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Approximation of functions using the Haar basis with scale 2 and 3

Y. N. Voronova [†]
Department of Higher Mathematics
Bauman Moscow State Technical University
Moscow 105005
Russian Federation

G. Arora *
Department of Mathematics
Lovely Professional University
Phagwara 144411
Punjab
India

O. V. Kravchenko [§]
Federal Research Center
Computer Science and Control of Russian Academy of Sciences
Moscow 119333
Russian Federation

Abstract

The orthonormal basis of Haar wavelets with both scale 2 and 3 is a useful tool in various scientific and engineering problems. Simultaneously, applications often require solutions that involve functions with extrema. In this study, a comparative analysis of approximation by Haar wavelets with scale 2 and 3 of functions with one and multiple extrema is considered. The properties and approximation capabilities of Haar wavelets at different levels of decomposition are analyzed, using the example of approximation of one-dimensional functions. Examples are also provided for the approximation of specific functions to illustrate the effectiveness of the approach. Also, the numerical analysis of errors of the considered approximations is carried out in the tables. Finally, numerical calculations are presented which are supplemented by graphical results.

Subject Classification: *26A06, 33F05, 65D20.*

Keywords: *Haar wavelets, Haar basis with scale 2, Haar basis with scale 3, Functions approximation.*

[†] *E-mail:* **julija.n.voronova@bmstu.ru**
* *E-mail:* **geetadma@gmail.com (Corresponding Author)**
[§] *E-mail:* **okravchenko@frcccsc.ru**

1. Introduction

The basis of Haar wavelets consists of piecewise constant functions and is mathematically the simplest of all wavelet families. An important feature of Haar wavelets is that they can be analytical mapped and integrals of any order can be analytically represented from the basis functions [1].

Haar wavelets are effective in detecting singularities and discontinuities due to the discontinuous nature of the initial basis functions. Thus, the Haar basis has been adapted to analyze dynamical systems with unit and distributed parameters, and an operational integration matrix and its application mechanisms have been presented [2]. Also, the methodology has been extended to optimize a dynamical system under different constraints and different conditions of time-invariant optimization problems [3]. The application of the Haar basis for solving linear stiff problems is discussed in [4], where an algorithm is presented that allows for the integration of stiff equations with highly accurate results for any time interval. In some cases, the Haar wavelet collocation method is not inferior to the classical Runge-Kutta method of order four. The Haar wavelet basis collocation approach was developed, improved, generalized, and adapted to address the challenges of approximation and the solution of initial, boundary value, and initial boundary value problems, as outlined in [5], [6], [8], [9]. A review of works on the application of wavelets to solve a wide class of linear and nonlinear reaction-diffusion equations is presented in [7]. It is noted that the Haar wavelet basis offers the advantage of representing functions using sparse matrices and the possibility of implementing fast approximation algorithms. The original Haar basis assumes a dyadic order between basis elements, that is, the next elements in basis by index are obtained from the previous ones with a scaling factor of two. An interesting generalization of the Haar wavelet basis with a scaling factor of 3 is presented in [10]. Similarly, the Haar wavelet basis with a scaling factor of 3 allows one to numerically solve the Cauchy problem, as well as problems with boundary conditions. Furthermore, research shows that the non-convexity of Haar wavelets of scale 3 is not as prominent than that of Haar wavelets of scale 2. A comparative analysis of these two bases functions for solving initial and initial boundary value problems for second-order ordinary differential equations is presented in [11].

The Haar wavelet collocation method, which employs a finite number of Haar series basis functions to approximate an unknown quantity, represents a field of ongoing development with a range of applications in applied mathematics. In [12], the Haar wavelet collocation method is applied to numerically solve the stationary problem of propagation of a pore flow of creeping fluid in a porous medium. This approach enables the acquisition of a convergent approximation to the porous medium equation with sufficient efficiency and reliability. The system of coupled nonlinear ordinary differential equations modeling the magnetohydrodynamic flow of a micropolar nanofluid with a stagnant point under the action of viscous dissipation and heat generation was solved numerically using the shifted Chebyshev collocation method and the Haar wavelet collocation method in [13]. The problem of beam deformation and the theory of plate deflection, as well as the deflection of a cantilever beam under a concentrated load, was solved numerically using the Haar wavelet collocation approach in [14]. The approach makes it possible to obtain a numerical solution to a problem on a grid with a relatively small number of nodes, and then refine the solution, increasing the accuracy by increasing the level of resolution, that is, the number of functions in the series. This methodology automates the incorporation of Neumann boundary conditions.

