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Certain subclasses of bi-univalent functions using generating functions of (*s, t*)-Lucas and balancing Lucas polynomials

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Abstract

This research article investigates two subclasses of bi-univalent functions associated with (*s, t*)-Lucas polynomials in $\mathbb{D}$. It intriguing since they are fundamental to geometric function theory. To understand the geometrical properties of these subclasses, the estimates of two initial Taylor-Maclaurin coefficients $|a_2|$ as well as $|a_3|$ are very essential. These play a vital role to explore a number of significant geometric features of subclasses. These initial coefficients provide significant information about the behavior of the function near the origin,

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revealing its functional characteristics, interaction with the boundary of the unit disk, and possible subclassifications. In particular, their distortion, growth behaviour, and mapping qualities. Additionally, we employ the Fekete-Szegö approximation, a technique frequently employed to derive precise constraints for these two subclasses, which traditionally gives upper bounds for $|a_3 - \tau a_2^2|$.

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1. Introduction

Let the open unit disc is defined as: $\mathbb{D} = \{\xi \in \mathbb{C}: |\xi| < 1\}$. The set of Schlichtly normalized function $f(\xi) \in \mathbb{A}$, expressed in the following form

$$f(\xi) = \xi + \sum_{k=2}^{\infty} a_k \xi^k, \tag{1.1}$$

where the complex coefficients are represented by a_k, and the subclass of functions inside $\mathbb{A}$ is represented by $\mathbb{S}$. In the unit disc, a f is consider to be bi-univalent if f, its inverse f^{-1} are analytic and one-to-one in $\mathbb{D}$. Using Equation (1.1), the set of bi-univalent functions that comprise the class Σ. This work was the beginning of systematic studies of coefficient problems for bi-univalent functions and provided a inspiration for many studies of estimates of the Taylor–Maclaurin coefficients for a number of subclasses of Σ. After Lewin [1] had attained the estimate for $|a_2|$, the problem of estimating the next coefficient $|a_3|$ became quite popular. Later a subclass of $f \in \Sigma$ was explored by Brannan and Taha [2] who attained bounds for the values of $|a_2|$ and $|a_3|$. They consider their work to be one of the earliest systematic investigations on the estimation of the third coefficient $|a_3|$ of subclasses.

Let $f \in \mathcal{A}$ is stated to be starlike in $\mathbb{D}$ if

$$\Re\left(\frac{\xi f'(\xi)}{f(\xi)}\right) > 0, \xi \in \mathbb{D},$$

whereas f is called convex in $\mathbb{D}$ whenever

$$\Re\left(1 + \frac{\xi f''(\xi)}{f'(\xi)}\right) > 0, \xi \in \mathbb{D}.$$

Using the principle of subordination, several generalized forms of starlike and convex function classes have been developed in geometric function theory. In this direction, Ma and Minda [3] introduced a unified approach for defining subclasses of starlike and convex functions through the subordinations

$$\frac{\xi f'(\xi)}{f(\xi)} \prec \phi(\xi)$$

and

$$1 + \frac{\xi f''(\xi)}{f'(\xi)} \prec \phi(\xi),$$

where the analytic function ϕ satisfies

$$\Re(\phi(\xi)) > 0, \xi \in \mathbb{D}.$$

These subclasses provide a unified structure for studying many eminent families of starlike and convex functions through suitable choices of the subordinating function $\phi(\xi)$. These generalized subclasses have motivated several researchers to investigate analogous subclasses in the family of $f \in \Sigma$ attain their initial coefficients.

Recent work has examined several fascinating subclasses belonging to the class Σ, numerous studies on bi-univalent functions have been conducted recently. Among these, Behera and Panda [4] presented a relatively new sequence called the Lucas numbers, and the past several decades many authors [5, 6, 7] investigated, Lee and Asci [8] and Ray [9] studied the geometrical properties of (p, q)- Lucas polynomials. These functions have very remarkable geometric properties which are relating to their mapping properties. Over the past few years, numerous researchers have resulted special families of Lucas polynomials and balancing Lucas polynomials etc. See [10, 11, 12, 13, 14, 15, 16, 17] and [18].

