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Bifurcation analysis of the Schamel Korteweg-de Vries equation with force term

Bamadev Sahoo [#]
Centre for Data Science
Siksha `O' Anusandhan (Deemed to be University)
Bhubaneswar 751030
Odisha
India

Shruti Tomar [†]
Department of Mathematics
School of Physical Science
DIT University
Dehradun 248009
Uttarakhand
India

Rohit Maheshwari [§]
Department of Civil Engineering
School of Engineering & Technology
DIT University
Dehradun 248009
Uttarakhand
India

Sunita Chand [‡]
Department of Mathematics
Siksha `O' Anusandhan (Deemed to be University)
Bhubaneswar 751030
Odisha
India

[#] *E-mail:* **bamadev90@gmail.com**
[†] *E-mail:* **tomer05shruti@gmail.com**
[§] *E-mail:* **rohit.maheshwari@dituniversity.edu.in**
[‡] *E-mail:* **sunitachand@soa.ac.in**

Ambit K. Pany [@]
Institute of Mathematics and Applications
Bhubaneswar 751029
Odisha
India

Sanjaya K. Mohanty *
Department of Mathematics
Siksha `O' Anusandhan (Deemed to be University)
Bhubaneswar 751030
Odisha
India

Abstract

In this investigation, we have studied the application of the bifurcation theory for the Schamel Korteweg-de Vries (SKdV) equation with force term. The equation has essential repercussions in engineering and the physical sciences. Furthermore, we study the equation incorporation of stepwise forcing elements, which are necessary in seismology for modeling seismic wave propagation and geological responses to seismic events. These terms contribute to the modeling of seismic waves. We investigate the dynamical system, stability, and phase portrait analysis for the SKdV equation with force term. Furthermore, we have studied the dynamic system's series analysis and found the system's quasi-periodic and weak quasi-periodic behavior.

Subject Classification (2020): *34C23, 34F10, 35B32.*

Keywords: *Schamel-Korteweg-de vries equation, Bifurcation analysis, Dynamical System, Phase portrait.*

1 Introduction

Bifurcation analysis is a mathematical technique used to explore changes in the qualitative properties of solutions to a system of equations as parameters are varied. In the context of wave dynamics, this analysis is vital for understanding phenomena like the transition between different wave patterns, the onset of instabilities, and the emergence of solitons. It is especially useful for studying how stable wave structures evolve into more complex behaviors, such as periodic or chaotic waves, under varying conditions [1-5]. In nonlinear wave theory, bifurcation analysis has been instrumental in investigating traveling waves, pattern formation, and instabilities in systems like the Korteweg-de Vries (KdV) equation

[@] *E-mail:* **ambit.pany@gmail.com**
* *E-mail:* **sanjayamohanty@soa.ac.in (Corresponding Author)**

and the nonlinear Schrödinger equation. For more details see [6-11] in which the authors utilized bifurcation theory within the context of dynamical systems to explore various nonlinear differential equations.

The study of bifurcation analysis for the nonlinear differential equations has gained prominence recently, as it provides an easy way to understand the behavior of nonlinear wave occurrences in physical sciences and engineering. The Schamel equation and its variations are a class of significant nonlinear evolution equations [12, 13]. The Schamel-KdV equation is essential for comprehending the behavior of nonlinear waves within the realm of plasma physics. Its significance extends to both theoretical investigations and practical applications in various contexts, including space and laboratory plasmas, electrostatic double layers, and fluid dynamics. This encompasses the analysis of internal waves in oceans, tsunami waves, and the transmission of high-speed signals in plasmas and fluids, among other areas.

In this study, we consider a step-wise forcing term whose application lies in the field of seismology such as for modeling seismic wave propagation and the response of geological structures to seismic events. By incorporating stepwise functions into the governing equations, we can effectively represent the sudden application of forces that simulate fault slip and other seismic sources. Additionally, the use of stepwise forcing terms aids in the calibration of models against observed seismic data, enhancing the accuracy of hazard assessments and the development of engineering solutions for earthquake-resistant structures. For instance, the work by Engelbrecht and Peipman [14] demonstrates how such models can be utilized to better understand the complex interactions between seismic waves and heterogeneous materials, ultimately improving earthquake preparedness and response strategies [15, 16].

Subsequently, this article proceeds to outline the stability analysis of the Schamel KdV equation with or without forcing term in section 2, further this section offers graphical representations of several dynamical systems explored. These representations highlight the influence of the nonlinear and dispersion variables. Finally, the section 3 presents the information regarding its conclusions.