In [15], the numerical solution of Lane-Emden type equations with different boundary conditions of the first, second, and third kind is discussed. The method is applied to the study of numerical solutions with heterogeneous boundary conditions, where the desired solution is approximated by the original Haar series with suitable derivatives or integrals. A novel numerical double integrals integration technique based on Legendre wavelets is proposed in [16]. In [17], the authors present a study on enhancing the precision and convergence order of the Haar wavelet collocation method. The principal advantages of this approach are its simplicity, stability, convergence, high-order accuracy, and efficiency under diverse boundary conditions. Finally, an improved approach for the numerical solution of elliptic differential equations using the Haar basis is given in [18].

In this case, a mixed derivative of the fourth order is approximated instead of an independent approximation of the derivatives with respect to x and y of order two. At the same time, the algorithm's efficiency increases. Furthermore, the approach can be generalized to various types of boundary conditions with minimal code changes. The results obtained in [19] using generalized Bernstein basis functions are promising. In the future, it seems promising to combine constructive properties of the Haar basis and generalized Bernstein polynomials.

In view of previous studies, the present work focuses on the comparative analysis of the approximation capabilities of Haar bases with scale 2 and 3. The graphical results are obtained using the MATLAB R2024 in-house code with Intel Core i9 notebook processor. To facilitate practical analysis, illustrative examples of approximation of functions with one and several extrema are also provided.

2. Haar wavelets of scales 2 and 3

The Haar wavelets with scale 2 and 3, and the requisite information from the theory, as set forth in [1], [2], [8-10], for the definition of functions and the construction of the basis is discussed in this section.

The scale 2 Haar wavelet basis functions are defined as follows

$$\psi_i^{(k,m)}(x) = \begin{cases} +1, & x \in [\xi_1, \xi_2], \\ -1, & x \in [\xi_2, \xi_3], \\ 0, & x \notin [\xi_1, \xi_3], \end{cases} \tag{1}$$

with

$$\xi_1 = \frac{k}{m}, \xi_2 = \frac{2k+1}{2m}, \xi_3 = \frac{k+1}{m}, m = 2^j, j = \overline{0, J}, k = \overline{0, m-1}.$$

The wavelet index $i \geq 2$ depends on current level of wavelet resolution and shift parameter $i = m + k + 1$. The explicit expression of the scaling function $\psi_1^0(x)$ is specified as

$$\psi_1^{(0,0)} = \begin{cases} 1, & x \in [0,1], \\ 0, & x \notin [0,1]. \end{cases} \tag{2}$$

The number of functions in wavelet family depends on maximal level of resolution equals $n = 2 \cdot 2^J = 2^{J+1}$. For the purpose of orthonormality the basis functions tailored by normalizing constants in the form of