This works focus to investigate the properties of new subclasses related to (s, t)-Lucas and balancing (s, t)-Lucas polynomials. The main aim of this work is to attain upper and lower bounds for the initial coefficients and to attain sharp bounds for $|a_3 - \tau a_2^2|$. This paper has three parts. In the Introduction section, some preliminary concepts $f \in \Sigma$ as well as a short discussion of (s, t)-Lucas and balancing (s, t)=Lucas polynomials are presented. In second and third parts new subclasses generated from these polynomial families are defined and studied.

2. Coefficient Bounds for the Subclass $\mathcal{G}_{s,t}^{\Sigma,G}(\mu)$

Take the Lucas polynomials $L_n(s(\rho), t(\rho); \rho)$ as a recurrence relation derived by the recurrence relation.

$$L_n(\rho) = s(\rho)L_{n-1}(\rho) + t(\rho)L_{n-2}(\rho), n \geq 2,$$

with the following basic conditions

$$L_0(\rho) = 2, L_1(\rho) = s(\rho).$$

These polynomials generalize the Lucas numbers for which the functions $s(\rho)$ and $t(\rho)$ are arbitrary. The sequence is generated recursively from the previous two terms and forms a large family of polynomials which have many applications in mathematics and related topics.

2.1 *Generating Function of Lucas Polynomials*

The generating function $G(\rho, \xi)$ of $L_n(s(\rho), t(\rho), \rho)$ is a power series in ξ, defined as:

$$G(\rho, \xi) = \frac{2 - s(\rho)\xi}{1 - s(\rho)\xi - t(\rho)\xi^2}.$$

This generating function can be expanded as:

$$G(\rho, \xi) = \sum_{n=0}^{\infty} L_n(s(\rho), t(\rho), \rho)\xi^n,$$

which encodes the entire sequence of Lucas polynomials $L_n(s(\rho), t(\rho), \rho)$. Specifically, the expansion for the first few terms is:

$$G(\rho,\xi) = 2 + s(\rho)\xi + (2t(\rho) + s^2(\rho))\xi^2 + (3s(\rho)t(\rho) + s^3(\rho))\xi^3 + \cdots.$$

This expansion gives a clear insight into how the Lucas polynomials are related to the coefficients of the generating function.

2.3 *Remarks on Specific Cases*

The section includes examples of specific forms of Lucas polynomials attained by assigning particular values to $s(\rho)$ and $t(\rho)$:

- **Pell-Lucas polynomials** $Q_n(s)$ arise when $s(\rho) = 2s$ and $t(\rho) = 1$.
- **Jacobsthal-Lucas polynomials** $j_n(s)$ are attained when $s(\rho) = 1$ and $t(\rho) = 2s$.
- **Fermat-Lucas polynomials** $f_n(s)$ are derived when $s(\rho) = 3s$ and $t(\rho) = -2$.
- **Chebyshev polynomials of the first kind** $T_n(s)$ arise when $s(\rho) = 2s$ and $t(\rho) = -1$.

These cases represent special types of Lucas polynomials, each with its own properties and applications.

2.3 *Subordination Condition and the Class* $\mathcal{G}_{s,t}^{\Sigma,G}(\mu)$

The subclass $\mathcal{G}_{s,t}^{\Sigma,G}(\mu)$ is defined by a **subordination condition** involving $f(\xi)$, which is belonging to the class Σ. This subordination condition relates $f(\xi)$ to the generating function $G(\rho,\xi)$. The key equations in the definition of this class are:

$$\left(\frac{\xi f'(\xi)}{f(\xi)}\right) + \mu\left(\frac{\xi f''(\xi)}{f(\xi)}\right) \prec G(\rho,\xi) - 1. \tag{2.1}$$

and

$$\left(\frac{\omega f'(\omega)}{f(\omega)}\right) + \mu\left(\frac{\omega f''(\omega)}{f(\omega)}\right) \prec G(\sigma,\omega) - 1. \tag{2.2}$$

Here, $\prec$ denotes a subordination relation between functions, This indicates that the left-hand function is subordinate to the right-hand function. This condition imposes constraints on the behavior of $f(\xi)$ within the context of the class $\mathcal{G}_{s,t}^{\Sigma,G}(\mu)$.