2. Stability analysis of the Schamel KdV equation with or without forcing term

In this section, we analyze the influence of the external forcing term on the dynamical behavior of the Schamel KdV equation. To illustrate this influence, we have employed a step-wise forcing term defined as

$f(u) = u(u-a)(u-b)$. The roots of the polynomial equation $f(u) = 0$ are identified as $u = 0$, a and b. The parameters a and b are designated as forcing parameters. Under specific conditions applied to these parameters, the forcing term in its current formulation may function as either an attenuator or an amplifier for solitary waves [17]. Such forcing terms have been utilized to investigate wave propagation in various media within the context of a perturbed KdV equation [18, 19].

Consider the Schamel KdV equation [12] given by

$$\frac{\partial u}{\partial t} + \left(Au^{\frac{1}{2}} + Bu\right)\frac{\partial u}{\partial x} + D\frac{\partial^3 u}{\partial x^3} = 0, \tag{1}$$

where A, B and D are arbitrary coefficients and $u = u(x,t)$. Consider the wave transformation $\eta = x - kt$, where k is the constant. The converted ordinary differential equation is given by

$$-K\frac{du}{d\eta} + \left(Au^{\frac{1}{2}} + Bu\right)\frac{du}{d\eta} + D\frac{d^3 u}{\partial \eta^3} = 0, \tag{2}$$

After integrating equation (2) with respect to η, we have

$$-ku + \frac{2}{3}Au^{3/2} + B\frac{u^2}{2} + D\frac{d^2 u}{d\eta^2} + c_1 = 0. \tag{3}$$

Let us substitute $u^{1/2}(\eta) = u(\eta)$ into the equation (3), and we get the following equation

$$-ku^2 + \frac{2}{3}Au^3 + \frac{Bu^4}{2} + 2D\left(\frac{du}{d\eta}\right)^2 + 2Du\frac{d^2 u}{d\eta^2} + c_1 = 0, \tag{4}$$

where c_1 is the integration constant. Corresponding dynamical system will be

$$u' = v;$$
$$v' = \frac{1}{2Du}\left(ku^2 - \frac{2}{3}Au^3 - \frac{B}{2}u^4 - 2Dv^2 - c_1\right); \tag{5}$$

Consider the forcing term $f(u) = u(u-a)(u-b)$, we get the following dynamical system

$$u' = v,$$
$$v' = \frac{1}{2Du}\left(ku^2 - \frac{2}{3}Au^3 - \frac{B}{2}u^4 - 2Dv^2 - c_1\right) + Eu(u-a)(u-b); \tag{6}$$

where E is constant.

After solving the dynamical systems (5), we get the solution set as the equilibrium point of the system say E_1. With the help of the forcing term, the corresponding equilibrium points of the dynamical systems (6) will be some E_2. With respect to all these equilibrium points, the corresponding behaviour of both the dynamical system is shown in Figure 1.

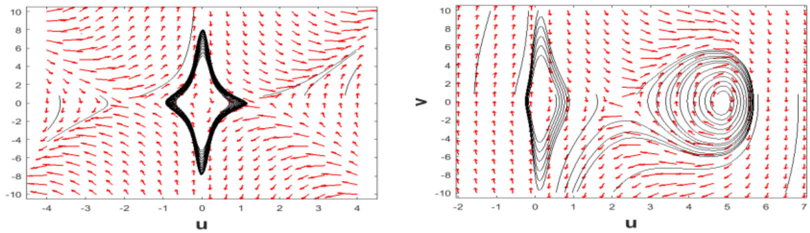

Figure 1

Quiver plots with respect to the model without and with forcing term. The value of the other parameters are $B = -0.8; k = -0.7; c_1 = 0.7; D = 0.5; A = 0.05$.