$$\psi_i^j(x) \triangleq \sqrt{2^j} \cdot \psi_i^j(x). \tag{3}$$

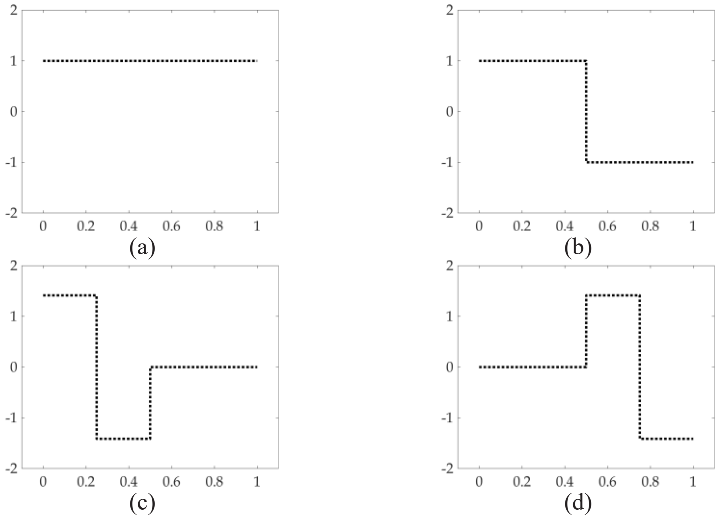


Figure 1
Haar scale 2 basis functions $\psi_i^{(k,m)}(x)$, $i = \overline{1,4}$; (a) scaling function $\psi_1^{(0,0)}(x)$, (b) wavelet $\psi_2^{(0,1)}(x)$, (c) wavelet $\psi_2^{(1,0)}(x)$, (d) wavelet $\psi_3^{(1,1)}(x)$

The graphs of the first four basis functions of the Haar series of scale 2, corresponding to decomposition level 1, are shown in Figure 1.

Similarly, according to [10] scale 3 Haar wavelet family is based on the same scaling function (2) and contains 2 types of wavelets so-called symmetric wavelet as follows

$$\psi_i^{(k,m)}(x) = \begin{cases} -1, & x \in [\xi_1, \xi_2], \\ 2, & x \in [\xi_2, \xi_3], \\ -1, & x \in [\xi_3, \xi_4], \\ 0, & x \in [\xi_1, \xi_4], \end{cases} \quad i = 2p,\ p > 0, \tag{4}$$

and asymmetric wavelet in the following form

$$\psi_i^{(k,m)}(x) = \begin{cases} 1, & x \in [\xi_1, \xi_2], \\ 0, & x \in [\xi_2, \xi_3], \\ -1, & x \in [\xi_3, \xi_4], \\ 0, & x \in [\xi_1, \xi_4], \end{cases} \quad i = 2p + 1,\ p > 0, \tag{5}$$

with

$$\xi_1 = \frac{k}{m}, \xi_2 = \frac{3k+1}{3m}, \xi_3 = \frac{3k+2}{3m}, \xi_4 = \frac{k+1}{m},$$

$m = 3^j, j = \overline{0, J}, k = \overline{0, m-1}$.

So, for each pair of $\boldsymbol{k}$ and $\boldsymbol{m}$ symmetric and asymmetric wavelets are defined, and the wavelet index $i \geq 2$ depends on current level of wavelet resolution and shift parameter as introduced in Table 1.

Table 1

Wavelet indexes

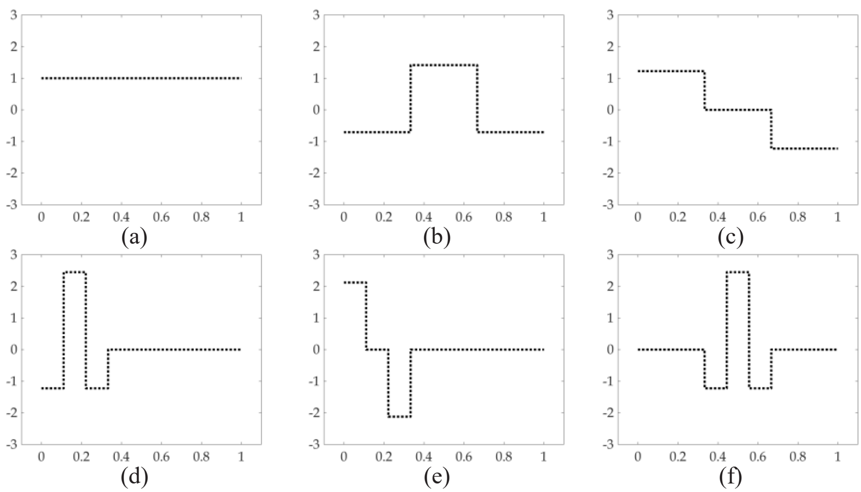
wavelet indexes	(k, m)
$i = 2p,\ p > 0$	$i = m + 2k + 1$
$i = 2p + 1,\ p > 0$	$i = m + 2k$

Also, to consider an orthonormal basis function one should adjust them by normalizing constants as follows

$$\psi_i^j(x) \triangleq \sqrt{\frac{3^j}{2}}\psi_i^j(x), \quad i = 2p,\ p > 0, \tag{6}$$

and

$$\psi_i^j(x) \triangleq \sqrt{\frac{3^{j+1}}{2}}\psi_i^j, \quad i = 2p + 1,\ p > 0. \tag{7}$$



