(\mathbb{X})$. We define $\overline{\mathrm{P}}_{\mu}^{q,h}$ for a set $E \subseteq \mathbb{X}$ to be the upper limit as r approaches zero of the product of $M_{\mu,r}^{q}(E)$ with $h(2r)$. The quantity $M_{\mu,r}^{q}(E)$ is defined as the supremum, ranging over all collections of closed balls $\{B(x_i,r)\}_i$ with centers x_i belongs to E satisfying $d(x_i,x_j) > 2r$ for $i \neq j$, of the sum of the q-*th* powers of their measures. It is readily observed that the set function $\overline{\mathrm{P}}_{\mu}^{q,h}$ is monotonic (i.e., increasing under set inclusion) and satisfies $\overline{\mathrm{P}}_{\mu}^{q,h}(\emptyset) = 0$. However, $\overline{\mathrm{P}}_{\mu}^{q,h}$ lacks σ-additivity. To resolve this, we define the outer measure $\mathrm{P}_{\mu}^{q,h}$ as illustrated below:

$$\mathrm{P}_{\mu}^{q,h}(E) = \inf_{\substack{E\subseteq\bigcup_i E_i\\ E_i\text{bounded}}} \sum_i \overline{\mathrm{P}}_{\mu}^{q,h}(E_i).$$

The measure $\mathrm{P}_{\mu}^{q,h}$ serves as a natural extension of the classical upper t-dimensional H-S measure P^t introduced in [6]. In analogy, we define $\overline{\mathrm{H}}_{\mu,0}^{q,h}$ of the set E to be the lower limit as r goes to zero of the product of $T_{\mu,r}^{q}(E)$ with $h(2r)$. The quantity $T_{\mu,r}^{q}(E)$ is defined as the infimum , ranging over all collections of closed balls $\{B(x_i,r)\}_i$ with centers x_i belongs to E satisfying $E\subseteq\bigcup_i B(x_i,r)$, of the sum of the q-*th* powers of their measures. Given that the set function $\overline{\mathrm{H}}_{\mu,0}^{q,h}(E)$ fails to be monotonic under set inclusion and due to its lack of countable subadditivity, a conventional adjustment is needed to define an outer measure. To this end, we redefine it as follows:

$$\mathrm{H}_{\mu}^{q,h}(E) = \inf_{\substack{E\subseteq\bigcup_i E_i\\ E_i\text{bounded}}} \sum_i \overline{\mathrm{H}}_{\mu,0}^{q,h}(E_i), \text{and } \mathrm{H}_{\mu}^{q,h}(E) := \sup\{\overline{\mathrm{H}}_{\mu}^{q,h}(F) | F\subseteq E\}.$$

The function $\mathrm{H}_{\mu}^{q,h}$ naturally extends the classical lower t-dimensional H-S measure H^t introduced in [6]. To further refine this framework, we define $\mathcal{H}_{\mu}^{q,h}$ and $\mathcal{P}_{\mu}^{q,h}$ as the multifractal Hausdorff measure and packing measure , respectively, whose precise constructions are detailed in Section 4.1. A useful relation between $\mathcal{H}_{\mu}^{q,h}$ and $\mathcal{P}_{\mu}^{q,h}$ and different measures defined above are given by [5, Theorem 2.1]

$$\mathcal{H}_{\mu}^{q,h}(E) \le \mathrm{H}_{\mu}^{q,h}(E) \le \xi\mathrm{P}_{\mu}^{q,h}(E) \le \xi\mathcal{P}_{\mu}^{q,h}(E),$$

where $\xi\in\mathbb{N}$. Moreover, in the case $q\le 0$, or if $q>0$ under the additional assumptions $\mu\in\mathcal{P}_D(\mathbb{X})$ and $h\in\mathcal{F}_0$, we get

$$\mathcal{H}_{\mu}^{q,h}(E) \le \mathrm{H}_{\mu}^{q,h}(E) \le \mathrm{P}_{\mu}^{q,h}(E) \le \mathcal{P}_{\mu}^{q,h}(E). \tag{2.1}$$

2.2 *Results concerning the density theorem*

Assume μ and ν are elements of $\mathcal{P}(\mathbb{X})$, together with a real parameter q, and let h belong to the class $\mathcal{F}$. choose a point x in the topological support of μ. We define the upper (q,h)-density of ν at x with respect to ν by the limit superior, as the radius r tends to zero, of the measure of the ball $B(x,r)$ under ν, divided by the product of the q-*th* power of $\mu(B(x,r))$ and the value of h at $2r$. We refer to this quantity as $\overline{d}_{\mu}^{q,h}(x,\nu)$. In an analogous way, the lower (q,h)-density is given by the limit inferior of the same ratio, and is written $\underline{d}_{\mu}^{q,h}(x,\nu)$. Whenever these two quantities coincide, their common value is referred to simply as the (q,h)-density of ν at x and is denoted $d_{\mu}^{q,h}(x,\nu)$.

Remark 1: There exists a fundamental connection between $\overline{d}_{\mu}^{q,h}$, $\underline{d}_{\mu}^{q,h}$ and the generalized H-S h-measures. As an illustrative case, we demonstrate that

$\mathrm{P}_{\mu}^{q,h}(E) < \infty$ if $\underline{d}_{\mu}^{q,h}(x,\nu) > 0$ uniformly over $x \in E$. Consider any $\gamma > 0$ such that $\gamma < \inf_{x\in E}\underline{d}_{\mu}^{q,h}(x,\nu)$. For each positive integer k, E_k consists of those point $x \in E$ with the property that every ball centered at x of radius at most $1/k$ carries a least the expected amount of mass under ν, more precisely, for every $r \in (0,1/k]$, the measure of $B(x,r)$ under ν is no less than $\gamma\, \mu(B(x,r))^q h(2r)$. Next, let $\{(x_i,r_i)\}_{i\geq 1}$ be a δ-packing of E_k, where $\delta < 2/k$. From the defining property of each $x_i \in E_k$, we know

$$\nu(B(x_i,r_i)) \geq \gamma\, \mu(B(x_i,r_i))^q\, h(2r_i).$$

This leads to the inequality stating that the total q-power weighted μ-measure of $B(x_i,r_i)$, scaled by the value of h at $2r_i$, does not exceed $\frac{1}{\gamma}$ times the corresponding total ν-measure of $B(x_i,r_i)$. Thus, the overall sum of $\mu(B(x_i,r_i))$ raised to the power q and multiplied by $h(2r_i)$, taken over all indices i, is therefore bounded above by $\frac{1}{\gamma}$ times the measure of $E_k^{(\delta)}$ under ν, where $E_k^{(\delta)}$ denotes the δ-neighborhood of E_k, defined by $E_k^{(\delta)} = \{x \in \mathbb{X} \mid d(x,E_k) \leq \delta\}$. Consequently, in the limit as δ approaches zero, we derive

$$\overline{\mathcal{P}}_{\mu}^{q,h}(E_k) \leq \frac{1}{\gamma}\, \nu(E_k) + \varepsilon.$$

Finally, since $E_k \nearrow E$ monotonically and $\mathrm{P}_{\mu}^{q,h} \leq \mathcal{P}_{\mu}^{q,h}$, the claim follows.

Example 1: Consider the unit interval $I = [0,1]$. Let $\{n_k\}_{k\geq 1}$ denote a strictly positive integer sequence and $\{r_k\}_{k\geq 1}$ a real-valued sequence such that

- $n_k \geq 1$ for all $k \geq 1$,
- $0 < n_k r_k < 1$ for every integer $k \geq 1$,
- The infimum of $\{r_k\}$ satisfies $\inf_k r_k > 0$.

