Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 29 (2026), No. 5-B, pp. 1713–1721
DOI : 10.47974/JIM-2284

Variational iteration method for solving two phases continuous casting moving boundary value problem

Radhi A. Zaboon *
Department of Mathematics
College of Science
Mustansiriyah University
Baghdad
Iraq

Abbas Al-Shimmary †
Department of Electrical Engineering
College of Engineering
Mustansiriyah University
Baghdad
Iraq

Abstract

This paper's primary goal is to find an approximate solution for the nonlinear continuous casting of liquids and solids in two phases. It is a type of moving boundary value problem, which is nonlinear, and the unknown moving interface position makes it hard to solve using analytical methods. The He's Variational Iteration Method (HVIM) procedure begins by initializing a solution with an initial guess in order to construct a correction functional based on optimal values of the Lagrange multiplier parameters. The approximate solution is then built up as a series of approximations, which quickly come together to the exact solution. As an operator equation, we have used the Stefan condition, along with the approximate solutions of the liquid and the solid phases, to build up a correction functional to the unknown moving boundary operator equation, which is obtained concurrently. It provides very accurate solutions to the unknown two-phase moving boundary values problem with an unknown moving interface, as indicated in the numerical values and comparisons (of the various initial guesses).

Subject Classification: *35R35, 35K20.*

Keywords: *Variational iteration method, Continuous casting, Stefan problem, Moving boundary problem.*

* *E-mail:* `r.zaboon@uomustansiriyah.edu.iq` (Corresponding Author)
† *E-mail:* `abbs.fadhil62@uomustansiriyah.edu.iq`

1. Introduction

Continuous casting is also required to make quality slabs, and one of the most popular studies in the past literature is the use of heat transfer models under controlled conditions [1-4]. Most methods used to solve this problem focus on simulating the growth of the solid metal phase inside the liquid molten casting mold. For instance, simplified formulations, including the finite element method to solve the thermal conductivity equation, have been explored in [5]. Continuous casting is intrinsically linked to moving-boundary problems, which are expressed as the nonlinear partial differential equation with initial, boundary, as well as Stefan equations, where the precise solution is hard to find. Various methods are suggested in order to overcome such issues, However, the techniques used to solve these equations include such as Domain decomposition method, Homogony perturbation method., Crank-Nicholson method, and semi-analytical methods, including the heat balance integral method and the variable space grid method [6-8].

This study focuses on the (HVIM), introduced by Chinese mathematician He and Latifizadeh [8] in 2020, as an improvement over the Lagrange multiplier technique for solving nonlinear partial differential equations (PDEs) and obtaining approximate solutions through iterative sequences [9]. HVIM formulates a correction function using a generalized Lagranges multiplier, that can be determine using variational theory [10, 11]. A methods is also apply to integral equations and non-linear ordinary differentials equation [12]. Additionally, the convergence of HVIM has been examined in [10].

The motivation for this work stems from the challenges associated with analytically solving such problems. To address this, the HVIM is employed to approximate solutions for the nonlinear PDEs governing the two-phase continuous casting process with an unknown moving boundary.

This paper has the following structure: Section 2 provides an explanation of the mathematical model. The variational iteration method's guiding principles are introduced in Section 3, and an example issue is used in Section 4 to illustrate the key findings. Section 5 concludes with a summary, conclusions, and suggestions for further study.