Let us define

$$G(\rho,\xi) - 1 = 1 + s(\rho)\xi + 2t(\rho) + s^2(\rho)\xi^2 + (3s(\rho)t(\rho) + s^3(\rho))\xi^3 + \cdots, \tag{2.3}$$

and

$$G(\sigma,\omega) - 1 = 1 + s(\sigma)\omega + 2t(\sigma) + s^2(\sigma)\omega^2 + (3s(\sigma)t(\sigma) + s^3(\sigma))\omega^3 + \cdots, \tag{2.4}$$

2.4 *Lemma*

The lemma introduced in the section helps establish an important property for the family of functions involved. It states that for any $p(\xi)$ in the class $\mathcal{P}$, the real part of $p(\xi)$ is positive and bounded $\mathrm{Re}(p(\xi)) > 0, p(\xi) = 1 + p_1\xi + p_2\xi^2 + \cdots$. This lemma is foundational for the analysis of functions in the class $\mathcal{G}_{s,t}^{\Sigma,G}(\mu)$ and their behavior within the unit disk $\mathbb{D}$, ensuring that these functions remain analytic and bounded.

Theorem 2.1: *If* $f(\xi)$ *exists in the class* $\mathcal{G}_{s,t}^{\Sigma,G}(\mu)$, *then the coefficients* a_2 *and* a_3 *are subject to the following inequalities:*

$$|a_2| \leq \sqrt{\frac{s^2(\rho)+2(t(\rho)+t(\sigma))+s^2(\sigma)}{2(1+4\mu)}}$$

and for a_3, *we have*

$$|a_3| \leq \frac{s^2(\rho)+s^2(\sigma)}{2(1+2\mu)^2} + \frac{2(t(\rho)-t(\sigma))+s^2(\rho)-s^2(\sigma)}{4(1+3\mu)}. \tag{2.5}$$

Proof: Let $f \in \mathcal{G}_{J,S}^{\Sigma,G}(\mu)$. Then there exist $f, g: \mathbb{D} \to \mathbb{D}$, given by (2.1) and (2.2),

$$\left(\frac{\xi f'(\xi)}{f(\xi)}\right) + \mu\left(\frac{\xi f''(\xi))}{(f(\xi)}\right) = 1 + (1+2\mu)a_2\xi$$
$$+(2(1+3\mu)a_3 - (1+2\mu)a_2^2)\xi^2 + \ldots \tag{2.6}$$

and

$$\left(\frac{\omega f'(\omega)}{f(\omega)}\right) + \mu\left(\frac{(\omega f''(\omega)}{f(\omega)}\right) = 1 - (1+2\mu)a_2\omega + (3+10\mu)a_2^2$$
$$-2(1+3\mu)a_3)\omega^2 + \ldots \tag{2.7}$$

From the equations (2.3) and (2.6),
Coefficient of ξ:

$$(1+2\mu)a_2 = s(\rho) \tag{2.8}$$

Coefficient of ξ^2:

$$2(1+3\mu)a_3 - (1+2\mu)a_2^2 = s^2(\rho) + 2t(\rho) \tag{2.9}$$

From the equations (2.4) and (2.7),
coefficient of ω:

$$-(1+2\mu)a_2 = s(\sigma) \tag{2.10}$$

coefficient of ω^2:

$$(3+10\mu)a_2^2 - 2(1+3\mu)a_3) = s^2(\sigma) + 2t(\sigma) \tag{2.11}$$

Now, adding the equations (2.8) and (2.10), we have

$$s(\rho) = -s(\sigma) \tag{2.12}$$

Equations (2.8) and (2.10) are squared and added to attain

$$a_2^2 = \frac{s^2(\rho)+s^2(\sigma)}{2(1+2\mu)^2} \tag{2.13}$$

subtracting the equations (2.9) and (2.11), we get

$$a_3 = a_2^2 + \frac{s^2(\rho)-s^2(\sigma)+2t(\rho)-2t(\sigma)}{4(1+3\mu)} \tag{2.14}$$