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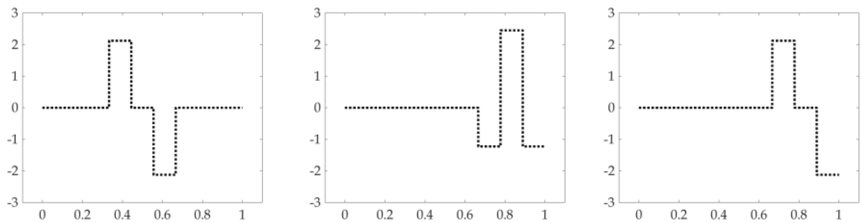

(g) (h) (i)

Figure 2

Haar scale 3 basis functions $\psi_i^{(k,m)}(x)$, $i = \overline{1,4}$; (a) scaling function $\psi_1^{(0,0)}(x)$; (b) wavelet $\psi_2^{(0,1)}(x)$; (c) wavelet $\psi_3^{(0,1)}(x)$ (d) wavelet $\psi_4^{(0,2)}(x)$; (e) wavelet $\psi_5^{(0,2)}(x)$; (f) wavelet $\psi_6^{(1,2)}(x)$; (g) wavelet $\psi_7^{(1,2)}(x)$; (h) wavelet $\psi_8^{(2,2)}(x)$; (i) wavelet $\psi_9^{(2,2)}(x)$

Similarly, the plots of the first four basis functions of the Haar series of scale 3, corresponding to decomposition level 1, are shown in Figure 2.

3. One-dimensional function approximation with Haar wavelets scale 2 and scale 3

Haar wavelets are useful for approximation of any function $f(x) \in L^2$. Consider $f(x)$ on interval $[0, 1]$ for simplicity and represent it with series based on Haar wavelets as follows

$$f(x) = c_1\psi_1(x) + \sum_{i=2}^{\infty} c_i\psi_i(x). \tag{8}$$

Here $c_1 = c_1^{(0,0)} = \int_0^1 f(x)dx$ and $c_i = c_i^{(k,m)} = \int_0^1 f(x)\psi_i(x)dx$ are Fourier-Haar coefficients.

To initialize on approximation, let's set the maximum level of wavelet resolution J and define $m = 2^j, j = \overline{0, J}, k = \overline{0, m-1}$. The partial sum of Haar wavelet scale 2 series approximates the original function as

$$f(x) \approx c_1\psi_1(x) + \sum_{i=2}^{2m} c_i\psi_i(x). \tag{9}$$

Repeating same steps for Haar wavelets scale 3 and defining

$$m = 3^j, j = \overline{0, J}, k = \overline{0, m-1}$$

gives another approximation of original function with scale 3 wavelets and according to Fourier-Haar coefficients, function results in the form

$$f(x) \approx c_1\psi_1(x) + \sum_{\substack{i=2\mathrm{k},\\ k>0}}^{3m} c_i\psi_i(x) + \sum_{\substack{i=2\mathrm{k}+1,\\ k>0}}^{3m} c_i\psi_i(x). \tag{10}$$

Let's choose specific functions such that it contains single or multiple extremes on a segment [0,1]. Hence,

$$f(x) = x(1 - x^{10}) \cdot 11\text{^}(11/10)/10,$$
$$g(x) = x(x-1)^4 \cdot 5\text{^}5/4\text{^}4,\ h(x) = \sin e^{3x}.$$

The primary difference between scale 2 and scale 3 is the number of basis functions at a given resolution level, either a power of two or three. We have considered maximum resolution levels $\boldsymbol{J = 2, 6, 8}$, for scale 2, and $\boldsymbol{J = 1, 3, 5}$, for scale 3, ensuring a comparable number of basis functions as introduced in Table 2.