2. The Mathematical Model

Consider the phenomenon in which the liquid medium moves during the casting process in a certain direction along the t-axis, where the time dimension is denoted by the variable t. Thus, the fundamental equations governing the thermal distribution and diffusion described this phenomenon are defined by:

$$\frac{\partial \mathcal{T}_s(\varkappa,\mathrm{t})}{\partial t} = \alpha_s \left(\frac{\partial^2 \mathcal{T}_s(\varkappa,t)}{\partial \varkappa^2}\right) + \mathcal{F}^s(\varkappa,\mathrm{t}) \tag{1}$$

$$\frac{\partial \mathcal{T}_l(\varkappa,t)}{\partial t} = \alpha_l \left(\frac{\partial^2 \mathcal{T}_l(\varkappa,t)}{\partial \varkappa^2}\right) + \mathcal{F}^l(\varkappa,\mathrm{t}), 0 \leqslant \mathrm{t} \leqslant \mathrm{t}_f \ \& \ \mathbb{S}_2(t) \leqslant \varkappa \leqslant \mathbb{S}_1(\mathrm{t}). \tag{2}$$

With initial conditions.

$$T\_s\ (\varkappa,0) = T\_(s,0), -L \leqslant \varkappa \leqslant S\_2\ (t)\ , S\_1\ (t) \leqslant \varkappa \leqslant L\ .$$

$$T\_l\ (\varkappa,0) = T\_(l,0),\ 〖S〗\_2\ (t) \leqslant \varkappa \leqslant S\_1\ (t). \tag{3}$$

A boundary condition.

T_x^s (-L,t)=u_2 (t), T_x^s (L,t)=u_1 (t),0⩽t⩽t_f . (4)

T_s (s_2 (t) ,t)=T_l (s_2 (t) ,t)=T_m^2 , 0⩽t⩽t_f. (5)

T_s (s_1 (t) ,t)=T_l (s_1 (t) ,t)=T_m^1 , 0⩽t⩽t_f . (6)

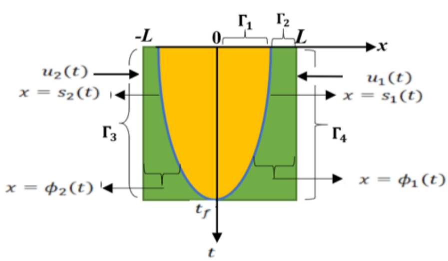
Additionally, the Stefan conditions at the moving interfaces are given by:

$$\left(K_s\frac{\partial \mathcal{T}_s}{\partial \varkappa}-K_l\frac{\partial \mathcal{T}_l}{\partial \varkappa}\right)_{\mathrm{x}=\mathbb{S}_1(t)}=\gamma_1\,\dot{\mathbb{S}}_1(t)\,,\left(K_l\frac{\partial \mathcal{T}_l}{\partial \varkappa}-K_s\frac{\partial \mathcal{T}_s}{\partial \varkappa}\right)_{\mathrm{x}=\mathbb{S}_2}=\gamma_2\,\dot{\mathbb{S}}_2(t)\,. \quad (7)$$

Where, $\mathcal{T}_s$ and $\mathcal{T}_l$ represent the temperature distributions of a solid phased (s) & liquid phased (l), respective, while $\mathcal{T}_{l,0}$ & $\mathcal{T}_{s,0}$ represent initials temperatures of these phases. The terms α_l and α_s indicate the thermal diffusion properties for liquid & solid, K_l & K_s indicate a thermal conductivities for liquid & solid phases, $\mathcal{T}_m^1$ and $\mathcal{T}_m^2$ stand for the melting temperatures. Furthermore, $u_2(t)$ and $u_1(t)$ describes the constant velocities of the cooling water placed on both sides of the casting device, $\mathbb{S}_2(t)$ and $\mathbb{S}_1(t)$ represents the right and left moving interfaces, The non-homogenous terms for solid & liquid phase represent as $\mathcal{F}^s(\mathrm{x},\mathrm{t})$ & $\mathcal{F}^l(\mathrm{x},\mathrm{t})$, $\mathrm{t_f}$ denoted final times of the process & finally, γ_1 and γ_2 are the latent heat of fusion. While, $\phi_2(t)$, $\phi_1(t)$ denotes a thickness for solidified layers given by

$$\phi_2(t)=\mathbb{S}_2(t)+\mathrm{L},\ \phi_1(t)=-\mathbb{S}_1(t)+\ \mathrm{L}. \quad (8)$$