Next, we construct a hierarchical indexing system as follows. For each integer $k \geq 1$, define the family of multi-indices D_k as the collection of all ordered k-tuples $(i_1, i_2, \dots, i_j)$ where each entry i_j is an integer satisfying $1 \leq i_j \leq n_j$ at position j, for every $1 \leq j \leq k$ with n_j denoting a fixed positive integer bound specific to the j-*th* coordinate. The total set of indices is given by $D = \bigcup_{k=0}^{\infty} D_k$, with $D_0 = \{\emptyset\}$ (the singleton empty index). Given two multi-indices $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ from D_k and $\beta = (\beta_1, \beta_2, \dots, \beta_m)$ from D_m, their concatenation operation combines them sequentially to produce a new composite index: The concatenated index $\alpha å \beta$ is formed by taking the ordered sequence of σ's components $(\alpha_1, \alpha_2, \dots, \alpha_k)$ followed immediately by τ's components $(\beta_1, \beta_2, \dots, \beta_m)$, resulting in $(\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_m)$. This new index belongs to D_{k+m}, the set of all multi-indices of length $k+m$. Now, we define two families of closed sub-intervals of I:

$$A_k = \{I_\alpha \mid \alpha \in D_k\} \text{ and } A = \{I_\alpha \mid \alpha \in D\}.$$

These families satisfy the following recursive properties:

(i) Root Interval: $I_{\emptyset} = I$.

(ii) Structure: For each $k \geq 1$:

- Sub-intervals $\{I_{\alpha\mathring{a}j}\}_{j=1}^{n_k}$ (children of I_α) are ordered left-to-right.
- The left endpoint of $I_{\alpha\mathring{a}j}$ matches that of I_α for $j = 1$.
- The right endpoint of $I_{\alpha\mathring{a}n_k}$ matches that of I_α.
- The spacing between consecutive sub-intervals is uniform, denoted by d_k.

(iii) Scaling Condition: For every $k \geq 1$, $\alpha \in D_k$, and $1 \leq j \leq n_k$:

$$\frac{\operatorname{diam}(I_{\alpha\mathring{a}j})}{\operatorname{diam}(I_\alpha)} = r_k,$$

where $\operatorname{diam}(A)$ denotes the diameter of set A.

Take E_k be the union of the intervals I_α for each $\alpha \in D_k$ and we define the homogeneous Cantor set, characterized by $\{n_k\}$ and $\{r_k\}$, by the set

$$E = \bigcup_{k\geq 0} E_k.$$

At this point, we introduce the function μ of set on $[0,1]$ by $\mu(I_\emptyset) = 1$ and $\mu(I_\alpha) = (n_1 \dots n_k)^{-1}$ for all $\sigma \in D_k$. Since

$$\mu(I_\alpha) = n_{k+1}\, \mu(I_{\alpha\mathring{a}j}), \alpha \in D_k, 1 \leq j \leq n_k,$$

We can derive a mass distribution on $[0,1]$, from μ. Let $s_k = \frac{\log(n_1 \dots n_k)}{\log(r_1 \dots r_k)}$ then,

$$\mu(I_\alpha) = \operatorname{diam}(r_1 \dots r_k)^{s_k} = \operatorname{diam}(I_\alpha)^{s_k}, \tag{2.2}$$

for all $\sigma \in D_k$. Now, for $x \in E$, a sequence $\{I_{\alpha_k}\}$ can be found with $\alpha_k \in D_k$, for all $k \geq 1$, and such that x lies in the intersection of the corresponding sets $I_{\alpha_k}(x)$. Consider the measure μ specified by equation and $h \in \mathcal{F}_0$ such that

$$\lim_{k\to\infty} \frac{(n_1 \dots n_{k+1})^{q-1}}{n_{k+1} h(r_1 \dots r_k)} \geq \gamma > 0 \tag{2.3}$$

Given any $r > 0$, one can choose an integer k so that $\operatorname{diam}(I_{\alpha_{k+1}}) \leq 2r < \operatorname{diam}(I_{\alpha_k})$. For each point x in E, the ratio of $\mu(B(x,r))$ to the product of $h(2r)$ and $\mu(B(x,r))$ raised to the power q, simplifies to the reciprocal of $h(2r)$ times $\mu(B(x,r))$ raised to the power $q-1$, and this quantity is bounded below by the expression $\frac{(n_1 \cdots n_{k+1})^{q-1}}{n_{k+1}\, h(r_1 \cdots r_k)}$. Therefore, using , we get $\underline{d}_\mu^{q,h}(x,\mu) > 0$ and then $\mathrm{P}_\mu^{q,h}(E) < \infty$.

The density result was established in terms of generalized Hausdorff and packing h-measures and within a metric space in [15]. More specifically, the next statement is as follows.

Lemma 1: *Take any Borel set* E *contained in* $supp(\mu)$. *Suppose that* μ *and* ν *are both probability measures on* $\mathbb{X}$, *a dimension function* $h \in \mathcal{F}$, *and fix a real exponent* q.

(1) *We have*

$$\left[\inf_{x\in E} \underline{d}_{\mu}^{q,h}(x,\nu)\right] \mathcal{P}_{\mu}^{q,h}(E) \leq \nu(E). \tag{2.4}$$

By convention, if one of the factors is zero, the entire left-hand side is taken to be zero.

(2) *We have*

$$\nu(E) \leq \left[\sup_{x\in E} \overline{d}_{\mu}^{q,h}(x,\nu)\right] \mathcal{H}_{\mu}^{q,h}(E). \tag{2.5}$$

By convention, if either factor is infinite then the right-hand side is set to ∞.

This lemma provides the foundation for proving Theorems 1 and 2. In particular, our density approach applies to the study of the generalized H-S h-measures $\mathrm{H}_{\mu}^{q,h}$ and $\mathrm{P}_{\mu}^{q,h}$ for product spaces, via a decomposition into the measures on each coordinate. Throughout, let $\mu_1 \in \mathcal{P}(\mathbb{X})$ and $\mu_2 \in \mathcal{P}(\mathbb{Y})$ be probability measures on $\mathbb{X}$ and $\mathbb{Y}$, respectively, and let $h, g \in \mathcal{F}$. We will then focus on Borel sets $E \subseteq \mathbb{X}$ and $F \subseteq \mathbb{Y}$ under the conditions listed below

$$\underline{d}_{\mu_1}^{q,h}(x,\nu) = \overline{d}_{\mu_1}^{q,h}(x,\nu),$$

for every $x \in E$ and $\nu \in \{\mathrm{H}_{\mu_1\,E}^{q,h}, \mathrm{P}_{\mu_1\,E}^{q,h}\}$ and

$$\underline{d}_{\mu_2}^{q,g}(x,\nu) = \overline{d}_{\mu_2}^{q,g}(x,\nu),$$

for every $x \in F$ and $\nu \in \{\mathrm{H}_{\mu_2\,F}^{q,h}, \mathrm{P}_{\mu_2\,F}^{q,h}\}$.

2.3 *Proof of Theorem 1*

First, using Lemma 1 we deduce, for a doubling measure μ together with a blanketed dimension function h, that

$$\left[\inf_{x\in E} \underline{d}_{\mu}^{q,h}(x,\nu)\right] \mathrm{H}_{\mu}^{q,h}(E) \leq \nu(E) \leq \left[\sup_{x\in E} \overline{d}_{\mu}^{q,h}(x,\nu)\right] \mathrm{H}_{\mu}^{q,h}(E). \tag{2.6}$$

Now, let $\nu_1 = \mathrm{H}_{\mu_1\,E}^{q,h}$ and $\nu_2 = \mathrm{H}_{\mu_2\,F}^{q,g}$. We set

$$\tilde{E} = \left\{x \in E \mid 1 \geq d_{\mu_1}^{q,h}(x,\nu_1)\right\} \text{ and } \tilde{F} = \left\{x \in F \mid 1 \geq d_{\mu_2}^{q,g}(x,\nu_2)\right\}.$$

We have $d_{\mu_1}^{q,h}(x,\nu_1) \geq 1$, $\mathrm{H}_{\mu_1}^{q,h}$-a.a. on E. Indeed, for every integer n, set

$$E_n = \left\{x \in E, d_{\mu_1}^{q,h}(x,\nu_1) < 1 - \tfrac{1}{n}\right\}.$$

Hence, apply (2.6) to ν_1 on E_n to get

$$\mathrm{H}_{\mu_1}^{q,h}(E_n) \leq \left(1 - \tfrac{1}{n}\right) \mathrm{H}_{\mu_1}^{q,h}(E_n).$$

This implies that $\mathrm{H}_{\mu_1}^{q,h}(E_n) = 0$. Therefore, we get the desired result. Then, $\nu_1(\tilde{E}) = \nu_1(E)$. Similarly, we get

$$d_{\mu_2}^{q,g}(x,\nu_2) \geq 1, \text{for } \mathrm{H}_{\mu_2}^{q,g} - \text{a.a.}\, x \in F.$$