Table 2

Number of basis functions for different level of resolution

resolution level	$\boldsymbol{J = 2}$	$\boldsymbol{J = 6}$	$\boldsymbol{J = 8}$
scale 2 basis (number of functions)	8	128	512
resolution level	$\boldsymbol{J = 1}$	$\boldsymbol{J = 3}$	$\boldsymbol{J = 5}$
scale 3 basis (number of functions)	9	81	729

Let us approximate chosen functions with Haar wavelets scale factor 2 and 3 for different level of resolution on interval [0, 1] with 1000 grid nodes and calculate the approximation error in L_1. Such a norm is considered since it has shown a similar trend to L_2 norm. Therefore, only L_1 results are presented to maintain brevity. Approximants are calculated using the collocation method as shown in Table 3.

Table 3

Approximation result for given functions

function	$f(x)$	$g(x)$	$h(x)$
scale 2 basis		L_1 norm	
$J = 2$	5.631e-01	5.388e-01	1.292e+00
$J = 6$	4.766e-02	4.192e-02	1.974e-01
$J = 8$	6.959e-03	6.085e-03	2.833e-02
scale 3 basis		L_1 norm	
$J = 1$	5.231e-01	4.971e-01	1.186e+00
$J = 3$	8.022e-02	7.090e-02	3.081e-01
$J = 5$	6.959e-03	6.085e-03	2.820e-02

4. Numerical experiment

Now let us evaluate and compare the Haar wavelet approximations of $f(x), g(x)$, and $h(x)$ using different parameter sets. Note that the coefficients of the basis decomposition can be obtained in different ways. Following [15], we will consider the collocation and Galerkin methods to compute the values of the coefficients of the Haar series, varying the number of nodes and observing how the approximation error changes. In addition to approximating the example functions on $[0,\ 1]$, these are also approximated on subintervals of monotonicity (different for each function).

The collocation method shows a clear dependency between calculation accuracy and the comparability of the number of nodes and basis functions. The most accurate approximation occurs when the number of nodes equals the number of basis functions (Table 4).

Table 4

Approximation for considered functions by collocation approach

function	$f(x)$	$g(x)$	$h(x)$
scale 2 basis	L_1 norm		
$J = 2$, 8 nodes	1.388e-16	1.110e-16	2.220e-16
$J = 6$, 128 nodes	5.551e-16	1.110e-15	9.992e-16
$J = 8$, 512 nodes	1.332e-15	2.220e-15	1.776e-15
scale 3 basis	L_1 norm		
$J = 1$, 9 nodes	2.220e-16	1.665e-16	7.772e-16
$J = 3$, 81 nodes	6.661e-16	8.882e-16	2.054e-15
$J = 5$, 729 nodes	4.059e-16	4.441e-16	1.221e-15

Once the number of nodes differs by a factor of two from the number of basis functions, accuracy drops dramatically to $10^{-1} - 10^{-2}$. Accuracy also degrades if the function has an extrema on the interval. Approximating the function on concatenated sub-intervals where the function is monotonic results in an error approximately 5 to 10 times smaller than approximating it over the entire interval (Table 5). Note that here and below the approximants are shown with color in different sections of monotonicity, and the black horizontal dashed lines show the approximant on the entire segment (Figure 3).

Table 5

Approximation for $f(x)$

function	segment		
$f(x)$	[0,1]	[0,0.8]	[0.8,1]
scale 2 basis	L_1 norm		
$J = 6$	5.308e-02	4.369e-03	1.086e-02
scale 3 basis	L_1 norm		
$J = 3$	8.250e-02	6.918e-03	1.710e-02

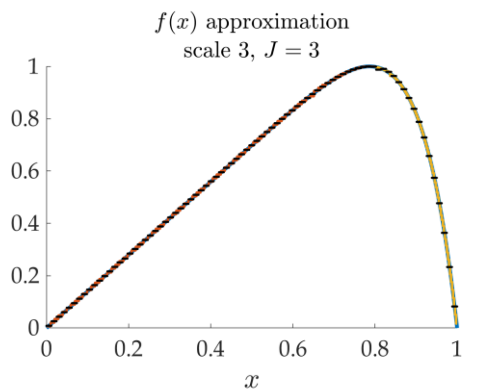

Figure 3

Approximation for $f(x)$

(Haar basis scale 3, $J = 3$, Galerkin)

Similar behavior is observed for the second function $g(x)$ (Figure 4), i.e., the accuracy of approximation running with the same parameters is better for subintervals of monotonicity and degrades on the entire interval (Table 6).

