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The integration on the symmetric t-approach : A Banach algebra

Maysoon A. Neamah
Department of Mathematics
Faculty of Open Education
General Directorate of Education Najaf
Najaf
Iraq

Abstract

This study aims to establish novel analytical findings concerning a class of Δ-symmetric Banach algebra structures. In this context, the space of continuous functions is rigorously demonstrated to possess both the Δ-symmetric property and the Banach algebra framework. Furthermore, the measurable function space is examined, where its compatibility with the Δ-symmetric structure and Banach algebra operations is analyzed and confirmed under appropriate conditions. In addition, a new functional space is constructed based on the interaction between the underlying Δ-symmetry and the algebraic structure of the considered spaces. The proposed space is shown to satisfy the defining requirements of a Δ-symmetric Banach algebra. Several original properties and auxiliary results are derived, providing further insight into the behavior of these spaces and expanding the theoretical understanding of Δ-symmetric Banach algebra systems.

Subject Classification: *46BXX, 46EXX.*

Keywords: *t^{ω}-approach space, t^{ω}-contraction, t^{ω}-approach group, t^{ω}-Δ-approach normed, t^{ω}-Δ-symmetric Banach algebra.*

1. Introduction

The notion of Banach algebra form is an essential part of functional analysis. Many concepts existing in classical analysis have the basic structure of a Banach algebra. Function algebra is an important part of this theory; moreover, it encompasses a wide range of concepts from analysis. The term of Banach algebra was defined by authors [8] in 2015. In 2008, The authors [4] introduced his book (ON ROOTS OF AN ELEMENT OF A BANACH ALGEBRA), in which he proved that any Banach algebra can be embedded isomorphically and isometrically in another Banach algebra, in such a way that any element in a Banach algebra is also an element in another Banach algebra. In 1974, the

E-mail: **maysoon.azeez@ec.edu.iq**

author [1] introduced the General Theory of Banach Algebras, the first book to provide a detailed account of the theory of Banach algebras. In 1945, a monograph on Banach algebras was published by Authors [7]. In 2022, The Authors [15] presented new results in the theory of real symmetric Banach algebras. In 1981, the Authors [13] and [12] developed spectral theory for elements in an ordered Kurepa [3] continued their studies of a spectral theory for positive elements in an ordered Banach algebra in 1961, so they proved a suitable version of the Kern-Rutman theorem, and got some results about the peripheral spectrum of positive elements. In 1973, $C(\chi)$ was studied as a Banach algebra by Authors [6], who showed that $C(\chi)$ was a universal commutative Banach algebra. In part Author [11] initiated the search with an algebraic cap A and endowed it with a topology to make the algebraic operations continuous; he also gave this topology a norm in 1996. In 1997, the author [2] found that the properties of the usual spectrum can be extended to the Ransford spectrum. In particular, he introduced the relationship between the spectrum and the multiplicative linear functional, and he defined the condition spectrum of an element in a complex Banach algebra. Also, he proved that the spectrum is a particular case of the Ransford spectrum. Two unital Banach algebras, cap A. and cap B, were studied by Authors [5] in 1998, such that B is semisimple. In 2023, some properties of the spectral radius in a complex ordered Banach algebra were established by authors [10, 14, 16] in his thesis. Moreover, in this study, some new definitions are introduced, so many spaces and an important space, which is called the state space of measurable functions, are defined on the space $L^0(\mu)$ have been proved to be symmetric t^ω - Δ- approach B-algebra. This work is divided into four sections: Section 1 shows the introduction of the research. In Section 2, preliminaries with basic definitions are introduced. In section 3, new results of symmetric t^ω - Δ- The approach to Banach algebras is studied, and some important propositions are proved. Section 4 discusses and explains the research's key conclusions.

2. Preliminaries

We start this work by defining an important notion, namely t^ω-approach distance.