where L refers to half slab thickness. As shown in Figure 1 below:



$$\begin{aligned}
\Gamma_1 &= \{(x,t): 0<x<s_1(z)\ ,0<\mathrm{t}<t_f\},\\
\Gamma_2 &= \{(x,t): s_1(z)<x<L\,,0<\mathrm{t}<t_f\},\\
\Gamma_3 &= \{(-L,t):\ \ 0<\mathrm{t}<t_f\},\\
\Gamma_4 &= \{(L,t):\ 0<\mathrm{t}<t_f\},\\
\Gamma_5 &= \{(s_1(t),t):\ 0<\mathrm{t}<t_f\},\\
\Gamma_6 &= \{(x,t_f):\ 0<x<L\ \},
\end{aligned}$$

Figure 1
Symmetric domain of the problem (1)-(7)

Remark 1: The continuous casting system features two controllers, $u_1(t)$ and $u_2(t)$, positioned on either side of the casting mold. At three separate positions, these controllers inject water into the mold to facilitate the heat dissipation of the molten alloy. For simplicity, the velocity of the water spray generated by the controllers is assumed to remain constant throughout the casting process.

3. He's Variational Iteration Method (HVIM)

This section aims to consider the fundamental concepts, basic requirements and methodological framework of the (HVIM), hence referred (VIM) for brevity.

3.1 *(V I M) for Liquid Phase in the t – Direction.*

This subsection applies an Variational Iterations Methods (VIM) to a liquid phase for equations (1) – (7) for more details one can follow [13]. Considering the temperature distribution along the time direction. Therefore, we start with the initial guesses for the temperature and the moving interface in the liquid phase are given by.

$$\mathcal{T}_0^1(\varkappa,t) \equiv \mathcal{T}_{l,0}(\varkappa,t_0)\,. \tag{9}$$

$$\mathcal{T}_{l,corr}(\varkappa,t_0) = \mathcal{T}_{l,0}(\varkappa,t_0) + \int_0^{t_0} \lambda_{1l}(\tau)\left[\left(\frac{\partial \mathcal{T}_{l,0}(\varkappa,\tau)}{\partial \tau} - \alpha_l \frac{\partial^2 \tilde{\mathcal{T}}_0^l(\varkappa,\tau)}{\partial \varkappa^2}\right) - \mathcal{F}^l(\varkappa,\tau)\right] d\tau. \tag{10.a}$$

The following is an expression for the liquid phase modified iteration method.

$$\mathcal{T}_{l,k+1}(\varkappa,t_0) = \mathcal{T}_{l,k}(\varkappa,t_0) + \int_0^{t_0} \lambda_{1l}(\tau)\left[\left(\frac{\partial \mathcal{T}_{l,k}(\varkappa,\tau)}{\partial \tau} - \alpha_l \frac{\partial^2 \tilde{\mathcal{T}}_k^l(\varkappa,\tau)}{\partial \varkappa^2}\right) - \mathcal{F}^l(\varkappa,\tau)\right] d\tau. \tag{10.b}$$

Here the subscript k refers to the k^{th} approximation and $\frac{\partial^2 \mathcal{T}_k^l(\varkappa,\tau)}{\partial \varkappa^2}$ is assumed to be restrict variations (i.e., $\frac{\delta \partial^2 \tilde{\mathcal{T}}_k^l(\varkappa,t)}{\partial \varkappa^2} = 0$). Because t_0 be arbitraries, t_0 is replace by t. The term $\lambda_{1l}(\tau)$ represents the general Lagrange multiplier associated with the liquid phase along the t-direction.

