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The linear dynamical systems with the fitting shadowing property

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Abstract

In dynamical systems theory, the concept of shadowing refers to the ability to approximate a pseudo-orbit (PO) by an actual orbit of the system. The accuracy of this approximation depends on the specific type of shadowing property being considered. We want to show the relationship between the hyperbolicity and the fitting shadowing property in the linear dynamical systems (DS) on $\mathbb{C}^n$(vector space).

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1. Introduction

In dynamically system theory, shadowing notion is fundamentally concerned with whether an approximate trajectory, known as a pseudo - orbits, is closely followed is exact orbits of system. In words, it examines the extent to which the long-term behavior suggested by approximate solutions reflects the true dynamics. Clearly, the resolution of this question depends on the approximation adopted. Among the various frameworks, the shadowing property

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stands out as one of the most robust and influential structural results. Notably, this property provides a powerful tool for establishing the structural stability of hyperbolic sets, thereby highlighting its central role in the qualitative analysis of dynamical systems.

The homeomorphism $\mathrm{P}: \mathrm{X} \to \mathrm{X}$ of metric spaces(X, d), with discrete time generated by P on X, assume that $\sigma > 0$, a sequence $\{x_n\}_{n\in\mathbb{Z}} \in \mathrm{X}$ be σ-pseudo orbits (PO) for the P where inequalities hold:

$$d(\mathrm{P}(x_n), x_{n+1}) < \sigma, n \in \mathbb{Z}$$

A dynamical systems P have shadowing properties (SH) provided that, for every tolerance $\rho > 0$ for which every σ - pseudo orbits $\{x_n\}_{n\in\mathbb{Z}} \in \mathrm{X}$ for the P be traceable by an actual orbit of the system. In other words, $\exists\omega \in \mathrm{X}$ s.t

$$d(\mathrm{P}^n(\omega), x_n) < \rho, n \in \mathbb{Z}$$

The knowledge of a pseudo-orbit by Birkhoff [1]. Substantial progress in shadowing theory emerged following the foundational contributions of Anosov [2] and Bonahon [3]. The main results obtained in the [4, 5, 6, 7]. By [3], each Axioms A diffeomorphism to non-wandering sets Λ have shadowing properties. A sequence $\{x_n\}_{n\in\mathbb{Z}} \in \mathrm{X}$ named σ-fitting pseudo orbits (FPO) for the P where there is positive integers $\mathrm{K} = \mathrm{K}(\sigma)$ s.t. for all $n \geq \mathrm{K}$ and $\mathrm{k} \in \mathbb{Z}$

$$\sum_{i=0}^{n-1} d(\mathrm{P}(x_{i+k}), x_{i+k+1}) < \sigma$$

The principal result of this paper deals with linear dynamical systems generated by nonsingular matrices. To establish this result, we begin by introducing the fundamental concepts in the general framework of discrete-time dynamical systems defined by homeomorphisms.

A dynamical system P named has fitting shadowing properties (FSP) where, every > 0 , $\exists\sigma > 0$ s.t each σ-fitting pseudo-orbits $\{x_n\}_{n\in\mathbb{Z}} \in \mathrm{X}$ be ρ-shadowed on fitting by some point $\omega \in \mathrm{X}$.

$$\lim_{n\to\infty} sup \sum_{i=0}^{n-1} d(\mathrm{P}^i(\omega), x_i) < \rho$$

A nonsingular matrix A is said to be hyperbolic when none of its eigenvalues lies on the unit circle. Easton introduced the following concept for a dynamical system ρ on a metric space (X, d): $\forall\rho > 0$, $\exists\sigma > 0$ s.t any bi-infinite sequences $\{x_n\}_{n\in\mathbb{Z}}$ satisfied the inequality

$$\sum_{n\in\mathbb{Z}} d(\mathrm{P}(x_n), x_{n+1}) < \sigma$$

Then there exists a point $\boldsymbol{\omega}$ s.t.

$$\sum_{n\in\mathbb{Z}} d(\mathrm{P}^n(\omega), x_n) < \rho$$

This is called strong shadowing property (SSP).

2. Main Results

Theorem 2.1: *Consider the linear discrete-time dynamical system $P: \mathbb{C}^n \to \mathbb{C}^n$, defined on the vector space $\mathbb{C}^n$ with a suitable norm $|.| = d(.,.)$. Suppose that P is generated by the mapping where Ais a nonsingular matrix. The system P* *possesses the fitting shadowing property (FSP) if and only if the matrix A is hyperbolic.*

To prove this theory, we need some facts and concepts.

Lemma 2.2: *Let $P, \mathcal{F}$ be two (DS) of a metric space X, there is a homeomorphism h on X s.t. h and h^{-1} are Lipschitz and $\mathcal{F}oh = hoP$. Then P has (FSP) if and only if $\mathcal{F}$ has (FSP).*

Proof: Assume P has (FSP), $\rho > 0$ arbitrary, and $\ell > 0$ a Lipschitz constant for h and h^{-1}.