Hence, $\nu_2(\tilde{F}) = \nu_2(F)$. For every (x,y) in $\tilde{E} \times \tilde{F}$, the quantity $d_{\mu_1\times\mu_2}^{q,hg}((x,y),\nu_1 \times \nu_2)$ is equal to the limit, as r tends to zero, of the product of two terms: the first quantity corresponds to the measure assigned by ν_1 to the ball $B(x,r)$, divided by h evaluated at $2r$, and by $\mu_1(B(x,r))$ raised to the power q, and the second being the analogous ratio involving $\nu_2(B(y,r))$, $g(2r)$, and $\mu_2(B(y,r))^q$. This limit coincides with the product $d_{\mu_1}^{q,h}(x,\nu_1)\, d_{\mu_2}^{q,g}(y,\nu_2)$, which is less than or equal to one. We get $\sup_{(x,y)\in\tilde{E}\times\tilde{F}} d_{\mu_1\times\mu_2}^{q,hg}((x,y),\nu_1 \times \nu_2) \leq 1 < \infty$, then by (2.6), it can be deduced that the measure $\mathrm{H}_{\mu_1\times\mu_2}^{q,hg}$ of the set $\tilde{E} \times \tilde{F}$ is bounded below by the product measure $\nu_1\,\nu_2$ of that set, which equals $\nu_1(E)\,\nu_2(F)$.

Now, we can also use Lemma 1 to obtain

$$\left[\inf_{x\in E} \underline{d}_{\mu}^{q,h}(x,\nu)\right] \mathrm{P}_{\mu}^{q,h}(E) \leq \nu(E) \leq \left[\sup_{x\in E} \overline{d}_{\mu}^{q,h}(x,\nu)\right] \mathrm{P}_{\mu}^{q,h}(E). \tag{2.7}$$

Following the same approach as inequality (1.1), we can establish inequality (1.2), using (2.7) instead of (2.6).

2.4 *Proof of Theorem 2*

Let $\nu_1 = \mathrm{H}_{\mu_1}^{q,h}{}_{E}$ and $\nu_2 = \mathrm{P}_{\mu_2}^{q,g}{}_{F}$. We set

$$\tilde{E} = \left\{x \in E \mid 1 \leq d_{\mu_1}^{q,h}(x,\nu_1)\right\} \text{and } \tilde{F} = \left\{x \in F \mid 1 \leq d_{\mu_2}^{q,g}(x,\nu_2)\right\}.$$

We have $d_{\mu_1}^{q,h}(x,\nu_1) \leq 1$, $\mathrm{H}_{\mu_1}^{q,h}$-a.a. on E. Let n be any integer, and introduce

$$E_n = \left\{x \in E, d_{\mu_1}^{q,h}(x,\nu_1) > 1 + \tfrac{1}{n}\right\}.$$

Then apply (2.6) to ν_1 on E_n we obtain,

$$\left(1 + \tfrac{1}{n}\right) \mathrm{H}_{\mu_1}^{q,h}(E_n) \leq \mathrm{H}_{\mu_1}^{q,h}(E_n).$$

This implies that $\mathrm{H}_{\mu_1}^{q,h}(E_n) = 0$. Therefore, we get the desired result. Then, $\nu_1(\tilde{E}) = \nu_1(E)$. Similarly, considering for $n \in \mathbb{N}$ the set

$$F_n = \left\{x \in F, d_{\mu_2}^{q,g}(x,\nu_2) > 1 + \tfrac{1}{n}\right\},$$

and using (2.7) instead of (2.6), we get $d_{\mu_2}^{q,g}(x,\nu_2) \leq 1$, $\mathrm{P}_{\mu_2}^{q,h}$-a.a. on F. Then, $\nu_2(\tilde{F}) = \nu_2(F)$. For every (x,y) in $\tilde{E} \times \tilde{F}$, the quantity $d_{\mu_1\times\mu_2}^{q,hg}((x,y),\nu_1 \times \nu_2)$ is equal to the limit, as r tends to zero, of the product of two terms: the first quantity corresponds to the measure assigned by ν_1 to the ball $B(x,r)$, divided by h evaluated at $2r$, and by $\mu_1(B(x,r))$ raised to the power q, and the second being the analogous ratio involving $\nu_2\big(B(y,r)\big)$, $g(2r)$, and $\mu_2(B(y,r))^q$. This limit coincides with the product $d_{\mu_1}^{q,h}(x,\nu_1)\, d_{\mu_2}^{q,g}(y,\nu_2)$, which is greater than or equal to one. We get $\inf_{(x,y)\in\tilde{E}\times\tilde{F}} d_{\mu_1\times\mu_2}^{q,hg}((x,y),\nu_1\times\nu_2)\geq 1>0$, then by (2.6), it can be deduced that the measure $\mathrm{H}_{\mu_1\times\mu_2}^{q,hg}$ of the set $\tilde{E}\times\tilde{F}$ is bounded above by the product measure $\nu_1\,\nu_2$ of that set, which equals $\nu_1(E)\,\nu_2(F)$. Now, following the same approach as inequality (1.3), we can establish inequality (1.4), using (2.7) instead of (2.6).

Example 2: We will show that no constant $\gamma>0$ can satisfy for every $A\subseteq\mathbb{X}$, $B\subseteq\mathbb{Y}$, every $\mu\in\mathcal{P}_D(\mathbb{X})$, $\nu\in\mathcal{P}_D(\mathbb{Y})$ and every $h,g\in\mathcal{F}_0$, the inequality

$$\mathrm{H}_{\mu\times\nu}^{q,h\,g}(A\times B)\ \leq\ \gamma\,\mathrm{H}_{\mu}^{q,h}(A)\,\mathrm{H}_{\nu}^{q,g}(B)$$

holds. It is enough to exhibit a case where the above inequality fails strictly. To this end, it suffices to construct explicit sets $A\subseteq\mathbb{X}$ and $B\subseteq\mathbb{Y}$ for which

$$\mathrm{H}_{\mu\times\nu}^{q,h\,g}(A\times B)\ >\ \mathrm{H}_{\mu}^{q,h}(A)\,\mathrm{H}_{\nu}^{q,g}(B).$$

Lemma 2 (Lipschitz-Type Mapping and Measure Comparison): *Consider a set $E\subseteq\mathbb{X}\times\mathbb{X}$ and another set $F\subseteq\mathbb{X}$. Let us suppose one can find a surjective map $\phi: E\to F$ and constants $c,\gamma>0$ such that for all $u=(x,y), v=(x',y')\in E$ and every $r>0$*

$$\begin{cases} d(\phi(u),\phi(v))\leq c\, d((x,y),(x',y')),\\ \mu(B(\phi(u),r))\leq\gamma\,\mu(B(x,r))\,\nu(B(y,r)),\end{cases}\tag{2.8}$$

Then, the following inequality holds:

$$\mathrm{H}_{\mu}^{q,h}(F)\leq c\gamma\cdot\mathrm{H}_{\mu\times\nu}^{q,h}(E).\tag{2.9}$$

Proof: Let $F_1\subseteq F_2\subseteq F$ and let $E_1\subseteq E_2\subseteq E$ such that $\phi(E_1)=F_1$ and $\phi(E_2)=F_2$. Starting with any collection of centered balls of radius r that covers E_1, one obtains a covering of by centered balls of radius cr. For that reason, for $\mu\in\mathcal{P}(\mathbb{X})$, $h\in\mathcal{F}$ and $q>0$, we have

$$T_{\mu,rc}^{q}(F_1)\leq\textstyle\sum_{i\geq1}\mu(B(\phi(u_i),cr))^q\leq\gamma\sum_{i\geq1}\mu(B(x_i,cr))^q\,\nu(B(y_i,cr))^q$$

It follows that

$$T_{\mu,rc}^{q}(F_1)\,h(2rc)\leq\gamma\,T_{\mu\times\nu,r}(E_1)\,h(2rc)\leq c\,\gamma\,T_{\mu\times\nu,r}(E_1)h(2r).$$