The first variation δ is taken with respect to the independent variable $\mathcal{T}_k^l$ for Eq. (10.b) and assuming that $\delta \frac{\partial^2 \tilde{\mathcal{T}}_k^l(\varkappa,t)}{\partial \varkappa^2} = 0$, with using the integration method by parts, the equation (10.b) becomes.

$$\delta \mathcal{T}_{l,k+1}(\varkappa,t) = \big(1+\lambda_{1l}(\tau)\big)\delta \mathcal{T}_{l,k}(\varkappa,t) - \delta \int_0^t \acute{\lambda}_{1l}(\tau)\mathcal{T}_k^l(\varkappa,t)d\tau = 0. \tag{10.c}$$

Setting equation (10.c) to zero, yields the following stationary condition Eqs.

$$1+\lambda_{1l}(\tau)|_{\tau=t} = 0, \acute{\lambda}_{1l}(\tau)|_{\tau=t} = 0. \tag{11}$$

A Lagranges multiplier be readily determined by $\lambda_{1l}(\tau) = -1$ from equation (11). Substituting this value into the correction equation (10.c) results in the recurrence relations:

$$\mathcal{T}_{l,k+1}(\varkappa,t) = \mathcal{T}_{l,k}(\varkappa,t) - \int_0^{t_f}\left[\left(\frac{\partial \mathcal{T}_k^l(\varkappa,\tau)}{\partial \tau} - \alpha_l \frac{\partial^2 \mathcal{T}_k^l(\varkappa,\tau)}{\partial \varkappa^2}\right) - \mathcal{F}^l(\varkappa,\tau)\right] d\tau. \tag{12}$$

So, the approximate solution of liquid phase $\mathcal{T}_{l,k}(\varkappa,t) \rightarrow \mathcal{T}_{l,exact}(\varkappa,t)$ be obtain $as\ k \rightarrow \infty$, for more details, one can see [9] and [10].

3.2. *(V I M) for solid phase in the t -direction.*

This subsection applies the same (VIM) to the solid phase for equations (1)– (7), where the temperature distribution is considered along the z-direction. The process begins with initial guesses for both the temperature and the moving interface in the solid phase, as specified by.

$\mathcal{T}_{s,0}(\varkappa,t) \equiv \mathcal{T}_s(\varkappa,t_0)$, $\mathbb{S}_{1,0}(t) \equiv \mathbb{S}_1(t_0)\ 0 \leqslant t \leqslant t_f$, $\mathbb{S}_1(t) \leqslant \varkappa \leqslant L$. (13)

Similarly, the approach used for the liquid phase in equations (9) -(12), the recurrence equation for the solid phase becomes:

$$\mathcal{T}_s(\varkappa,t) = \mathcal{T}_{s,k}(\varkappa,t) - \int_0^{t_f}\left[\left(\frac{\partial \mathcal{T}_k^S(\varkappa,\tau)}{\partial \tau} - \alpha_s \frac{\partial^2 \mathcal{T}_k^S(\varkappa,\tau)}{\partial \varkappa^2}\right) - \mathcal{F}^s(\varkappa,\tau)\right] d\tau\ . \qquad (14)$$

And, the approximate solution $\mathcal{T}_{s,k}(\varkappa,t) \rightarrow\ \mathcal{T}_{s,e\varkappa act}(\varkappa,t)\ as\ k \rightarrow \infty$ is obtained.

The moving interface $\mathbb{S}_1(t)$ is finally, to approximated via the Stefan equation.

$$\dot{\mathbb{S}}_1(t) = \alpha_s \frac{\partial \mathcal{T}^s(\mathbb{S}_1(t),t)}{\partial \varkappa} - \alpha_l \frac{\partial \mathcal{T}^l(\mathbb{S}_1(t),t)}{\partial \varkappa}. \qquad (15)$$

the approximate solutions $\mathcal{T}_{l,k+1}(\varkappa,t), \mathcal{T}_{s,k+1}(\varkappa,t)$ that be obtain earlier froms Eq. (12) &(14) substitute into (15) to obtain an expression for $\mathbb{S}_1(t)$, starting with $\mathbb{S}_1(0) = \mathbb{S}_{1,0}(t) = 1.5$, then for the next iterations use the recurrence formula.