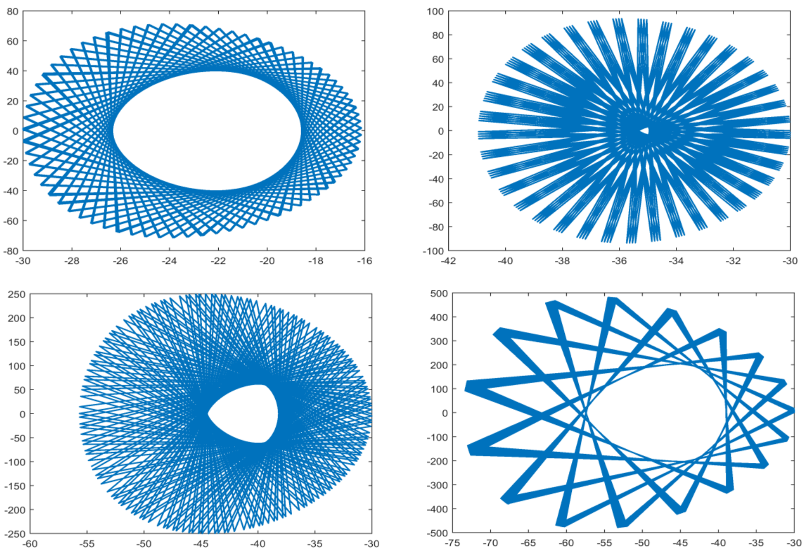

Figure 2

Phase portrait of the dynamical system equation (5) for the range of the parameter $A \in [1, 2.33]$. The corresponding values of the parameters are $A = [1, 1.4, 1.8, 2.333]$,$B = -0.8; k = -0.7$,$c_1 = 0.7$, $D = 0.5$. The time interval is $0: 6\pi/100: 10\pi$ and the equilibrium point is $[-30, 0]$.

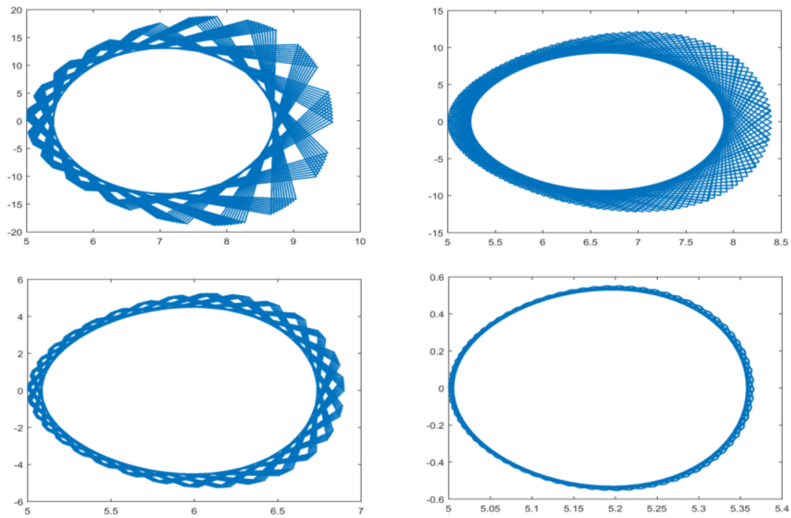

Figure 3

Phase portrait of the dynamical system of equation (6) after introducing the forcing term $f(u) = u(u-a)(u-b)$ for the range of the parameter $A \in [-2.7, 2.7]$. The corresponding values of the parameters are $A = [-2.7, -0.5, 1.6, 2.7]$, $B = -0.8$, $k = -0.7$, $c_1 = 0.7$, $D = 0.5$ and $E = -3$, $a = 4$, $b = 5$. The time interval is $0: 6\pi/100: 10\pi$ and the equilibrium point is [5, 0].

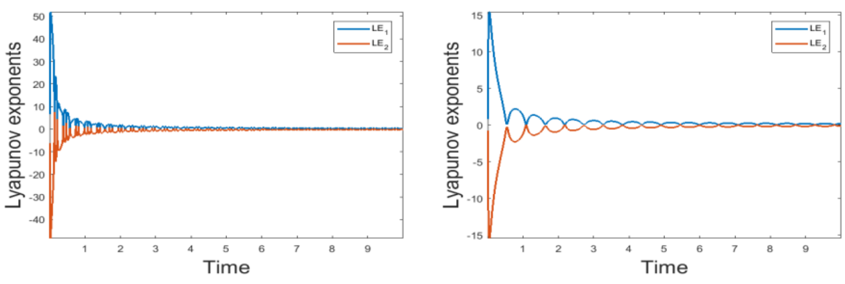

Figure 4

Lyapunov Exponent plot with respect to both the dynamical system equations (5) and (6). The corresponding values of the parameters are $A = 1.4$, $B = -0.8$, $k = -0.7$, $c_1 = 0.7$, $D = 0.5$ and $E = -3$, $a = 4$, $b = 5$. The time interval is $0: 6\pi/100: 10\pi$.