Thus,

$$\overline{\mathrm{H}}_{\mu,0}^{q,h}(F_1)\leq c\,\gamma\,\overline{\mathrm{H}}_{\mu\times\nu,0}^{q,h}(E_1)\leq c\,\gamma\,\overline{\mathrm{H}}_{\mu\times\nu}^{q,h}(E_1)\leq c\,\gamma\,\overline{\mathrm{H}}_{\mu\times\nu}^{q,h}(E_2).$$

One concludes that

$$\overline{\mathrm{H}}_{\mu}^{q,h}(F_2)\leq c\,\gamma\,\overline{\mathrm{H}}_{\mu\times\nu}^{q,h}(E_2).$$

Assume $E\subseteq\bigcup_i E_i$. Define the family $\{F_i\}$ via

$$F_i=\phi(E_i\cap E).$$

Then,

$$\mathrm{H}_{\mu}^{q,h}(F) \le c\,\gamma\,\overline{\mathrm{H}}_{\mu\times\nu}^{q,h}(E_i).$$

Because $\{E_i\}$ was chosen as an arbitrary cover of E E, it follows that

$$\mathrm{H}_{\mu}^{q,h}(F) \le c\,\gamma\,\mathrm{H}_{\mu\times\nu}^{q,h}(E).$$

Fix a strictly decreasing sequence $\{s_j\}_{j\ge1}$ tending to zero, and choose an increasing sequence of integers $(m_j)_{j\ge0}$ with $m_0 = 0$, growing so rapidly that for every $j \ge 1$

$$\sum_{k=0}^{j-1} (m_{2k+1} - m_{2k}) \le s_j\, m_{2j} \text{ and } \sum_{k=1}^{j} (m_{2k} - m_{2k-1}) \le s_j\, m_{2j}. \quad (2.10)$$

Now define two Cantor-like subsets of $[0,1]$ by prescribing long runs of zeros in their decimal expansions. The set A consists of all decimals whose r-*th* digit is zero whenever r is an odd integer in the closed interval $[m_j + 1, m_{j+1}]$. Dually, the set B consists of all decimals whose r-*th* digit is zero whenever r is an even integer in the closed interval $[m_j + 1, m_{j+1}]$. Because of the way we've arranged the digit-blocks, one stage $2j$ is complete the sum of all free (nonzero) digits contributed by the odd-numbered intervals, namely the stretches from m_0 up to m_1, from m_2 to m_3 and so through m_{2j-2} to m_{2j-1}, cannot exceed s_j times the length of the stage $2j$ block, m_{2j}. So A can be covered by at most $10^{\sum_{k=0}^{j-1}(m_{2k+1}-m_{2k})} \le 10^{s_j\, m_{2j}}$ intervals of length $10^{-m_{2j}}$. Now, for $x \in A$, assume that

$$\underline{d}_{\mu}^{q,h}(x,(2r)^s) > 0 \text{ and } \overline{d}_{\mu}^{q,h}(x,(2r)^s) < \infty, \quad (2.11)$$

for some $s > 0$. It follows from (2.10), that $\overline{\mathrm{H}}_{\mu,0}^{q,h}(A) = 0$. This is true for all set $\tilde{A} \subseteq A$, then $\mathrm{H}_{\mu}^{q,h}(A) = 0$, and similarly we have $\mathrm{H}_{\mu}^{q,h}(B) = 0$. Consider the diagonal line $L\colon y = x$ in $\mathbb{R}^2$, and let $\Phi\colon \mathbb{R}^2 \to L$ denote the orthogonal projection map. For any point (x,y), $\Phi(x,y)$ corresponds to the unique point on L located at a distance of $\frac{x+y}{\sqrt{2}}$ from the origin. By leveraging the structure of the sets A and B, every $v \in [0,1]$ can be written in exactly one way as $v = x + y$ with $x \in A$ and $y \in B$. This decomposition ensures that each digit of v is contributed exclusively by either x or y, avoiding overlap due to their complementary zero constraints. The projection $\Phi(A \times B)$ maps the Cartesian product onto L, forming a line segment of length $\frac{1}{\sqrt{2}}$. Applying Lemma 2, which relates measures under Lipchitz mappings, we have

$$\mathrm{H}_{\mu\times\mu}^{q,h}(A\times B) \ge \gamma'\,\mathrm{H}_{\mu}^{q,h}(\phi(A\times B)) \ge \gamma'\,\mathcal{H}_{\mu}^{q,h}(\phi(A\times B))$$

Therefore, by

$$\mathrm{H}_{\mu\times\mu}^{q,h}(A\times B) \ge 1 > \mathrm{H}_{\mu}^{q,h}(A)\,\mathrm{H}_{\mu}^{q,h}(B).$$

3. Main results: Product inequalities using different approach

Consider two separable metric space $(\mathbb{X}, d)$ and $(\mathbb{Y}, d')$. Their product space $\mathbb{X}\times\mathbb{Y}$ is endowed with a metric, defined for any two points

$(x_1, y_1), (x_2, y_2) \in \mathbb{X} \times \mathbb{Y}$ by

$$d \times d'((x_1, y_1), (x_2, y_2)) := \max\{d(x_1, x_2), d'(y_1, y_2)\}.$$

This metric generates the standard product topology on $\mathbb{X} \times \mathbb{Y}$. Under the maximum metric $d \times d'$, closed balls in the product space admit a straightforward characterization: Under the max-metric on $\mathbb{X} \times \mathbb{Y}$, a closed ball centered at (x, y) with radius r is the Cartesian product of the corresponding closed balls in $\mathbb{X}$ and $\mathbb{Y}$. As previously stated, our objective is to refine the inequalities established in [6]. Specifically, we will examine these inequalities within a separable metric space, without assuming the doubling condition. To this end, we introduce a new fractal measure of packing type, which is greater than the standard packing measure. This new measure, called the pseudo generalized H-S h-measure and denoted by $\mathrm{R}_{\mu}^{q,h}$, is constructed through the pseudo-packing of balls of equal radius, analogous to the pseudo-packing h-measure $\mathcal{R}_{\mu}^{q,h}$ (see definition in Section 4.1).

Remark 2: Note that the density approach previously used is not valid for the generalized pseudo-packing h-measure $\mathcal{R}_{\mu}^{q,h}$. In other words, inequality (2.4) reminds false using $\mathcal{R}_{\mu}^{q,h}$ instead of $\mathcal{P}_{\mu}^{q,h}$.

In the upcoming example, we will build a set E belong to the Davies's $\mathcal{X}$ [15] and does not satisfy (2.4).

Example 3: Consider the uniform probability measure μ established on the Davies space $\mathcal{X} = \prod_{n=1}^{\infty} G(N_n)$, equipped with its canonical metric (see [15]). Fix a point $u = (u_1, u_2, \dots) \in \mathcal{X}$ and a radius $r \in (0,1)$. The set $B(u, r)$ as a closed r-ball given by

$$B(u, r) = \{v \in \mathcal{X} \,|v_k = u_k \text{ for all } k < n \text{ and } u_n : v_n\},$$

where n is the only integer for which $\left(\frac{1}{2}\right)^n \leq r < \left(\frac{1}{2}\right)^{n-1}$. When u_n corresponds to a peripheral vertex, the ball $B(u, r)$ decomposes into two disjoint cylinders, one central and one peripheral, yielding $\mu(B(u, r)) = 2\gamma_n$. A point $u \in \mathcal{X}$ is termed eventually peripheral if its components u_k are peripheral for all sufficiently large k. Let $E \subset \mathcal{X}$ denote the subset of eventually peripheral points, then the measure $\mu(E) = 1$ (see [13]). Indeed, write the set E_m as the collection of all points points $u = (u_1, u_2, \dots) \in \mathcal{X}$ satisfying the condition that u_k is peripheral vertex whenever k is an integer not less than m. For the graph $G(N_n)$, the total vertex count is $N_n(N_n + 1)$, with exactly N_n^2 vertices designated as peripheral. Consequently, the measure of E_m under μ is given by

$$\mu(E_m) = \prod_{n=m}^{\infty} \frac{N_n^2}{N_n(N_n+1)} = \prod_{n=m}^{\infty} \frac{N_n}{N_n+1}.$$

Suppose the infinite product $\prod_{n=m}^{\infty} \frac{N_n - 1}{N_n + 1}$ converges. For each n, the radio of $N_n - 1$ to $N_n + 1$ is no larger than the ratio of N_n to $N_n + 1$, and that latter ratio never exceeds one. Thus, we deduce that $\prod_{n=m}^{\infty} \frac{N_n}{N_n+1}$ must also converge.