$$\mathbb{S}_{1,k+1}(t) = \mathbb{S}_{1,k}(t) + \int_0^{\tau} \alpha_s \frac{\partial \mathcal{T}^s(\xi_{1,k}(\tau),\tau)}{\partial \varkappa} - \alpha_l \frac{\partial \mathcal{T}^l(\xi_{1,k}(\tau),\tau)}{\partial \varkappa}\, d\,\tau. \qquad (16)$$

To enhance the results of $\mathbb{S}_{1,k}(t)$, substitute the previous iteration at each step to achieve a more accurate approximation, and continue this process.

3. Illustration Problem

In this study, the analysis was performed within the subdomain [0, L] due to symmetry about the t-axis, focusing on the half-slab thickness (L), as illustrated in Figure 2 below. The physical parameters and assumptions used in the analysis were outlined in Table 1, providing the foundation for the modeling approach.

Table 1

The physical parameters of the illustration problem.

Nomenclature			
$\boldsymbol{L = 3}$	Length of the domain.	$\gamma = 0.8$	Latent heat of fusion
$\boldsymbol{t_f = 5}$	End time of process.	$\mathcal{F}^l(\varkappa,t) = 0$	Nonhomogeneous part of $\mathcal{T}^l$
$\boldsymbol{T_m}$ =1	Phase change temperature	$\mathcal{F}^s(\varkappa,t) = 0$	Nonhomogeneous part of $\mathcal{T}^s$
$\boldsymbol{K_l = 6}$	Thermal conductivity of phase I	$\varphi_1(\varkappa,0) = e^{(3-2\varkappa)/10}$	initial condition of $\mathcal{T}^l$
$\boldsymbol{K_s = 2}$	Thermal conductivity of phase II	$\varphi_2(\varkappa,0) = e^{(3-2\varkappa)/5}$	initial condition of $\mathcal{T}^s$
$\boldsymbol{\alpha_l}$ =2.5	Thermal diffusivity of phase I	$\psi_1(0,t) = e^{(3+t)/10}$	liquid temp. at $\varkappa = 0$
$\boldsymbol{\alpha_s = 1.25}$	Thermal diffusivity of phase I	$\psi_2(3,t) = e^{(t-3)/5}$	solid temp. at $\varkappa = 3$

Therefore, the mathematical model (1) - (7) becomes as follows:

$$\frac{\partial \mathcal{T}_l(\varkappa,t)}{\partial t} = \alpha_l \frac{\partial^2 \mathcal{T}_l(\varkappa,t)}{\partial \varkappa^2}, 0 \leqslant \mathrm{t} \leqslant \mathrm{t}_f, 0 \leqslant \varkappa \leqslant \mathbb{S}_1(t) . \tag{17}$$

$$\frac{\partial \mathcal{T}_S(\varkappa,t)}{\partial t} = \alpha_s \frac{\partial^2 \mathcal{T}_S(\varkappa,t)}{\partial \varkappa^2}, 0 \leqslant \mathrm{t} \leqslant \mathrm{t}_f, \mathbb{S}_1(t) \leqslant \varkappa \leqslant L . \tag{18}$$

Subject to initials & boundary condition.

$$\mathcal{T}_s(\varkappa,0) = \mathcal{T}_{s,0} , \mathcal{T}_l(\varkappa,0) = \mathcal{T}_{l,0} , , 0 \leqslant \varkappa \leqslant L . \tag{19}$$

$$\mathcal{T}_\varkappa^s(L,t) = u_1(t), \tag{20}$$

$$\mathcal{T}_s(\mathbb{S}_1(t), t) = \mathcal{T}_l(\mathbb{S}_1(t), t) = \mathcal{T}_{1,m}. \tag{21}$$

Additionally, Stefan conditions at move interfaces $\mathbb{S}_1(t)$ is:

$$\left(K_s \frac{\partial \mathcal{T}^s}{\partial \varkappa} - K_l \frac{\partial \mathcal{T}^l}{\partial \varkappa}\right)_{\varkappa=\mathbb{S}_1(t)} = \gamma_1 \dot{\mathbb{S}}_1(t). \tag{22}$$

The problem domain is partitioned into two subdomains, denoted as D_l and D_s, representing the domain of liquid and solid phases, respectively. The domain boundaries were previously illustrated in Figure 2.