Definition 2.1 [15]: Support that χ be a non-empty set. A collection $(t^\omega)_{\omega<\infty}$ of functions $t^\omega : 2^\chi \times 2^\chi \to [0,\infty]$ is known as t^ω-approach distance on χ if this function satisfies the following properties:

(t_1) $\forall\omega \in \mathbb{R}^+, \forall A, B \in 2^\chi : t^\omega(A,B) = 0 \Rightarrow A = B$,

(t_2) $\forall\omega \in \mathbb{R}^+, \forall A \in 2^\chi$, then $t^\omega(A,\emptyset) = \infty$,

(t_3) $\forall\omega \in \mathbb{R}^+, \forall A, B, C \in 2^\chi : t^\omega(A, B \cap C) = max\{t^\omega(A,B), t^\omega(A,C)\}$,

(t_4) $\forall\omega \in \mathbb{R}^+, \forall A, B \in 2^\chi, \forall\varepsilon < \omega : t^\omega\left(A^\omega_{(\varepsilon)}, B\right) \leq t^\omega(A,B) + \varepsilon$ where

$$A^\omega_{(\varepsilon)} = \{x \in \chi | t^\omega(\{x\}, B) \leq \omega - \varepsilon\}.$$

A pair (χ, t^ω) where the function t^ω is a distance and this pair is called t^ω- approach space and denoted by t^ω-app-spaces.

Definition 2.2 [9]: Support that (χ, t^{ω}) and $(Y, t^{\omega'})$ are app- spaces. A function $\varrho: \chi \to Y$ is known as $t^{\omega}-$ contraction if for all $\omega < \infty$, and for all $A, B \in 2^{\chi}, t^{\omega'}(\varrho((A), \varrho(B)) \leq t^{\omega}(A, B)$.

Definition 2.3: Support that χ is a t^{ω}-An Approach to L-Spaces over a Field F. Δ- norm on χ s.t $\|.\|_{\Delta}: \chi \to R^{+}$ $\forall x, y \in \chi$, t^{ω} is an approach distance on χ, then $(\chi, \|.\|_{\Delta}, t^{\omega}{}_{\|.\|_{\Delta}})$ is called t^{ω}- Δ- approach N-space if the following conditions satisfy:

1) $\| x \|_{\Delta} > 0$ for $x \neq 0, x \in \chi$;
2) $\| \alpha x \|_{\Delta} \leq \| x \|_{\Delta}$, $|\alpha| \leq 1, x \in \chi$;
3) $\lim_{\alpha \to 0} \| \alpha x \|_{\Delta} = 0$ for $x \in \chi$;
4) $\| x + y \|_{\Delta} \leq c \max(\| x \|_{\Delta}, \| y \|_{\Delta})$ for all $x, y \in \chi$ where c is some constant such that $c > 1$;
5) $t^{\omega}{}_{\|.\|_{\Delta}}(A, B) = sup_{A,B \in 2^{\chi}} sup_{x \in B} \inf_{a \in A} \|x - a\|_{\Delta}$.

Definition 2.4: Support that χ be a t^{ω} - Δ- approach N-space, A space is complete if every Cauchy sequence converges within it.

Definition 2.5: t^{ω}-approach linear space χ is said to be t^{ω} - Δ- app- normed linear algebra if :

1) χ is t^{ω} - Δ- approach N-space;
2) χ is t^{ω}-app- linear algebra;
3) For all $a, b \in \chi$, $\|ab\| \leq \|a\| \|b\|$;
4) If χ with identity, then $\|e\| = 1$.

Definition 2.6: Support that χ be a t^{ω} - Δ- approach N-algebra, then χ is a complete t^{ω} - Δ-approach algebra if every t^{ω} -Cauchy sequence is convergent.

3. New Results on Δ-Symmetric Approach in Banach Algebra

Proposition 3.1: Support that χ be a locally compact H-space and $C_0(\chi) = \{f \mid f: \chi \to \mathbb{R}, \text{continuous}, f(x) \geq 0, \text{for all } x \in \chi\}$. Then, $C_0(\chi)$ is a symmetric t^{ω}- Δ- approach B-algebra if $\| f \|_{\Delta} = \sup |f(x)|$.

Proof: Let $\omega < \infty$, $f_1, f_2 \in C_0(\chi), x \in \chi, A, B \in 2^{C_0(\chi)}$.

The usual distance $t^{\omega}: 2^{C_0(\chi)} \times 2^{C_0(\chi)} \to [0, \infty]$ is defined on $C_0(\chi)$ by

$$t^{\omega}(A, B) = \begin{cases} inf_{f_1 \in A, f_2 \in B} |f_1(x) - f_2(x)| & A \neq \emptyset, B \neq \emptyset \\ \infty & A \text{ or } B = \emptyset \end{cases},$$

$+, .$ are addition and multiplication binary operations such that

$$(f_1 + f_2)(x) = f_1(x) + f_2(x);$$
$$(f_1 . f_2)(x) = f_1(x) . f_2(x).$$

First:

(a) We must prove that $(C_0(\chi), t^{\omega})$ be an app-apace

(1) If $A, B \neq \emptyset, t^{\omega}(A, B)$
$= inf_{f_1 \in A, f_2 \in B}|f_1(x) - f_2(x)| = 0 \Rightarrow f_1(x) = f_2(x)$, then $A = B, \forall\ A, B \in 2^{C_0(\chi)}$

If $A\ or\ B = \emptyset$,(the proof is trivial).