When we replace equation (2.9) with equation (2.11), we attain

$$a_2^2 = \frac{t(\rho)+t(\sigma)}{(1+4\mu)-2(1+2\mu)^2} \tag{2.15}$$

By replacing equations (2.13) in equation (2.14), we attain

$$|a_3| \leq \frac{s^2(\rho)+s^2(\sigma)}{2(1+2\mu)^2} + \frac{s^2(\rho)-s^2(\sigma)+2t(\rho)-2t(\sigma)}{4(1+3\mu)} \tag{2.16}$$

When we replace (2.14) with (2.15), we get

$$a_3 = \frac{s^2(\rho)+s^2(\sigma)+2(t(\rho)+t(\sigma))}{2(1+4\mu)} + \frac{s^2(\rho)-s^2(\sigma)+2t(\rho)-2t(\sigma)}{4(1+3\mu)}$$

Hence,

$$|a_2| \leq \sqrt{\frac{s^2(\rho)+s^2(\sigma)+2(t(\rho)+t(\sigma))}{2(1+4\mu)}}$$

and

$$|a_3| \leq \frac{s^2(\rho)+s^2(\sigma)}{2(1+2\mu)^2} + \frac{s^2(\rho)-s^2(\sigma)+2t(\rho)-2t(\sigma)}{4(1+3\mu)}$$

The following corollaries are attained by replacing the values $\mu = 0{,}1$ in 2.5, correspondingly.

Corollary 2.2: *If $f(\xi)$ is a member of the class $\mathcal{G}_{s,t}^{\Sigma,G}(\mu)$ and is provided by (1.1), then*

$$|a_2| \leq \sqrt{\frac{s^2(\rho)+2(t(\rho)+t(\sigma))+s^2(\sigma)}{2}}$$

and

$$|a_3| \leq \frac{2(t(\rho)-t(\sigma))+s^2(\rho)-s^2(\sigma)}{4} + \frac{s^2(\rho)+s^2(\sigma)}{2}. \tag{2.17}$$

Corollary 2.3: *If $f(\xi) \in \mathcal{G}_{s,t}^{\Sigma,G}(\mu)$ and is provided by (1.1), then*

$$|a_2| \leq \sqrt{\frac{s^2(\rho)+2(t(\rho)+t(\sigma))+s^2(\sigma)}{10}}$$

and

$$|a_3| \leq \frac{2(t(\rho)-t(\sigma))+s^2(\rho)-s^2(\sigma)}{16} + \frac{s^2(\rho)+s^2(\sigma)}{32}. \tag{2.18}$$

2.5 *Fekete-Szego Inequality*

The Fekete-Szego inequality is a fundamental result in complex analysis, specifically in the study of univalent and analytic functions. It provides a bound on the coefficients of a univalent function, which are often critical in various applications of geometric function theory.

Theorem 2.4: *If $f(\xi) \in \mathcal{G}_{s,t}^{\Sigma,G}(\mu)$, then*

$$|a_3 - \tau a_2^2| \leq \begin{cases} \frac{t(\rho)}{(1+3\mu)} & if\ 0 \leq h_1(\tau) \leq \frac{1}{2(1+3\mu)} \\ t(\rho)\left|\frac{(1-\tau)}{(1+4\mu)-2(1+2\mu)^2}\right| & if\ h_1(\tau) \geq \frac{1}{2(1+3\mu)} \end{cases} \tag{2.19}$$

where $h_1(\tau) = \frac{1-\tau}{(1+4\mu)-2(1+2\mu)^2}$.