3.1 *The case in which the temperature distribution of the liquid and solid phases in time- direction using (HVIM)*

In this subsection, we will analyze the problem defined by equations (17)–(22) using (VIM). For the first iteration, we use the initial conditions as the initial guesses:

$$\mathcal{T}_{l,0}(\varkappa,t) = e^{0.3-0.2\varkappa}, \mathcal{T}_{s,0}(\varkappa,t) = e^{0.6-0.4\varkappa}, \mathbb{S}_1^0 = 1.5 . \tag{23}$$

Applying the correctional functional, with $\alpha_l = 2.5$ and $\alpha_s = 1.25$, and the Lagrange multiplier $\lambda_{1l}(\tau) = \lambda_{1s}(\tau) = -1$ as found in (11).

$$\mathcal{T}_{l,1}(\varkappa,t) = \mathcal{T}_{l,0}(\varkappa,t) - \int_0^{t_f} \left[\left(\frac{\partial \mathcal{T}_0^l(\varkappa,\tau)}{\partial \tau} - (2.5)\frac{\partial^2 \mathcal{T}_0^l(\varkappa,\tau)}{\partial \varkappa^2}\right)\right] d\tau, \tag{24}$$

$$\mathcal{T}_{s,1}(\varkappa,t) = \mathcal{T}_{s,0}(\varkappa,t) - \int_0^{t_f} \left[\left(\frac{\partial \mathcal{T}_0^S(\varkappa,\tau)}{\partial \tau} - (1.25)\frac{\partial^2 \mathcal{T}_0^S(\varkappa,\tau)}{\partial \varkappa^2}\right)\right] d\tau, \tag{25}$$

Therefore, the first iteration for the approximate $\mathcal{T}_1^l(\varkappa,t)$ and $\mathcal{T}_1^s(\varkappa,t)$ given by

$$\mathcal{T}_{l,1}(\varkappa,t) = e^{0.3-0.2\varkappa} - \int_0^{t_f}(-0.1)(e^{0.3-0.2\varkappa})d\tau = e^{0.3-0.2\varkappa}(1+0.1\,t). \tag{26.a}$$

$$\mathcal{T}_{s,1}(\varkappa,t) = e^{0.6-0.4\varkappa} - \int_0^{t_f}(-0.2)(e^{0.6-0.4\varkappa})d\tau = e^{0.6-0.4\varkappa}(1+0.2\,t). \tag{26.b}$$

It is clear that, the two expressions must be consistent for the liquid and solid phase at $\mathbb{S}_1^1(t)$ by equate them $\mathcal{T}_{l,1}(\mathbb{S}_1^1(t),t) = \mathcal{T}_{s,1}(\mathbb{S}_1^1(t),t) = 1$, hence;

$$\mathbb{S}_{1,1}(t) = \frac{0.3+ln(1+0.1\,t)}{0.2}. \tag{27}$$

Applying the same procedure and the fourth iteration can be found as;

$$\mathcal{T}_{l,4}(\varkappa,t) = e^{0.3-0.2\varkappa}(1+0.1\,t+0.005\,t^2+0.00167t^3+.0002087t^4). \tag{28.a}$$

$$\mathcal{T}_{s,4}(\varkappa,t) = e^{0.6-0.4\varkappa}(1+0.2\,t+0.02\,t^2+0.00133t^3+0.0000665t^4). \tag{28.b}$$

$$\mathbb{S}_{1,4}(t) \approx \frac{0.3+\ln(1+0.1\,t+0.005\,t^2+0.00167t^3+0.00020875t^4)}{0.2}. \tag{28.c}$$