Let $\{x_n\}_{n\in\mathbb{Z}} \in$ X be $\frac{\sigma}{\ell}$- (FPO) of P. Then for all k $\in \mathbb{Z}$

$$\sum_{i=0}^{n-1} d\,(\,\mathrm{P}(x_{i+k}), x_{i+k+1}) < \frac{\sigma}{\ell}$$

And

$$\lim_{n\to\infty} sup \sum_{i=0}^{n-1} d\,(\,\mathrm{P}^i(\omega), x_i) < \frac{\rho}{\ell}$$

Since $\mathcal{F}oh = ho$P and h is Lipschitz, so:

$$\sum_{i=0}^{n-1} d\,(\,\mathcal{F}(h(x_{i+k}), h(x_{i+k+1})) = \sum_{i=0}^{n-1} d\,(\,\mathrm{P}(h(x_{i+k}), h(x_{i+k+1}))$$
$$\leq \ell \sum_{i=0}^{n-1} d\,(\,\mathrm{P}(x_{i+k}), x_{i+k+1}) \,<\, \sigma$$

So $\{h(x_n)\}_{n\in\mathbb{Z}} \in \chi$ is σ-fitting pseudo-orbit of $\mathcal{F}$

We obtain:

$$\lim_{n\to\infty} sup \sum_{i=0}^{n-1} d\,(\,\mathcal{F}^i(h(\omega), h(x_i)) =$$
$$\lim_{n\to\infty} sup \sum_{i=0}^{n-1} d\left(h(\mathrm{P}^i(\omega)\right)(h(x_i)) \leq \lim_{n\to\infty} sup\, \ell \sum_{i=0}^{n-1} d\,(\,\mathrm{P}^i(\omega), x_i) <\, \rho$$

Now $\mathcal{F}$ has (FSP), since the point $h(\omega) \in$ X is the ρ-fitting shadowed point of $\mathcal{F}$.

Lemma 2.3 [8]: *Assume A non hyperbolic matrix ($|\lambda| = 1$). Then there exists a nonsingular matrix E s.t.* $\mathrm{J} = E^{-1}AE$ *is a Jorden form* $\begin{pmatrix} B & 0 \\ 0 & D \end{pmatrix}$*of A and* J, *where B is the nonsingular $\mathrm{n} \times \mathrm{n}$ complex matrix with the form*

$$\begin{pmatrix} \lambda & 0 & \dots & 0 & 0 \\ 1 & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & \lambda \end{pmatrix}$$

And D is the hyperbolic matrix.

Lemma 2.4: *Let $P: \mathbb{C}^n \to \mathbb{C}^n$ be as previously defined. If P has (SSP), then it has (FSP).*

Proof: Suppose that P has (SSP), to prove it has (FSP). For any $\sigma > 0$, if $\{x_n\}_{n\in\mathbb{Z}}$ the σ $-$strong pseudo-orbits for the P s.t.

$$\sum_{n\in\mathbb{Z}} d(\mathrm{P}(x_n), x_{n+1}) = \sum_{n\in\mathbb{Z}} |A(x_n) - x_{n+1}| < \sigma$$

Will be the σ −strong pseudo orbit $\{x_n\}_{n\in\mathbb{Z}}$ is a σ −FPO of P since

$$\sum_{i=0}^{n-1} |A(x_{i+k}) - x_{i+k+1}| < \sum_{n\in\mathbb{Z}} d(\mathrm{P}(x_n), x_{n+1})$$
$$= \sum_{n\in\mathbb{Z}} |A(x_n) - x_{n+1}| < \sigma$$

Because P has (SSP), there is a point $\omega \in \mathbb{C}^n$ s.t.

$$\sum_{n\in\mathbb{Z}} d(\mathrm{P}^n(\omega), x_n) < \rho$$

Then

$$\lim_{n\to\infty} sup \sum_{i=0}^{n-1} d\,(\,\mathrm{P}^i(\omega), x_i) < \sum_{n\in\mathbb{Z}} d(\mathrm{P}^n(\omega), x_n) < \rho$$

This means ω is a ρ −shadowed point in $\mathbb{C}^n$. By [9] if A is hyperbolic then P has (SSP).

Lemma 2.5: *Given as previously specified, the hyperbolicity of the nonsingular matrix A guarantees that P exhibits the fitting shadowing property (FSP).*

Proof: To prove P has (FSP), by [9] and Lemma (2.4), P has (FSP).

Proof Theorem (2.1): Suppose that P has the (FSP), to prove by contradiction. Assume A is nonhyperbolic, ($|\lambda| = 1$)

By Lemma (2.3), there is a nonsingular matrix E s.t. $\mathrm{J} = \mathrm{E}^{-1}A\mathrm{E}$ is a Jorden form of A and the Jorden form $\mathrm{J}=\begin{pmatrix} B & 0 \\ 0 & D \end{pmatrix}$, where B, D are previously defined.