Proof: From the equation (2.14) and (2.15), we get

$$a_3 - \tau a_2^2 = t(\rho)\left(h_1(\tau) + \frac{1}{2(1+3\mu)}\right) + t(\sigma)\left(h_1(\tau) - \frac{1}{2(1+3\mu)}\right).$$

Hence,

$$|a_3 - \tau a_2^2| \leq \begin{cases} \frac{t(\rho)}{(1+3\mu)} & \text{if } 0 \leq h_1(\tau) \leq \frac{1}{2(1+3\mu)}, \\ t(\rho)\left|\frac{(1-\tau)}{(1+4\mu)-2(1+2\mu)^2}\right| & \text{if } h_1(\tau) \geq \frac{1}{2(1+3\mu)}. \end{cases}$$

The suitable choice of parameter values like $\mu = 0,1$ in 2.19 are now specialized. As a result, the following corollaries follow:

Corollary 2.5: *Assume that $f(\xi)$ is a member of the class $\mathcal{G}_{s,t}^{\Sigma,G}(\mu)$ and is provided by (1.1). Then, the inequality that follows is true:*

$$|a_3 - \tau a_2^2| \leq \begin{cases} t(\rho) & \text{if } 0 \leq h_1(\tau) \leq \frac{1}{2}, \\ t(\rho)|\tau - 1| & \text{if } h_1(\tau) > \frac{1}{2}, \end{cases} \tag{2.20}$$

where $h_1(\tau)$ is defined as: $h_1(\tau) = 1 - \tau$.

Corollary 2. 6: *Assume that $f(\xi)$ is a member of the class $\mathcal{G}_{s,t}^{\Sigma,G}(\mu)$ and is provided by (1.1). Then, the inequality that follows is true:*

$$|a_3 - \tau a_2^2| \leq \begin{cases} \frac{t(\rho)}{4} & \text{if } 0 \leq h_1(\tau) \leq \frac{1}{4}, \\ t(\rho)\left|\frac{1-\tau}{13}\right| & \text{if } h_1(\tau) > \frac{1}{4}, \end{cases} \tag{2.21}$$

where $h_1(\tau)$ is again given by: $h_1(\tau) = \frac{1-\tau}{13}$.

3. Coefficient Bounds for the Subclass $\mathcal{G}^{\Sigma,G}(\mu)$GÎ£,G(Î¼)

Definition 3.1: If the subordination below is true, then a function, $f \in \mathcal{G}^{\Sigma,G}(\mu)$

$$\left(\frac{\xi f'(\xi)}{f(\xi)}\right) + \mu\left(\frac{\xi f''(\xi))}{(f(\xi)}\right) \prec G(\rho, \xi) - 1 \tag{3.1}$$

and

$$\left(\frac{\omega f'(\omega)}{f(\omega)}\right) + \mu\left(\frac{(\omega f''(\omega)}{f(\omega)}\right) \prec G(\sigma, \omega) - 1 \tag{3.2}$$

Theorem 3.2: If $f(\xi) \in \mathcal{G}^{\Sigma,G}(\mu)$ and is provided by (1.1), then

$$|a_2| \leq \sqrt{\frac{\rho^2+\sigma^2+4}{4(1+2\mu)}} \tag{3.3}$$

and

$$|a_3| \leq \frac{\rho^2+\sigma^2}{2(1+2\mu)^2} + \frac{\rho^2-\sigma^2}{4(1+3\mu)} \tag{3.4}$$

Theorem 3.3: *If* $f(\xi)$ *corresponds to* $\mathcal{G}^{\Sigma,G}(\mu)$, *then*

$$|a_3 - \tau a_2^2| \leq \begin{cases} \frac{\rho^2}{2(1+3\mu)} & \text{if } 0 \leq h_1(\tau) \leq \frac{1}{4(1+3\mu)}, \\ 2\rho^2 \left| \frac{(1-\tau)}{2(1+2\mu)^2} \right| & \text{if } h_1(\tau) \geq \frac{1}{4(1+3\mu)}, \end{cases} \tag{3.5}$$

where $h_1(\tau) = \frac{(1-\tau)}{2(1+2\mu)^2}$.

4. Conclusion

In conclusion, two subclasses have explored within this prism of geometric function theory. Furthermore, extremes of coefficients $|a_2|$, $|a_3|$ and Fekete-Szegö sharpness are obtained. Thus, the present investigation enriches the theory of bi-univalent functions and opens new directions for further research involving special polynomial structures and related subclasses.