And so on for the $(k+1)^{th}$ iteration can be derived from the formulas.

$$\mathcal{T}_{l,k+1}(\varkappa,t) = \mathcal{T}_{l,k}(\varkappa,t) - \int_0^{t_f}\left[\left(\frac{\partial \mathcal{T}_k^l(\varkappa,\tau)}{\partial \tau} - (2.5)\frac{\partial^2 \mathcal{T}_k^l(\varkappa,\tau)}{\partial \varkappa^2}\right)\right] d\tau. \tag{29.a}$$

$$\mathcal{T}_{s,k+1}(\varkappa,t) = \mathcal{T}_{s,k}(\varkappa,t) - \int_0^{t_f}\left[\left(\frac{\partial \mathcal{T}_k^s(\varkappa,\tau)}{\partial \tau} - (1.25)\frac{\partial^2 \mathcal{T}_k^s(\varkappa,\tau)}{\partial \varkappa^2}\right)\right] d\tau. \tag{29.b}$$

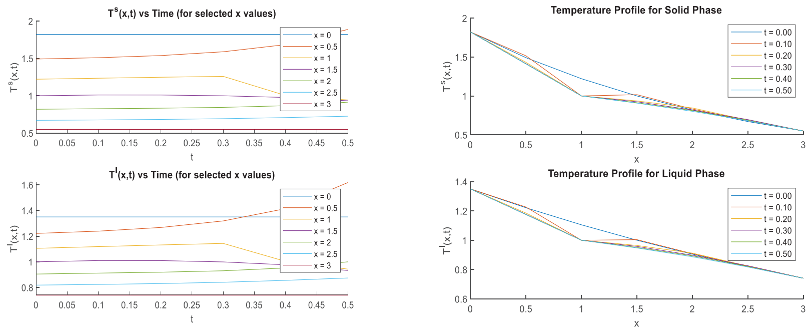


Figure 2
The evolution of the solid & liquid phases at different times t

The following Figure 3 is the comparison between the approximate solutions for $\mathcal{T}_{VIM}^{l,4}(\varkappa,t), \mathcal{T}_{VIM}^{s,4}(\varkappa,t)$ with the exact solutions, for the values $t = 0, 0.1, 0.2$.

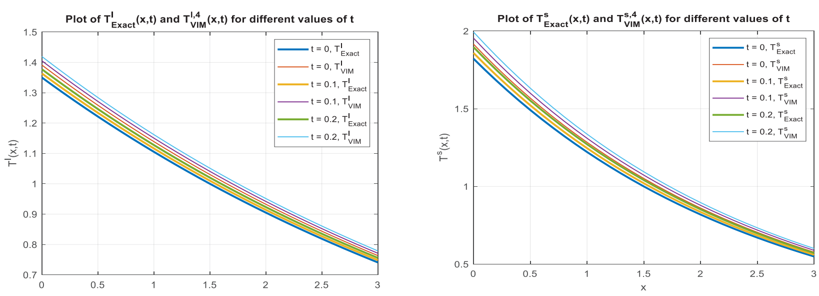


Figure 3
Comparison between the approximate with the exact solutions for
$\mathcal{T}_{VIM}^{l,4}(\varkappa,t), \mathcal{T}_{VIM}^{s,4}(\varkappa,t)$

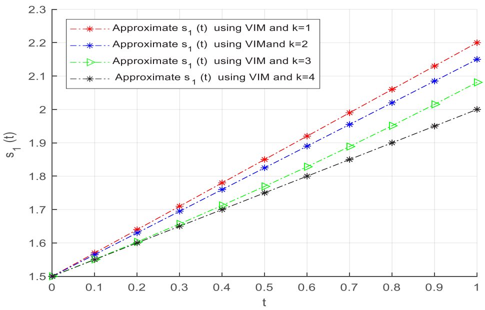


Figure 4

The moving interface $\mathbb{S}_1^k(t)$ of different values for iteration k

Remark 2: As one can see, even for a small number of iterations $k = 1,2,3,4$. The obtained result is very accurate compared with the exact solutions. For a more accurate solution, one can increase the iteration k as shown in Figure 4.