(2) $t^{\omega}(A, B) = t^{\omega}(A, \emptyset) = inf_{\emptyset}|f_1(x) - f_2(x)| = \infty, \forall\ A \in 2^{C_0(\chi)}$

(3) Let $A, B, C \in 2^{C_0(\chi)}, t^{\omega}(A, B \cap C) = inf_{f_1 \in A, f_2 \in B \cap C}|f_1(x) - f_2(x)| = inf_{f_1 \in A}|f_1(x) - f_2(x)| \vee inf_{f_2 \in B \cap C}|f_1(x) - f_2(x)|$,

$$= max\left(inf_{f_1 \in A}|f_1(x) - f_2(x)|, inf_{f_2 \in B \cap C}|f_1(x) - f_2(x)|\right)$$
$$= max\left(t^{\omega}(A, B), t^{\omega}(A, C)\right), \forall\ A, B, C \in 2^{C_0(\chi)}.$$

(4) (a) Support that $\omega < \infty, A, B \in 2^{C_0(\chi)}$. In case that $A^{\omega}_{(\varepsilon)} \cap B^c \neq \emptyset$ for all $\varepsilon < \omega, so, t^{\omega}(A, B) \leq \omega - \varepsilon$, for all $\varepsilon < \omega, f \in A^{\omega}_{(\varepsilon)}$, then by t_4

$$t^{\omega}\left(A^{\omega}_{(\varepsilon)}, B\right) = inf_{f_1 \in A, f_2 \in B}|f_1(x) - f_2(x)|$$
$$\leq inf_{f_1 \in A, f_2 \in B}|f_1(x) - f_2(x)|$$
$$+ \ inf\{\varepsilon < \omega : \{f \in C_0(\chi) | t^{\omega}(A, B) \leq \omega - \varepsilon\}$$
$$\leq t^{\omega}(A, B) + \varepsilon, \forall f_1, f_2 \in C_0(\chi), A, B \in 2^{C_0(\chi)}.$$

Then $(C_0(\chi), t^{\omega})$ is t^{ω}-Anapproach space.

(b) Clearly($C_0(\chi), +)$ is a group.

(c) We now establish $+: C_0(\chi) \oplus C_0(\chi) \rightarrow C_0(\chi): (f_1, f_2) \rightarrow f_1 + f_2\ is\ t^{\omega}$ – contraction for all $\omega < \infty, f_1, f_2, f_3, f_4 \in C_0(\chi), A, B, C, D \in 2^{C_0(\chi)}: f_1 \in A, f_2 \in B, f_3 \in C, f_4 \in D$.

$$t^{\omega\prime}\left(\varrho(A, B), \varrho(C, D)\right) = inf_{f_1 \in A, f_2 \in B, f_3 \in C, f_4 \in D}|(f_1 + f_2)(x) - (f_3 + f_4)(x)|)$$
$$\leq inf_{f_1 \in A, f_3 \in C}\ (|f_1(x) - f_3(x)|) + inf_{f_2 \in B, f_4 \in D}(|(f_2(x) - f_4(x)|)$$

for some $k \in [0,1]$. Then, $(C_0(\chi), t^{\omega}, +)$ is t^{ω}- approach group.

Second: We must prove $(C_0(\chi), t^{\omega}, +, .)$ is a t^{ω}-approach L-algebra.