Since the tail products $\prod_{n=m}^{\infty} \frac{N_n}{N_n+1}$ converge to 1, we have $\lim_{m\to\infty}\mu(E_m) = 1$. By the continuity of μ under increasing limits ($E_m \nearrow E$), it follows that $\mu(E) = 1$. Now, we consider

$$\mathcal{F}^q = \{h \in \mathcal{F}, h(2^{-n+2}) = \gamma_n^{1-q}\},$$

where $q \in [0,1)$. We proved in [15] that for $h \in \mathcal{F}^q$, we have $\mathcal{R}_\mu^{q,h}(\mathcal{X}) = 1$. Our goal is to show that inequality (2.4) fails on the subset E consisting of eventually peripheral points. To do so, choose an integer n for which

$$r \in [(1/2)^n, (1/2)^{n-1}],$$

so,

$$\frac{\mu(B(u,r))^{1-q}}{h(2r)} = \frac{(2\gamma_n)^{1-q}}{h(2r)} > \frac{(2\gamma_n)^{1-q}}{h(2^{-n+2})} = \frac{(2\gamma_n)^{1-q}}{\gamma_n^{1-q}} = 2^{1-q}.$$

Therefore $\inf_{u\in E} \underline{d}_\mu^{q,h}(x,\mu) > 2^{1-q}$ and we obtain

$$\mathcal{R}_\mu^{q,h}(E) \inf_{u\in E} \underline{D}_\mu^{q,h}(x,\nu) > 2^{1-q}.$$

On the other hand, we have $\mu(E) = 1$. So, we obtain $2^{1-q} < 1$ which is impossible.

Now, we define the generalized pseudo H-S h-measure $\mathrm{R}_\mu^{q,h}$. For a probability measure μ, a dimension function h, and a real number q, we define $\overline{\mathrm{R}}_\mu^{q,h}$ of E to be the upper limit as r approaches zero of the product of $N_{\mu,r}^q(E)$ with $h(2r)$. The value of $N_{\mu,r}^q(E)$ quantifies the supremum of the weighted sums over any collections of closed r-balls centered in E, where the centers are at pairwise distances exceeding r. We next set $\mathrm{R}_\mu^{q,h}(E)$ to be the pseudo generalized H-S h-measure, specified as follows

$$\mathrm{R}_\mu^{q,h}(E) = \inf_{\substack{E\subseteq \bigcup_i E_i \\ E_i \text{bounded}}} \sum_i \overline{\mathrm{R}}_\mu^{q,h}(E_i).$$

Next, we will investigate how the pseudo generalized H-S h-measure $\mathrm{R}_\mu^{q,h}$ is related to both measures $\mathrm{P}_\mu^{q,h}$ and $\mathrm{H}_\mu^{q,h}$. Since every true packing also meets the criteria for a pseudo-packing, it follows that $\mathrm{P}_\mu^{q,h} \le \mathrm{R}_\mu^{q,h}$. We present a proposition that gives a sufficient condition for the equivalence between $\mathrm{P}_\mu^{q,h}$ and $\mathrm{R}_\mu^{q,h}$.

Proposition 1: Let $q \in \mathbb{R}$. We can find a constant $\gamma > 0$ for which, for any $E \subseteq \mathbb{X}$, the following holds

$$\mathrm{R}_\mu^{q,h}(E) \le \gamma\, \mathrm{P}_\mu^{q,h}(E),$$

in the case $q \le 0$, or if $q > 0$ under the additional assumptions $\mu \in \mathcal{P}_D(\mathbb{X})$ and $h \in \mathcal{F}_0$.

Proof: Let $E \subseteq \mathbb{X}$ and r be small. For any r-pseudo-packing $\{(x_i, r)\}_i$ of E, the collection $\{(x_i, r/2)\}_i$ forms an $r/2$- packing of E, since the separation condition $d(x_i, x_j) > r$ for pseudo-packing implies $d(x_i, x_j) > 2\,(r/2)$, satisfying the disjointness requirement for packing. Now, given $h \in \mathcal{F}_0$ and $\mu \in \mathcal{P}_D(\mathbb{X})$, one can choose a constant $\gamma > 0$, whose value depends on μ, h and q for which

$$N^q_{\mu,r}(E) \leq \gamma M^q_{\mu,r/2}(E).$$

Multiplying both sides by $h(2r)$ and and taking the limsup as $r \to 0$, the inequality $\overline{\mathrm{R}}^{q,h}_{\mu}(E) \leq \gamma \overline{\mathrm{P}}^{q,h}_{\mu}(E)$ follows. Now, Consider any countable sequence of bounded subsets $(E_i)_i$ of $\mathbb{X}$ for which $E \subseteq \cup_i E_i$. Applying the inequality above to each E_i, we obtain

$$\overline{\mathrm{R}}^{q,h}_{\mu}(E_i) \leq \gamma \overline{\mathrm{P}}^{q,h}_{\mu}(E_i).$$

Applying the infimum over every covering $(E_i)_i$ of this form produces the desired outcome.

In addition, if $h \in \mathcal{F}_0$ and $\mu \in \mathcal{P}_D(\mathbb{X})$, we have using (2.1) that $\mathrm{H}^{q,h}_{\mu} \leq \mathrm{R}^{q,h}_{\mu}$. Later, in Proposition 2, we show that this inequality remains valid even when $h \notin \mathcal{F}_0$ and $\mu \notin \mathcal{P}_D(\mathbb{X})$.

Proposition 2: Consider a measure $\mu \in \mathcal{P}(\mathbb{X})$, a function $h \in \mathcal{F}$ and a real number $q \in \mathbb{R}$. Hence, for each set $E \subseteq \mathbb{X}$, one obtains

$$\mathrm{H}^{q,h}_{\mu} \leq \mathrm{R}^{q,h}_{\mu}.$$

Proof: Consider a set $E \subseteq \mathbb{X}$ and suppose that the pseudo-measure $\mathrm{R}^{q,h}_{\mu}$ of the set E is finite. For any $r > 0$ and any subset $F \subseteq E$, we can using Lemma 3, select a maximal pseudo-packing $(x_k, r)_{k\geq 1}$ constructed using pairs (x, r) with $x \in F$ in such a way that their union forms a cover of F. This choice ensures that

$$T^q_{\mu,r}(F)\, h(2r) \leq \sum_{k\geq 1} h(2r)\, \mu(B(x_k, r))^q \leq h(2r)\, N^q_{\mu,r}(F).$$

From the definitions, it follows immediately that

$$\overline{\mathrm{H}}^{q,h}_{\mu,0}(F) \leq \overline{\mathrm{R}}^{q,h}_{\mu}(F) \text{ and } \overline{\mathrm{H}}^{q,h}_{\mu}(F) \leq \mathrm{R}^{q,h}_{\mu}(F) \leq \mathrm{R}^{q,h}_{\mu}(E).$$

We therefore conclude,

$$\mathrm{H}^{q,h}_{\mu}(E) \leq \mathrm{R}^{q,h}_{\mu}(E).$$

3.1 *First inequality of Theorem 3*

We begin by assuming that $\mathrm{H}^{q,h}_{\mu}(E)$ and $\mathrm{R}^{q,g}_{\nu}(F)$ are both finite. Fix an integer $j \geq 1$, and let $\{B_i\}_{i\geq 1} = \{B(x_i, r)\}_{i\geq 1}$ denote a covering of E at scale r. For any subset $\tilde{F} \subseteq F$, Lemma 3 guarantees the existence of a maximal r-pseudo-packing $\{(y_k, r)\}_k$ centered in F which fully covers $\tilde{F}$. Consequently,

$$T^q_{\mu\times\nu,r}(B_i\times\tilde F)h(2r)g(2r)\ \le \textstyle\sum_{k=1}^{\infty}\mu(B_i)^q h(2r)\nu(B(y_k,r))^q g(2r)$$
$$\le \mu(B_i)^q h(2r) N^q_{\nu,r}(\tilde F) g(2r)$$

and then,

$$T^q_{\mu\times\nu,r}(E\times\tilde F)\le\ g(2r)\ N^q_{\nu,r}(\tilde F)\textstyle\sum_i\ h(2r)\ \mu(B_i)^q.$$