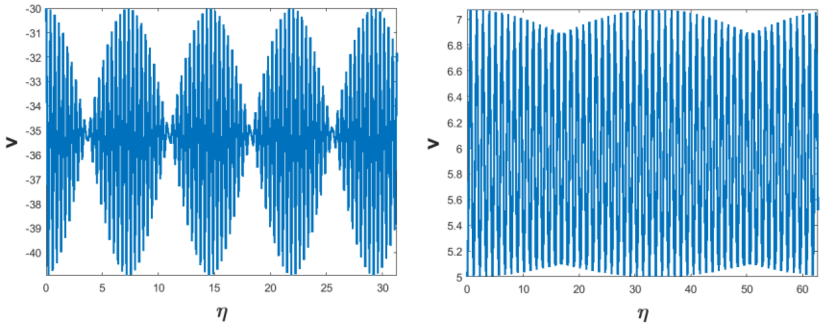

Figure 5
Time series plot with respect to both the dynamical system equations (5) and (6). The corresponding values of the parameters are $\boldsymbol{A = 1.4, B = -0.8, k = -0.7, c_1 = 0.7, D = 0.5}$ and $\boldsymbol{E = -3, a = 4, b = 5}$. The time interval is $\boldsymbol{0: 6\pi/100: 10\pi}$.

The Figure 1 shows the unstable behaviour of the trajectories moving in the neighborhoods of the equilibrium points of the dynamical system (5). The instability into the system can be viewed with the help of the opposite directions of the direction lines along the trajectories shown in the figure. But when we introduce the forcing term into the dynamical system (6), this converts the system into stable one. This force term generates a stable point in the system around which we may get the periodic solutions of the equation (1). This region of the stable solution can be visible in the given second figure between the range value of $u \in (2.5, 6)$. The interesting fact in the system is the stable behaviour of the Schamel KdV equation after introducing the forcing term into its dynamical system. Figure 2 shows the quasi-periodic cyclic behaviour with respect to the exact equilibrium point $[30, 0]$ which is calculated by the evaluated equilibrium point from dynamical system (5). These figures represent the range of the parameter $A \in [1, 2.33]$ with respect to the four represented values of this parameter. The break point with respect to the range of the parameters is $A = 2.33$ at which the system becomes completely unstable. When we introduce the step-wise forcing term into the dynamical system then system does not show any representative behaviour of the system with respect to the same range of the parameter A. Therefore, we have perturbed the values of this parameter to get the desired behaviour of the system. The corresponding behaviour of the dynamical system is shown in Figure 3 for the range of the parameter $A \in [-2.7, 2.7]$. The values of the other parameters are $B = -0.8$; $k = -0.7$; $c_1 = 0.7$; $D = 0.5$ and $E = -3, a = 4, b = 5$. The conversion in the behaviour of the dynamical system before and after introducing the forcing term is visible in these two figures from unstable to stable. To explain more about the effect of the forcing term onto the dynamical system, we have utilized the Lyapunov exponent Figure 4. The corresponding chaos is visible in the first figure for the the dynamical system (5) and smooth behaviour is shown in second figure with respect to the dynamical system (6). The corresponding strong quasi-periodic and weak quasi-periodic behaviour of the systems are shown with the help of the time series plots in Figure 5.

3. Conclusion

This research highlights the relevance of bifurcation analysis in the process of comprehending the dynamics of nonlinear waves, notably in systems such as the Schamel Korteweg-de Vries (KdV) equation. We reveal critical insights into wave pattern transitions, soliton generation, and system instabilities by applying bifurcation theory to nonlinear differential equations such as the Schamel KdV equation. These discoveries have significant implications for engineering and the physical sciences. Further, incorporation of step-wise forcing factors, particularly for seismological models, helps the simulation of abrupt forces like fault slides, enhancing the accuracy of seismic wave models. This technique assists in better calibration with observed data, contributing to enhanced seismic hazard assessments and earthquake-resistant engineering solutions. Overall, this research adds to our understanding of complex wave dynamics and gives essential tools for solving issues in earthquake modeling and preparation. The Schamel equation characterizes ion-acoustic wave propagation influenced by trapped electrons. It models nonlinear interactions between these electrons and ion-acoustic waves in plasma, allowing researchers to study the formation of solitons under these conditions.

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