Table 6

Approximation results for $g(x)$

$g(x)$	[0,1]	[0,0.2]	[0.2,1]
scale 2 basis	L_1 norm		
$J = 6$	4.670e-02	9.497e-03	7.738e-03
scale 3 basis	L_1 norm		
$J = 3$	7.291e-02	1.294e-02	1.497e-02

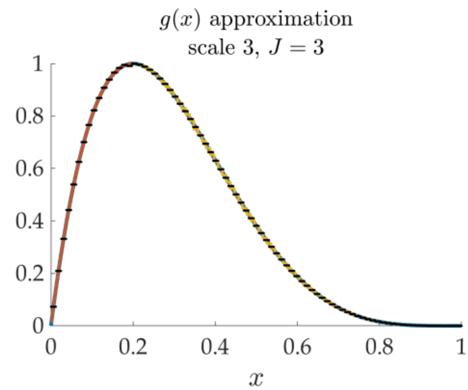

Figure 4
Approximation for $g(x)$
(Haar basis scale 3, $J = 3$, Galerkin)

The third example for $h(x)$ shows the most significant degradation of calculation accuracy across the entire interval (Figure 5). The function has seven extrema and the accuracy of calculation on each interval of increasing or decreasing is better than the overall accuracy (approximately ten times) (Table 7).

Table 7

Approximation results for $h(x)$

$h(x)$	[0,1]	$\left[0, \frac{\ln\frac{\pi}{2}}{3}\right]$	$\left[\frac{\ln\frac{\pi}{2}}{3}; \frac{\ln\frac{3\pi}{2}}{3}\right]$	$\left[\frac{\ln\frac{3\pi}{2}}{3}; \frac{\ln\frac{5\pi}{2}}{3}\right]$	$\left[\frac{\ln\frac{5\pi}{2}}{3}; \frac{\ln\frac{7\pi}{2}}{3}\right]$	$\left[\frac{\ln\frac{7\pi}{2}}{3}; \frac{\ln\frac{9\pi}{2}}{3}\right]$	$\left[\frac{\ln\frac{9\pi}{2}}{3}; \frac{\ln\frac{11\pi}{2}}{3}\right]$	$\left[\frac{\ln\frac{11\pi}{2}}{3}; 1\right]$
scale 2 basis	L_1 norm							
$J = 6$	2.124e-01	9.525e-04	1.301e-02	1.238e-02	1.220e-02	1.208e-02	1.200e-02	1.087e-02
scale 3 basis	L_1 norm							
$J = 3$	3.243e-01	7.675e-03	2.204e-02	1.977e-02	1.966e-02	1.955e-02	1.947e-02	1.744e-02

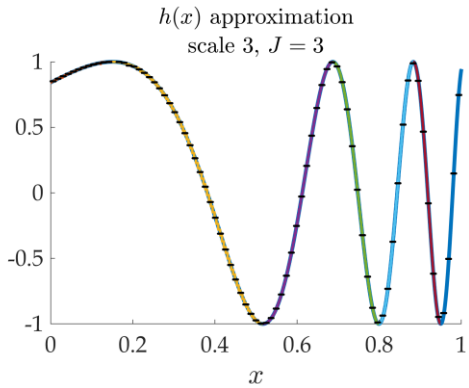

Figure 5
Approximation for $h(x)$
(Haar basis scale 3, $J = 3$, Galerkin)

Numerical experiments demonstrated that increasing the number of nodes and the wavelet level of resolution can improve accuracy. It is important that these values are equal or at least comparable. To overcome the impact of function extrema, future research could potentially yield better results by introducing non-uniform wavelets.

Conclusions

In the present study, approximation by Haar wavelets with scale 2 and 3 is considered. The properties and approximation capabilities of Haar wavelets are analyzed, and examples of approximation of functions with one and several extrema are considered. The extremum point divides the approximation segment into two, each with a different rate of change of the function. The analysis of the errors of the approximations under consideration is also carried out numerically. Furthermore, the numerical calculations are supplemented by the graphical results. The obtained quantitative result requires the need to study the approximation capabilities of non-uniform Haar wavelets in further works.

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