Taking the infimum across all r-covers of E and passing to the lower limit as $r\to 0$ leads to

$$\overline{\mathrm{H}}^{q,hg}_{\mu\times\nu,0}(E\times\tilde F)\le\overline{\mathrm{H}}^{q,h}_{\mu,0}(E)\ \overline{\mathrm{R}}^{q,g}_{\mu}(\tilde F).$$

Suppose that E can be covered by $\cup_i E_i$ and F by $\cup_j F_j$. Then, the Cartesian product satisfies $E\times\tilde F\subseteq E\times F\subseteq\cup_{i,j}E_i\times F_j$. As a result,

$$\overline{\mathrm{H}}^{q,hg}_{\mu\times\nu}(E\times\tilde F)\ \le\ \textstyle\sum_{i,j}\overline{\mathrm{H}}^{q,hg}_{\mu\times\nu,0}(E_i\times F_j)$$
$$\le\ \textstyle\sum_{i,j}\overline{\mathrm{H}}^{q,h}_{\mu,0}(E_i)\ \overline{\mathrm{R}}^{q,g}_{\nu}(F_i)$$
$$\le\ \left(\textstyle\sum_i\overline{\mathrm{H}}^{q,h}_{\mu,0}(E_i)\right)\left(\textstyle\sum_j\overline{\mathrm{R}}^{q,g}_{\nu}(F_i)\right).$$

Since E_i and F_j are arbitrarily chosen, we deduce that

$$\overline{\mathrm{H}}^{q,hg}_{\mu\times\nu}(E\times\tilde F)\le\overline{\mathrm{H}}^{q,h}_{\mu}(E)\ \mathrm{R}^{q,g}_{\nu}(F)\le\mathrm{H}^{q,h}_{\mu}(E)\ \mathrm{R}^{q,g}_{\nu}(F),$$

Since this is valid for any $E\times\tilde F\subseteq E\times F$, it implies that

$$\mathrm{H}^{q,hg}_{\mu\times\nu}(E\times F)\le\mathrm{H}^{q,h}_{\mu}(E)\ \mathrm{R}^{q,g}_{\nu}(F).$$

3.2 *Second inequality of Theorem 3*

Assume that the product measure $\mathrm{R}^{q,hg}_{\mu\times\nu}(E\times F)$ is finite, and the lower H-S h-measure $\mathrm{H}^{q,g}_{\nu}(F)$ strictly positive. Fix $\varepsilon>0$ and let $\{\Lambda_i\}_i$ denote a collection of subsets in $\mathbb{X}\times\mathbb{Y}$ satisfying the following properties

$$E\times F\subseteq\textstyle\bigcup_{i=1}^{\infty}\Lambda_i\text{ and }\sum_{i=1}^{\infty}\overline{\mathrm{R}}^{q,hg}_{\mu\times\nu}(\Lambda_i)\le\mathrm{R}^{q,hg}_{\mu\times\nu}(E\times F)+\varepsilon.\tag{3.1}$$

Select a constant η satisfying $\eta<\mathrm{H}^{q,g}_{\nu}(F)$, and fix a scale $r>0$ such that $T^q_{\nu,r}(F)\ g(2r)>\eta$. Next, For each index i and each point $x\in E$, let us define

$$\Lambda_i^F(x)=\pi_F(\Lambda_i\cap(\{x\}\times F)),$$

where $\pi_F(x,y)$, the projection onto the second coordinate, satisfies$\pi_F(x,y)=y$. Next, let us define

$$E_n=\{\,x\in E{:}\textstyle\sum_{i=1}^{n}T^q_{\nu,r}(\Lambda_i^F(x))\ g(2r)\ge\eta\}.$$

By construction, whenever $x\in E_n$, we have

$$\textstyle\sum_{i=1}^{n}T^q_{\nu,r}(\Lambda_i^F(x))\ h(2r)\ge T^q_{\nu,r}(F)\ h(2r)>\eta.$$

Hence for every $x\in E_n$ the desired lower bound holds. Thus, let $n\in\mathbb{N}^*$ and $\gamma\in(0,r)$. Fix a maximal r-pseudo-packing $\{(x_j,r)\}_{j\ge1}$ within E_n. For each $i\in\mathbb{N}^*$, define the active index set of Λ_i as

$$\mathcal{L}(i){:}=\{j\ge1:\ \Lambda_i^F(x_i)\text{ isnon}-\text{empty}\}.$$

This ensures the projected collection $\{(x_j, r)\}_{j\in\mathcal{L}(i)}$ forms an r-pseudo-packing of the canonical projection $\pi_{\mathbb{X}}(\Lambda_i)$. For fixed $i \in \mathbb{N}^*$, iterate over $j \in \mathcal{L}(i)$. By virtue of Lemma 3, we may select from the family $\{(y,r) \mid y \in \Lambda_i^F(x_j)\}$ a maximal r-pseudo-packing that still covers $\Lambda_i^F(x_j)$. The union $\cup_{j\in\mathcal{L}(i)} \mathcal{H}(i,j)$ then forms a pseudo-packing of Λ_i. By the definition of $\mathcal{H}(i,j)$ and $T^q_{\nu,r}$, we directly infer

$$\underbrace{\sum \nu(B(y,r))^q\, g(2r)}_{\text{Sumover}\mathcal{H}(i,j)} \geq \underbrace{T^q_{\nu,\gamma}(\Lambda_i^F(x_j))\, g(2r)}_{\text{Lowerboundatscale }\gamma} \geq \underbrace{T^q_{\nu,r}(\Lambda_i^F(x_j))\, g(2r)}_{\text{Lowerboundatscale }r}.$$

Finally, define the intersection index class for x_j up to n

$$\mathcal{I}(j,n) = \{i \in \mathbb{N}^* \mid x_j \in \Lambda_i \text{ and } i \leq n\}.$$

The expression represents a nested triple summation process:

(1) First, sum over the integer index i from 1 to n.

(2) Second, for each fixed i, sum over all elements j in the set $\mathcal{L}(i)$.

(3) Third, for each fixed pair (i,j) , sum over all ball-center-radius pairs (y,r) in the slice-packing collection $\mathcal{H}(i,j)$.

For every such triple $(i,j,(y,r))$, the term added to the sum is:

$$\mu(B(x_j,r))^q\, \nu(B(y,r))^q\, h(2r)\, g(2r).$$

By construction, each inner sum over $(y,r) \in \mathcal{H}(i,j)$ is bounded below by

$$T^q_{\nu,r}(\Lambda_i^F(x_j))\, g(2r) \;\geq\; \eta,$$

Hence, when we sum $h(2r)\,\mu(B(x_j,r))^q\, g(2r)\,\nu(B(y,r))^q$ first over $(y,r) \in \mathcal{H}(i,j)$, then over $i \in \mathcal{I}(j,n)$, and finally over $j \geq 1$, the total is at least η times the simpler double sum. Finally, observing that this latter double sum is exactly $N^{q,h}_{\mu}(E_n)\, h(2r)$ and the supremum across every r-pseudo-packing of E_n gives

$$\sum_{i=1}^{\infty} N^{q,hg}_{\mu\times\nu}(\Lambda_i)\, h(2r)\, g(2r) \;\geq\; \eta\, N^{q,h}_{\mu}(E_n)\, h(2r).$$

Therefore, we obtain

$$\sum_{i=1}^{n} \overline{\mathrm{R}}^{q,hg}_{\mu\times\nu}(\Lambda_i) \geq \eta\, \overline{\mathrm{R}}^{q,h}_{\mu}(E_n).$$

Now, taking $n \to +\infty$, we obtain using (3.1), that

$$\mathrm{R}^{q,hg}_{\mu\times\nu}(E\times F) + \varepsilon \geq \eta\, \mathrm{R}^{q,h}_{\mu}(E).$$

Because our estimates hold for every choice of $\varepsilon > 0$ and any $\eta < \mathrm{H}^{q,g}_{\nu}(F)$, the claimed inequality follows.