4. Conclusions and Future Works

In this research, the semi-analytic variational iteration approach was used to solve the continuous casting process as an application of the MBVP. The temperature distributions in the liquid and solid phases for various initial guess solutions were examined, and it was shown that the variational iteration approach produces results that nearly resemble the precise solution. The convergence of the method was shown to depend on the selection of an appropriate initial guess. Additionally, a comparison table and figure were provided to illustrate the closeness of the results to the exact solution, further validating the effectiveness of the proposed approach.

References

[1] Z. Chen, J. Bentsman, and B. G. Thomas, “Optimal control of free boundary of a stefan problem for metallurgical length maintenance in continuous steel casting,” in *2019 American Control Conference (ACC)*, *IEEE*, pp. 3206-3211 (2019).

[2] Z. Chen, J. Bentsman, B. G. Thomas, and A. Matsui, “Study of spray cooling control to maintain metallurgical length during speed drop in steel continuous casting,” *Iron Steel Technol*, vol. 14, no. 10, pp. 92-103 (2017).

[3] R. Grzymkowski, M. Pleszczyński, and E. Hetmaniok, “Problem of the moving boundary in continuous casting solved by the analytic-numerical method,” *Archives of Foundry Engineering*, vol. 13, no. 1, pp. 33-38 (2013).

[4] G. Abd and R. Ali, "Parametrization approach for solving index-4 linear differential-algebraic control systems," *International Journal of Mathematics and Computer Science*, vol. 17, no. 2, pp. 815-825 (2022).

[5] A. Hemeda, "Variational iteration method for solving non-linear partial differential equations," *Chaos, Solitons & Fractals*, vol. 39, no. 3, pp. 1297-1303 (2009).

[6] G. F. Abd and R. A. Zaboon, "Parameterization technique for solving variational formulation of index three linear time-varying singular control systems," *Journal of Interdisciplinary Mathematics*, vol. 24, no. 7, pp. 1935–1945 (2021), doi: 10.1080/09720502.2021.1966949.

[7] H. Ahmad, T. A. Khan, P. S. Stanimirović, Y.-M. Chu, and I. Ahmad, "Modified Variational Iteration Algorithm-II: Convergence and Applications to Diffusion Models," *Complexity*, vol. 2020, no. 1, p. 8841718 (2020).

[8] J.-H. He and H. Latifizadeh, "A general numerical algorithm for nonlinear differential equations by the variational iteration method," *International Journal of Numerical Methods for Heat & Fluid Flow*, vol. 30, no. 11, pp. 4797-4810 (2020).

[9] X. Wang, Q. Xu, and S. N. Atluri, "Combination of the variational iteration method and numerical algorithms for nonlinear problems," *Applied Mathematical Modelling*, vol. 79, pp. 243-259 (2020).

[10] Z. M. Odibat, "A study on the convergence of variational iteration method," *Mathematical and Computer Modelling,* vol. 51, no. 9-10, pp. 1181-1192 (2010).

[11] D. Słota, "Homotopy perturbation method for solving the two-phase inverse Stefan problem," *Numerical Heat Transfer, Part A: Applications*, vol. 59, no. 10, pp. 755-768 (2011).

[12] L. Sowa and A. Bokota, "Numerical model of thermal and flow phenomena in the process of growing the CC slab," *Archives of metallurgy and materials*, vol. 56, pp. 359-366 (2011).

[13] J. Singh, K. P. Gupta, and N. K. Rai, "Variational iteration method to solve moving boundary problem with temperature dependent physical properties," *Thermal Science*, vol. 15, no. suppl. 2, pp. 229-239 (2011).

Received May, 2025