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References

[1] M. Lewin, "On a coefficient problem for bi-univalent functions," *Proceedings of the American Mathematical Society*, vol. 18, pp. 63-68 (1967).

[2] D. A. Brannan and T. S. Taha, "On some classes of bi-valent functions," *Studia Universitatis Babeş-Bolyai Mathematica*, vol. 31, pp. 70-77 (1986).

[3] W. C. Ma and D. Minda, "A unified treatment of some special classes of univalent functions," in *Proceedings of the Conference on Complex Analysis*, Tianjin, China, pp. 157-169 (1992).

[4] A. Behera and G. K. Panda, "On the square roots of triangular numbers," *The Fibonacci Quarterly*, vol. 37, pp. 98-105 (1999).

[5] B. A. Frasin and M. K. Aouf, "New subclasses of bi-univalent functions," *Applied Mathematics Letters*, vol. 24, pp. 1569-1573 (2011).

[6] A. Y. Lashin, "On certain subclasses of analytic and bi-univalent functions," *Journal of the Egyptian Mathematical Society*, vol. 24, pp. 220-225 (2016).

[7] K. A. Jassim, R. O. Rasheed, and R. H. Jassim, "Generalized subclass of analytic bi-univalent functions defined by differential operator," *Journal of Interdisciplinary Mathematics*, vol. 24, no. 4, pp. 961–970 (2021), doi: 10.1080/09720502.2021.1885814.

[8] G. Lee and M. Asci, “Some properties of the (p,q)-Fibonacci and (p,q)-Lucas polynomials,” *Journal of Applied Mathematics,* vol. 2012, Art. no. 264842 (2012).

[9] P. K. Ray, “Certain matrices associated with balancing and Lucas-balancing numbers,” *Matematika,* vol. 28, pp. 15-22 (2012).

[10] T. Stalin, S. Thalapathiraj, A. Manoj, and R. Balakumar, “Certain subclasses of bi-univalent functions connected with polylogarithm function,” *Journal of Interdisciplinary Mathematics*, vol. 28, no. 5, pp. 1881-1891 (2025).

[11] E. H. Abd, “New bi-univalent function subclasses described by convolution,” *Journal of Interdisciplinary Mathematics*, vol. 25, no. 6, pp. 1875-1884 (2022).

[12] S. R. Swamy, A. K. Wanas, and Y. Sailaja, “Some special families of holomorphic and Salagean type bi-univalent functions associated with (m,n)-Lucas polynomials,” vol. 11, no. 4, pp. 563-574 (2020).

[13] N. Magesh, C. Abirami, and Ş. Altınkaya, “Initial bounds for certain classes of bi-univalent functions defined by the (p,q)-Lucas polynomials,” *TWMS Journal of Applied and Engineering Mathematics*, vol. 11, no. 1, pp. 282–288 (2021).

[14] A. K. Wanas and L. I. Cotîrlă, “Applications of (M,N)-Lucas polynomials on a certain family of bi-univalent functions,” *Mathematics,* vol. 10, no. 4, Art. no. 595 (2022).

[15] A. Hussen and M. Illafe, “Coefficient bounds for a certain subclass of bi-univalent functions associated with Lucas-balancing polynomials,” *Mathematics*, vol. 11, Art. no. 4941 (2023).

[16] I. Aktaş and İ. Karaman, “On some new subclasses of bi-univalent functions defined by balancing polynomials,” *Karamanoglu Mehmetbey University Journal of Engineering and Natural Sciences*, vol. 5, pp. 25-32 (2023).

[17] A. Hussen, M. S. A. Madi, and A. M. M. Abominjil, “Bounding coefficients for certain subclasses of bi-univalent functions related to Lucas-balancing polynomials,” *AIMS Mathematics*, vol. 9, no. 7, pp. 18034-18047 (2024).

[18] M. Buyankara and M. Çağlar, “Coefficient inequalities for two new subclasses of bi-univalent functions involving Lucas-balancing polynomials,” *Eastern Anatolian Journal of Science*, vol. 10, no. 2, pp. 5-11 (2024).

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