3.3 *Third inequality of Theorem 3*

We begin by introducing an auxiliary fractal measure, called the weighted H-S h-measure and denoted by $\mathrm{Q}^{q,h}_{\mu}$, that will be pivotal in establishing the third inequality of Theorem 3. This measure is defined using weighted packing of balls with the same radius. Given a set $E \subseteq \mathbb{X}$ together with positive weights c_i, centers $x_i \in E$, and radii $r_i > 0$, these data constitute a weighted packing of E exactly when each point $x \in E$ lies in balls whose combined weight does not exceed one; that is,

$$\sum_{\{i:\, d(x_i,x) \le r_i\}} c_i \le 1.$$

We define $\overline{\mathrm{Q}}_\mu^{q,h}$ of E to be the upper limit as r approaches zero of the product $S_{\mu,r}^q(E)$ with $h(2r)$. The value of $S_{\mu,r}^q(E)$ quantifies the the supremum of $\sum_i c_i\, \mu(B(x_i,r))^q$, ranging over all collections of weighted packing of E. Hence, the weighted version of the generalized H–S h-measure, denoted $\mathrm{Q}_\mu^{q,h}$, through the formula below

$$\mathrm{Q}_\mu^{q,h}(E) = \inf_{\substack{E \subseteq \bigcup_i E_i \\ E_i \text{bounded}}} \sum_i \overline{\mathrm{Q}}_\mu^{q,h}(E_i).$$

Proposition 3: Consider a measure $\mu \in \mathcal{P}(\mathbb{X})$, a function $h \in \mathcal{F}$ and a real number $q \in \mathbb{R}$. Hence, for each set $E \subseteq \mathbb{X}$, one obtains

$$\mathrm{P}_\mu^{q,h} \le \mathrm{Q}_\mu^{q,h} \le \mathrm{R}_\mu^{q,h} \tag{3.2}$$

Proof: By definition, a weighted r-packing is a generalization of an ordinary r-packing, where each ball can be assigned a weight. This means any r-packing can be viewed as a weighted r-packing with all weights equal to one. Consequently, this immediately gives

$$\mathrm{P}_\mu^{q,h}(E) \le \mathrm{Q}_\mu^{q,h}(E),$$

for any set $E \subseteq \mathbb{X}$. Let us now establish the second inequality. Start by fixing a small scale $r > 0$ and setting

$$l \le N_{\mu,r}^q(E)h(2r).$$

This means that there exists an r-packing of E whose total weighted sum is close to l. To ensure the sum remains above l, we construct a weighted r-packing as follows: Choose a family of triples $\{(c_i, x_i, r)\}_i$ that forms a weighted r-packing of E. Approximating each c_i by rational value α_i/N with N sufficiently large, we have

$$\sum_{i=1}^{\infty} \frac{\alpha_i}{N}\, \mu(B(x_i,r))^q h(2r) > l.$$

We further truncate the sum to finite number n of terms, ensuring

$$\sum_{i=1}^{n} \frac{\alpha_i}{N}\, \mu(B(x_i,r))^q h(2r) > l. \tag{3.3}$$

Next, we construct a sequence of integers $m_0(i) = \alpha_i$ for $i = 1, \dots, n$. We use these to create a covering process: Define the set J_1 of indices where $m_0(i) \ge 1$ and for each $j \ge 1$, select a subset $I_j \subseteq J_j$ such that $\{(x_i, r) \mid i \in I_j\}$ is a maximal r-pseudo-packing among the points with positive multiplicity. Then, adjust the multiplicities using the rule

$$m_j(i) = \begin{cases} m_{j-1}(i) - 1 & \text{if } i \in I_j \\ m_{j-1}(i) & \text{otherwise} \end{cases}$$

Therefore, we repeat this process until no points with positive multiplicity remain. Then introduce for each $j \geq 0$ the count

$$\zeta_j(x) = \sum_{\{i:\, d(x_i,x) \leq r\}} m_j(i)$$

Because I_j covers every x_i with $m_{j-1}(i) \geq 1$, one sees that for each such i

$$\zeta_j(x_i) \leq \zeta_{j-1}(x_i) - 1.$$

If the process never terminated by round N, then choosing $k \in J_N$ (and then k belongs to every set $J_1, \ldots J_N$ since $\{J_i\}_{i=1}^{N}$ is a decreasing sequence) yields

$$\zeta_0(x_k) \geq N + \zeta_N(x_k) \geq N + 1, \tag{3.4}$$

while the packing condition forces $\zeta(x_k) \leq N$. This contradiction shows $J_N = \emptyset$ and hence

$$\sum_{j=1}^{N} (m_{j-1}(i) - m_j(i)) = \alpha_i.$$

Since each round I_j is itself a pseudo-packing of E, its contribution is bounded bey $N_{\mu,r}^{q}(E)\, h(2r)$. Summing over $j = 1, \ldots, N$ and dividing by reproduces the left-hand sum, showing

$$l \leq N_{\mu,r}^{q}(E)\, h(2r)$$

Therefore, $S_{\mu,r}^{q}(E)\, h(2r) \leq N_{\mu,r}^{q}(E)\, h(2r)$. Hence,

$$\overline{\mathrm{Q}}_{\mu}^{q,h}(E) \leq \overline{\mathrm{R}}_{\mu}^{q,h}(E). \tag{3.5}$$

Finally, let $\varepsilon > 0$. Pick a countable cover $E \subseteq \bigcup_i E_i$ by bounded sets E_i so that

$$\sum_i \overline{\mathrm{R}}_{\mu}^{q,h}(E_i) \;\leq\; \mathrm{R}_{\mu}^{q,h}(E) + \varepsilon.$$

Since we already know $\overline{\mathrm{Q}}_{\mu}^{q,h}(E_i) \leq \overline{\mathrm{R}}_{\mu}^{q,h}(E_i)$ for each i, it follows that

$$\mathrm{Q}_{\mu}^{q,h}(E) \leq \sum_i \overline{\mathrm{R}}_{\mu}^{q,h}(E_i) \;\leq\; \mathrm{R}_{\mu}^{q,h}(E) + \varepsilon.$$

Letting $\varepsilon \to 0$ yields the desired inequality

$$\mathrm{Q}_{\mu}^{q,h}(E) \;\leq\; \mathrm{R}_{\mu}^{q,h}(E).$$

Proof of the third inequality in Theorem 3. We aim to show

$$\mathrm{P}_{\mu\times\nu}^{q,hg}(E \times F) \;\leq\; \mathrm{Q}_{\mu}^{q,h}(E)\, \mathrm{P}_{\nu}^{q,g}(F),$$

and then invoke Proposition 3. To begin, assume both $\mathrm{Q}_{\mu}^{q,h}(E)$ and $\mathrm{P}_{\nu}^{q,g}(F)$ are finite. Choose countable covers

$$E \;\subseteq\; \bigcup_{i=1}^{\infty} E_i, F \;\subseteq\; \bigcup_{j=1}^{\infty} F_j,$$

so that each E_i and F_j approximates the corresponding quasi-measure arbitrarily well. Next, let $\{(x_k, y_k), r\}$ be any packing of $E \times F$ satisfying

$$\sum_{k=1}^{\infty} \mu(B(x_k,r))^q \, \nu(B(y_k,r))^q \, h(2r) \, g(2r) \; > \; l.$$

Let each index k be associated with

$$a_k \; = \; N \, \mu(B(x_k,r))^q \, h(2r) \; - \; \eta, b_k \; = \; N \, \nu(B(y_k,r))^q \, g(2r) \; - \; \eta,$$

where N is taken large enough and $\eta > 0$ small enough so that

$$\sum_{k=1}^{\infty} \frac{a_k \, b_k}{N^2} \; > \; l.$$

By truncating to the first n terms if necessary, we may assume $a_k, b_k > 0$ and

$$\sum_{k=1}^{n} \frac{a_k \, b_k}{N^2} \; > \; l.$$

Fix any $x \in E$. Then those indices k for which $d(x_k, x) \leq r$ give an r-packing of F, so

$$\sum_{d(x_k,x)\leq r} b_k \leq \sum_{d(x_k,x)\leq r} N \, \nu(B(y_k,r))^q \, g(2r) \leq N \, M^q_{\nu,r}(F) \, g(2r).$$

Hence the family $\{(x_k, r), b_k/N\}_{k=1}^{n}$ constitutes a weighted packing of E, yielding

$$M^q_{\mu\times\nu,r}(E\times F) \, h(2r) \, g(2r) \; \leq \; S^q_{\mu,r}(E) \, h(2r) \, M^q_{\nu,r}(F) \, g(2r).$$

Dividing by N^2 and letting $r \to 0$ shows for every $i, j \geq 1$,

$$\overline{\mathrm{P}}^{q,hg}_{\mu\times\nu}(E_i \times F_j) \; \leq \; \overline{\mathrm{Q}}^{q,h}_{\mu}(E_i) \, \overline{\mathrm{P}}^{q,g}_{\nu}(F_j).$$

Finally, summing over i, j and optimizing over the arbitrary covers of E and F gives the desired inequality

$$\mathrm{P}^{q,hg}_{\mu\times\nu}(E\times F) \; \leq \; \mathrm{Q}^{q,h}_{\mu}(E) \, \mathrm{P}^{q,g}_{\nu}(F).$$

This completes the proof.