Assume $\mathcal{F}(x) = \mathrm{J}(x) = \mathrm{E}^{-1}A\mathrm{E}(x)$, and $h(x) = E(x)$ for $x \in \mathbb{C}^n$. So, $\mathcal{F}oh = ho\mathrm{P}$ from the fitting shadowing property (FSP) of P and $\sigma > 0$ of the fitting shadowing property (FSP) of $\mathcal{F}$.

Denoted by $x^{(i)}$the i −th component of a vector $x \in \mathbb{C}^n$. We construct a σ −pseudo orbit (PO) as follows:

$$x_{i+k+1}^{(1)} = \lambda x_{i+k}^{(1)} \left(1 + \frac{\sigma}{2\left|x_{i+k}^{(1)}\right|}\right),$$

And

$$x'_{i+k+1} = \left(x_{i+k+1}^{(2)}, x_{i+k+1}^{(3)}, \dots, x_{i+k+1}^{(n)}\right)$$
$$= (\mathrm{J}x_{i+k}^{(2)}, \mathrm{J}x_{i+k}^{(3)}, \dots, \mathrm{J}x_{i+k}^{(n)}) \text{ for all } i, k \in \mathbb{Z}$$

Since

$$\mathcal{F}(x_{i+k}) = \mathrm{J}x_{i+k} = (\lambda x_{i+k}^{(1)}, (\mathrm{J}x_{i+k})^{(2)}, (\mathrm{J}x_{i+k})^{(3)}, \dots, (\mathrm{J}x_{i+k})^{(n)}$$
$$= (\lambda x_{i+k}^{(1)}, x'_{i+k+1})$$

We know that if $\lambda = 1$, then

$$d(\mathcal{F}(x_{i+k}), x_{i+k+1}) = \left| x_{i+k}^{(1)} - x_{i+k}^{(1)} - \frac{x_{i+k}^{(1)}\sigma}{2\left|x_{i+k}^{(1)}\right|} \right| = \frac{\sigma}{2} < \sigma, \forall i, k \in \mathbb{Z}$$

Thus $\{x_n\}_{n\in\mathbb{Z}} \in \mathrm{X}$ named $\sigma-$pseudo-orbits (PO) for the $\mathcal{F}$, and this sequence is a $\sigma-$fitting pseudo-orbit (FPO) of $\mathcal{F}$. Since $\mathcal{F}$ has (FSP), ∃ a point $\omega \in \mathbb{C}^n$s.t.

$$\lim_{n\to\infty} sup \sum_{i=0}^{n-1} d\,(\,\mathcal{F}^i(\omega), x_i) < \rho$$

Now, we can take two possibilities:

If $\omega = (0,0,..,0)$, then $d(\,\mathcal{F}^i(\omega), x_i) = |x_i| > 0$. This possibility leads us to:

$$\lim_{n\to\infty} sup \sum_{i=0}^{n-1} d\,(\,\mathcal{F}^i(\omega), x_i) > \left|x_j\right|$$

Where $\left|x_j\right| = \min\{|x_i|:\text{for all } i \in \mathbb{Z}\}$. This is contradiction.

Let $\omega = (\omega^{(1)}, \omega^{(2)}, \omega^{(3)}, \dots, \omega^{(n)})$ then:

$$\mathcal{F}^i(\omega) = (\omega^{(1)}, (\mathrm{J}^i\omega)^{(2)}, (\mathrm{J}^i\omega)^{(3)}, \dots, (\mathrm{J}^i\omega)^{(n)})$$

And

$$d(\,\mathcal{F}^i(\omega), x_i) \geq \left|\left|\mathcal{F}^i(\omega)\right| - |x_i|\right|$$

Using this formula:

$$\lim_{n\to\infty} sup \sum_{i=0}^{n-1} d\,(\,\mathcal{F}^i(\omega), x_i) \geq \left||\mathcal{F}^k(\omega)| - |x_k|\right|$$

Where $\left||\mathcal{F}^k(\omega)| - |x_k|\right| = \min\{\left|\left|\mathcal{F}^i(\omega)\right| - |x_i|\right|$ for all $i \in \mathbb{Z}\}$. Here, we take $\rho_2 = \frac{min\{\left||\mathcal{F}^i(\omega)|-|x_i|\right|\}}{2}$, this is contradiction. Assume that $\rho = \min\{\rho_1, \rho_2\}$, where $\rho_1 = \frac{\min\{|x_i|\}}{2}$

We cannot get ω that is $\rho-$fitting shadowed. So, if P has (FSP), then A is hyperbolic. Now we want to prove the opposite direction. Suppose that the matrix A is hyperbolic.

By Lemma (2.5), P has the fitting shadowing property (FSP).

3. Conclusion

In general, hyperbolic dynamical systems do not satisfied fitting shadowing properties (FSP), except for Anosov systems in the context of linear dynamical models. The paper introduces several related concepts, which may be further explored and re-evaluated on other important classes of spaces.

4. Future Work

Future investigations may also be carried out on hyperbolic dynamically system use notion for the strongly fitting shadowing properties. Such studies could highlight the differences between the two approaches and assess their influence on identifying dynamical characteristics that might be applied to the solution of various mathematical problems.

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