Support that $f_1, f_2 \in C_0(\chi), \alpha \geq 0$, such that $f_1 \geq 0, f_2 \geq 0$, for all $x \in \chi$.

a) $f_1(x) + f_2(x) = (f_1 + f_2)(x) \geq 0$;

b) $(\alpha f_1)(x) = \alpha(f_1(x)) \geq 0$, for all $\alpha \geq 0$;

c) $f_1(x).f_2(x) = (f_1.f_2)(x) \geq 0$;

d) $t^{\omega}\ (\{f(e)f(x)\}, A) = t^{\omega}\ (\{If(x)\}, A) = t^{\omega}\ (\{f(x)\}, A) \geq 0$;

e) $\alpha(f_1.f_2)(x) = \left(\alpha.f_1(x)\right).f_2(x) = f_1(x).(\alpha.f_2(x))$;

f) $\left(f_1(x).f_2(x)\right)f_3(x) = f_1(x).\left(f_2(x).f_3(x)\right)$;

g) $\left(f_1(x) + f_2(x)\right).f_3(x) = \left(f_1(x).f_3(x) + f_2(x).f_3(x)\right)$, and $f_1(x)\left(f_2(x) + f_3(x)\right) = \left(f_1(x).f_2(x) + f_1(x).f_3(x)\right)$.

Then, $(C_0(\chi), t^{\omega}, +, .)$ is t^{ω}-approach L-algebra.

Third: To prove $C_0(\chi)$ is t^{ω}- Δ- approach N-algebra must prove $\|.\|_{\Delta}$ is norm on $C_0(\chi)$.

1) Support that $f \in C_0(\chi)$ such that $f \neq 0,$, then $f > 0$ and $|f(x)| > 0$ for all $x \in \chi$.
Therefore, $\sup |f(x)| > 0$. Hence, $\| f \|_\Delta > 0$.

2) Support that $f \in C_0(\chi)$ and $|\alpha| \leq 1$,
$|\alpha||f(x)| \leq |f(x)|$, then, $\sup |\alpha f(x)| \leq \sup|f(x)|$.
Thus, we get that $\| \alpha f \|_\Delta \leq \| f \|_\Delta$.

3) Support that $f \in C_0(\chi)$, $lim_{\alpha \to 0} \| f \|_\Delta = lim_{\alpha \to 0} \sup|f(x)| = lim_{\alpha \to 0} \sup|\alpha f(x)| = \sup lim_{\alpha \to 0} |\alpha f(x)| = 0$.

4) Support that $f_1, f_2 \in C_0(\chi)$,

$$\max\{\| f_1 \|_\Delta, \| f_2 \|_\Delta\} = \begin{cases} \| f_1 \|_\Delta & if \ \| f_1 \|_\Delta \geq \| f_2 \|_\Delta \\ \| f_2 \|_\Delta & if \ \| f_2 \|_\Delta \geq \| f_1 \|_\Delta \end{cases},$$

If $\| f_2 \|_\Delta \geq \| f_1 \|_\Delta$, then, $\max\{\| f_1 \|_\Delta, \| f_2 \|_\Delta\} = \| f_2 \|_\Delta$ and

$$\| f_1 \|_\Delta + \| f_2 \|_\Delta \leq 2 \| f_2 \|_\Delta = 2\max\{\| f_1 \|_\Delta, \| f_2 \|_\Delta\},$$

If $\| f_1 \|_\Delta \geq \| f_2 \|_\Delta$, then, $\max\{\| f_1 \|_\Delta, \| f_2 \|_\Delta\} = \| f_1 \|_\Delta$ and

$$\| f_1 \|_\Delta + \| f_2 \|_\Delta \leq 2 \| f_1 \|_\Delta = 2\max\{\| f_1 \|_\Delta, \| f_2 \|_\Delta\},$$

Since, $\| f_1 + f_2 \|_\Delta \leq \| f_1 \|_\Delta + \| f_2 \|_\Delta$, thus

$\| f_1 + f_2 \|_\Delta \leq 2\max\{\| f_1 \|_\Delta, \| f_2 \|_\Delta\}$, for all $f_1, f_2 \in C_0(\chi)$,

5) $t^{\omega}{}_{\|.\|_\Delta}(A,B) = inf_{f \in A, f_1 \in B} d_{t^{\omega}{}_{\|.\|_\Delta}}(f, f_1) = inf_{f \in A, f_1 \in B} \| f - f_1 \|_\Delta = inf_{f \in A, f_1 \in B} sup \left\{ \frac{|f(x) - f_1(x)|}{1 + |f(x) - f_1(x)|} \right\} \leq sup_{A,B \in 2^{C_0(\chi)}} \inf_{f \in A, f_1 \in B} sup \left\{ \frac{|f(x) - f_1(x)|}{1 + |f(x) - f_1(x)|} \right\} \leq sup_{A,B \in 2^{C_0(\chi)}} sup_{f_1 \in B} \inf_{f \in A} sup \left\{ \frac{|f(x) - f_1(x)|}{1 + |f(x) - f_1(x)|} \right\} = sup_{f_1 \in B} \inf_{f \in A} \| f - f_1 \|_\Delta$.