4. Appendix

4.1 *Generalized packing and Hausdorff measures*

Considering a probability measure μ over the space $\mathbb{X}$, a dimension function h, and a real exponent q, our first step is to define the generalized packing h-measure $\mathcal{P}^{q,h}_{\mu}$ of a subset E of $\mathbb{X}$. Given $\delta > 0$, a countable family of pairs $\{(x_i, r_i)\}_i$ with $x_i \in \mathbb{X}$ and $0 < r_i \leq \delta$ constitutes a δ-packing of E if the following inequality holds for every pair of distinct indices i and j

$$\rho(x_i, x_j) > r_i + r_j.$$

The symbol $\Upsilon_\delta(E)$ will stand for the collection of every δ€'packing of E. Assuming $E \neq \emptyset$, for each $\delta > 0$ we set

$$\mathcal{P}^{q,h}_{\mu,\delta}(E) := \sup_{(x_i,r_i)_i\in\Upsilon_\delta(E)} \sum_i \mu(B(x_i,r_i))^q \, h(2r_i),$$

and then define

$$\mathcal{P}^{q,h}_{\mu,0}(E) := \inf_{\delta>0} \mathcal{P}^{q,h}_{\mu,\delta}(E) = \lim_{\delta\to 0} \mathcal{P}^{q,h}_{\mu,\delta}(E).$$

Although $\mathcal{P}^{q,h}_{\mu,0}$ is monotone with respect to set inclusion, it is not

σ-additive. Hence, by the usual “outer-measure” extension [29, 30, 24], this leads to the definition of the generalized packing h-measure

$$\mathcal{P}_{\mu}^{q,h}(E) = \inf_{E \subseteq \bigcup_{i=1}^{\infty} E_i} \sum_i \mathcal{P}_{\mu,0}^{q,h}(E_i),$$

with the convention $\mathcal{P}_{\mu}^{q,h}(\emptyset) = 0$. In particular, $\mathcal{P}_{\mu}^{q,h}$ extends both the classical packing measure $\mathcal{P}^t$ [14, 25, 20] and the packing h-measure $\mathcal{P}^h$ [12, 17, 10]. Now, assuming $E \neq \emptyset$, for each $\delta > 0$ we set

$$\widetilde{\Upsilon}_{\delta}(E) = \{(x_i, r_i)_i : x_i \in E, 0 < r_i \leq \delta, \rho(x_i, x_j) > \max(r_i, r_j) \ \forall i \neq j\},$$

and define

$$\mathcal{R}_{\mu,\delta}^{q,h}(E) := \sup_{(x_i,r_i)_i \in \widetilde{\Upsilon}_{\delta}(E)} \sum_i \mu(B(x_i, r_i))^q \, h(2r_i).$$

Then

$$\mathcal{R}_{\mu,0}^{q,h}(E) := \inf_{\delta > 0} \mathcal{R}_{\mu,\delta}^{q,h}(E) = \lim_{\delta \to 0} \mathcal{R}_{\mu,\delta}^{q,h}(E),$$

and this gives rise to the concept of the generalized pseudo-packing h€‘measure

$$\mathcal{R}_{\mu}^{q,h}(E) = \inf_{E \subseteq \bigcup_{i=1}^{\infty} E_i} \sum_i \mathcal{R}_{\mu,0}^{q,h}(E_i),$$

] with the convention $\mathcal{R}_{\mu}^{q,h}(\emptyset) = 0$. In particular, $\mathcal{R}_{\mu}^{q,h}$ extends the classical pseudo-packing h€‘measure $\mathcal{R}^h$ [17, 10]. We next turn to the centered Hausdorff approach: For a nonempty set E and any scale $\delta > 0$, we set

$$\mathcal{C}_{\delta}(E) = \{(x_i, r_i)_i : x_i \in E, 0 < r_i \leq \delta, E \subseteq \bigcup_i B(x_i, r_i)\}.$$

Then one writes

$$\mathcal{H}_{\mu}^{q,h}(E) = \inf_{(x_i,r_i)_i \in \mathcal{C}_{\delta}(E)} \sum_i \mu(B(x_i, r_i))^q \, h(2r_i).$$

and then

$$\mathcal{H}_{\mu,0}^{q,h}(E) := \sup_{\delta > 0} \mathcal{H}_{\mu,\delta}^{q,h}(E) = \lim_{\delta \to 0} \mathcal{H}_{\mu,\delta}^{q,h}(E).$$

Finally, the generalized centered Hausdorff h-measure is obtained by considering the supremum of the quantity as F ranges through all subsets of E:

$$\mathcal{H}_{\mu}^{q,h}(E) = \sup_{F \subseteq E} \mathcal{H}_{\mu,0}^{q,h}(F).$$

Since $\mathcal{H}_{\mu}^{q,h}$ satisfies the properties of a metric outer measure, It descends in a straightforward way to the Borel σ-algebra of $\mathbb{X}$, yielding a measure. This function naturally extends the classical centered Hausdorff measure $\mathcal{C}^t$ [14], as well as, the generalized Hausdorff measure $\mathcal{H}_{\mu}^{q,t}$ [25, 22], providing a multifractal generalization that captures a broader range of scaling behaviors.

4.2 *Maximal pseudo-packing*

Definition 1 (Maximal pseudo-packing): Consider a set $\mathbb{X}$ equipped with the distance function ρ, making it a metric space. Consider the collection

$$\mathcal{A} = \{(x, r)\} \text{ with } x \in \mathbb{X}, r > 0,$$

consisting of every center x and positive radius r. A subset $G \subseteq \mathcal{A}$ is called *maximal pseudo-packing* if:

(1) **Separation**: If (x, r) and (y, s) are two different elements of G, then their centers are farther apart than either radius:

$$\rho(x, y) > \max\{r, s\}.$$

(2) **Covering**: Let (x, y) be an arbitrary ball in $\mathcal{A}$, one can pick some $(y, s) \in G$ whose center y lies within the distance given by the larger radius:

$$\rho(x, y) \leq \max\{r, s\}.$$

(3) **Maximality**: No super-set of G within $\mathcal{A}$ satisfies both the separation and covering properties.

This ensures G is the largest possible subset of $\mathcal{A}$ where distinct elements are non-overlapping in their metric neighborhoods while still covering all of $\mathcal{A}$.

Lemma 3 (Existence of maximal pseudo-packing [17]): *For a fixed $\delta > 0$ and a chosen subset $E \subseteq \mathbb{X}$, the associated family of pairs is given by*

$$\mathcal{A}_\delta = \{(x, \delta) | x \in E\}.$$

Hence one can choose $G \subseteq \mathcal{A}_\delta$ to be a maximal pseudo-packing which provides a cover of E.

Lemma 4 (Finite maximal pseudo-packing [17]); *Consider a subset E of $\mathbb{X}$ and a finite collection F of ordered pairs consisting of a point in $\mathbb{X}$ and a positive radius. These are chosen so that every element of E appears as the first entry of at least one pair in F, equivalently, for each point in E there exists some positive radius for which the corresponding pair belongs to F. Hence one can choose $G \subseteq F$ to be a maximal pseudo-packing which provides a cover of E.*

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Received November, 2024