2) $[\alpha f_1(x) + \beta f_2(x)]^* = \bar{\alpha}\overline{f_1(x)} + \bar{\beta}\overline{f_2(x)} = \bar{\alpha} f_1^*(x) + \bar{\beta} f_2^*(x)$;

3) $(f_1.f_2)^*(x) = \overline{(f_1.f_2)(x)} = \overline{f_2(x)}.\overline{f_1(x)} = f_2^*(x) f_1^*(x)$.

Now, to prove The function $*: C_0(\chi) \to C_0(\chi): f \to f^* is\ t^{\omega}$ - contraction for all $\omega < \infty$, $f_1, f_2 \in C_0(\chi), A, B \in 2^{C_0(\chi)}: f_1 \in A, f_2 \in B$.

$$\begin{aligned} &t^{\omega\prime}(\varrho(A), \varrho(B)) \\ &= inf_{f_1 \in A, f_2 \in B} |f_1^*(x) - f_2^*(x)| = inf_{f_1 \in A, f_2 \in B} \left| \overline{(f_1(x))^*} - \overline{(f_2(x))^*} \right| \\ &= inf_{f_1 \in A, f_2 \in B} \left| \overline{\overline{(f_1(x))}} - \overline{\overline{(f_2(x))}} \right| \\ &= inf_{f_1 \in A, f_2 \in B} |f_1(x) - f_2(x)| = t^{\omega}(A,B) \\ &\leq k t^{\omega}(A,B), \text{ for some } k \in [0,1]. \end{aligned}$$

Therefore, $C_0(\chi)$ is symmetric t^{ω} - Δ- approach N-algebra.

Fifth: Support that $\{A_n\}_{n=1}^{\infty}$ be a t^{ω}-Cauchy sequence in $(C_0(\chi), t^{\omega})$. Then,

$lim_{n \to \infty} \inf_{f_i \in A_i, f \in B} t^{\omega}(\{A_n\}, B) = 0$, for all $B \in \Gamma\left(2^{(C_0(\chi))}\right)$, $i = 1, \ldots, n \in Z^+$.

$$0 = lim_{n\to\infty}\inf_{f_i\in A_i, f\in B} t^{\omega}(A_n, B) = \inf_{A_n, A_m\in\Gamma\left(2^{(C_0(\chi))}\right)}\inf_{f_i\in A_i, f_j\in A_j\subseteq B} t^{\omega}(A_n, A_m) = \inf_{A_n, A_m\in\Gamma\left(2^{(C_0(\chi))}\right)} \inf_{f_i\in A_i, f_j\in A_j\subseteq B}|f_i(x) - f_j(x)|,$$

Support that $\{A_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $(C_0(\chi), t^{\omega})$, $A_n, B \in 2^{C_0(\chi)}$ and Support that $f \in B$, this means, for all $B \in \Gamma\left(2^{C_0(\chi)}\right)$,

$$lim_{n\to\infty} inf_{f_i\in A_i, f\in B} t^{\omega}(A_n, B) = 0, i = 1, \dots, n \in Z^+$$

$$0 = lim_{n\to\infty} inf_{f_i\in A_i, f\in B} t^{\omega}(A_n, B) = lim_{n\to\infty} inf_{f_i\in A_i, f\in B}|f_i(x) - f(x)| = lim_{n\to\infty}\inf_{f\in B}\inf_{f_i\in A_i}|f_i(x) - f(x)| = lim_{n\to\infty}\inf_{f\in B}\inf_{f_i\in A_i}||f_i - f||_{\Delta}; i = 1, \dots, n \in Z^+;$$

that is, $lim_{n\to\infty}\inf_{f_i\in A_i, f\in B}||f_i - f||_{\Delta} = lim_{n\to\infty}\inf_{f_i\in A_i, f\in B}\sup|f_i(x) - f(x)|$

$$\leq lim_{n\to\infty}\inf_{f_i\in A_i, f\in B}\sup|f_i(x)| + lim_{n\to\infty}\inf_{f_i\in A_i, f\in B}\sup|f(x)| = 0.$$

Then, for all $f_i, f \in C_0(\chi)$, $lim_{n\to\infty}\inf_{f_i\in A_i, f\in B}||f_i - f||_{\Delta} = 0$.

Theorem 3.2*: Support that* $S_{\mathbb{C}}\left(L^0(\mu)\right)$ *be the set of all state functional in* $L^0(\mu)$*, and* $S_{\mathbb{C}}\left(L^0(\mu)\right) = \{\wp \mid \wp: L^0(\mu) \to \mathbb{C}\ is\ a\ state\ functional\}$

With $\| \wp(f)_{(x)} \|_{\Delta} = \int \frac{|\wp(f)_{(x)}|}{1+|\wp(f)_{(x)}|}\ d\mu(x)$.

Then, $S_{\mathbb{C}}\left(L^0(\mu)\right)$ *is symmetric* t^{ω} *- Δ- approach B-algebra.*

Proof: We define the distance on $S_{\mathbb{C}}\left(L^0(\mu)\right)$ by $t^{\omega}: 2^{S_{\mathbb{C}}\left(L^0(\mu)\right)} \times 2^{S_{\mathbb{C}}\left(L^0(\mu)\right)} \to [0, \infty]$ and for all

$$\forall\omega < \infty, \wp_1, \wp_2 \in S_{\mathbb{C}}\left(L^0(\mu)\right), A, B \in 2^{S_{\mathbb{C}}\left(L^0(\mu)\right)},$$

$$t^{\omega}(A, B) = \begin{cases} inf_{\wp_1\in A, \wp_2\in B, x\neq 0}\dfrac{|\wp_1(f)_{(x)} - \wp_2(f)_{(x)}|}{|x|} & if\ A\ and\ B\ \neq \emptyset \\ \infty & if\ A\ or\ B\ \neq \emptyset \end{cases}$$

+,. are addition and multiplication binary operations such that

$$(\wp_1 + \wp_2)(f_{(x)}) = \wp_1(f)_{(x)} + \wp_2(f)_{(x)}.$$

$$(\wp_1.\wp_2)(f_{(x)}) = \wp_1(f)_{(x)}.\wp_2(f)_{(x)}.$$

First: By definition [15], we prove that $(S_{\mathbb{C}}\left(L^0(\mu)\right), t^{\omega})$ is t^{ω}- app. space and $(S_{\mathbb{C}}\left(L^0(\mu)\right), t^{\omega}, +)$ is a t^{ω}- app. group.

Second: By definition8, we prove $(S_{\mathbb{C}}\left(L^0(\mu)\right), t^{\omega}, +, .)$ is t^{ω}-approach L-algebra.

Third: By the of definition of t^{ω}- Δ- approach N-algebra we prove that $S_{\mathbb{C}}\left(L^0(\mu)\right)$ is t^{ω}- Δ- approach N-algebra.

Fourth: To prove $*: S_{\mathbb{C}}\left(L^0(\mu)\right) \to S_{\mathbb{C}}\left(L^0(\mu)\right)$ defined by $*\left(\wp(f)_{(x)}\right) = \overline{\wp(f)_{(x)}}$,

Then, $*$ is satisfying involution axioms such that

$\| \wp \|_{\Delta} = \int | \wp(f)_{(x)} | d\mu(x)$

1) $\left(\wp(f)_{(x)}\right)^{**} = (\wp(f)_{(x)})$, since $\left(\wp(f)_{(x)}\right)^{**} = ((\wp(f)_{(x)})^{*})^{*}$,

We obtain $\wp^{*}(f)_{(x)} = (\overline{(\wp(f)_{(x)})})^{*} = \overline{\overline{(\wp(f)_{(x)})}} = \left(\wp(f)_{(x)}\right)$;

2) $[\alpha\wp_1(f)_{(x)} + \beta\wp_2(f)_{(x)}]^{*} = \overline{\alpha\wp_1(f)_{(x)}} + \bar{\beta}\,\overline{\wp_2(f)_{(x)}}$
$= \bar{\alpha}\,\wp_1^{*}(f)_{(x)} + \bar{\beta}\wp_2^{*}(f)_{(x)}$;

3) $[\wp_1.\wp_2]^{*}(f)_{(x)} = \frac{1}{2}\left[\overline{(\wp_1 + \wp_2)^2(f)_{(x)}} - \overline{\wp_1{}^2(f)_{(x)}} - \overline{\wp_2{}^2(f)_{(x)}}\right]$
$= \frac{1}{2}\left[\overline{\left(\wp_1(f)_{(x)} + \wp_2(f)_{(x)}\right)^2} - \overline{\wp_1{}^2(f)_{(x)}} - \overline{\wp_2{}^2(f)_{(x)}}\right] =$
$\wp_1^{*}(f)_{(x)}.\wp_2^{*}(f)_{(x)}$.

Now, we must prove The function $*: S_{\mathbb{C}}\left(L^0(\mu)\right) \to S_{\mathbb{C}}\left(L^0(\mu)\right): \wp \to \wp^{*} is\ t^{\omega}$- contraction for all $\omega < \infty, f \in L^0(\mu), A, B \in 2^{S_{\mathbb{C}}\left(L^0(\mu)\right)}$.

$$t^{\omega\prime}\left(\varrho(A), \varrho(B)\right) = t^{\omega\prime}(A^{*}, B^{*}) = inf_{\wp_1 \in A, \wp_2 \in B} |\wp_1^{*}(f)_{(x)} - \wp_2^{*}(f)_{(x)}| =$$
$$= inf_{\wp_1 \in A, \wp_2 \in B}\left|\overline{(\wp_1)^{*}(f)_{(x)}} - \overline{(\wp_2)^{*}(f)_{(x)}}\right|$$
$$= inf_{\wp_1 \in A, \wp_2 \in B}\left|\overline{\overline{(\wp_1(f)_{(x)})}} - \overline{\overline{(\wp_2(f)_{(x)})}}\right|$$
$$= inf_{\wp_1 \in A, \wp_2 \in B}\left|\wp_1(f)_{(x)} - \wp_2(f)_{(x)}\right| = t^{\omega}(A, B)$$
$$\leq t^{\omega}(A, B).$$

Thus, $(S_{\mathbb{C}}(L^0(\mu)), \|.\|_{\Delta}, t^{\omega}{}_{\|.\|_{\Delta}})$ is symmetric t^{ω} - Δ- approach N-algebra.

Fifth: Support that $A_n, B \in 2^{S_{\mathbb{C}}\left(L^0(\mu)\right)}$ and Support that $\{A_n\}_{n=1}^{\infty}$ be a t^{ω}-Cauchy sequence in $(S_{\mathbb{C}}(L^0(\mu)), t^{\omega})$, t^{ω}-approach space by $\Gamma\left(S_{\mathbb{C}}\left(L^0(\mu)\right)\right)$, $B \in \Gamma\left(2^{S_{\mathbb{C}}\left(L^0(\mu)\right)}\right)$ and Support that

$$t^{\omega}(A_n, B) = \begin{cases} inf_{\wp_i(f) \in A_i, \wp(f) \in B, x \neq 0} \dfrac{|\wp_i(f)_{(x)} - \wp(f)_{(x)}|}{|x|}, i = 1, \dots n \in Z^{+} & if\ \{A_n\}\ and\ B \neq \emptyset \\ \infty & if\ \{A_n\} = \emptyset\ or\ B = \emptyset \end{cases}$$

This means,

for all $B \in \Gamma\left(2^{S_{\mathbb{C}}\left(L^0(\mu)\right)}\right)$, $lim_{n\to\infty} \inf_{\wp_i(f) \in A_i, \wp(f) \in B, x \neq 0} t^{\omega}(A_n, B) = 0, i = 1, \dots, n \in Z^{+}$ so, by definition 3,

$$lim_{n\to\infty} \inf_{B \in \Gamma\left(2^{\left(S_{\mathbb{C}}\left(L^0(\mu)\right)\right)}\right)} \inf_{\wp_n(f) \in A_n, \wp(f) \in B, x \neq 0} \frac{|\wp_n(f)_{(x)} - \wp(f)_{(x)}|}{|x|} = 0,$$

$$lim_{n\to\infty} \sup_{B \in \Gamma\left(2^{\left(S_{\mathbb{C}}\left(L^0(\mu)\right)\right)}\right)} \inf_{\wp_n(f) \in A_n, \wp(f) \in B, x \neq 0} \frac{|\wp_n(f)_{(x)} - \wp(f)_{(x)}|}{|x|} = 0.$$

Thus, $\{A_n\}_{n=1}^{\infty}$ is convergent in $\left(S_{\mathbb{C}}\left(L^0(\mu)\right), t^{\omega}\right)$.

Now, we must prove that $(S_{\mathbb{C}}(L^0(\mu)), \|.\|_{\Delta}, t^{\omega}{}_{\|.\|_{\Delta}})$ is complete t^{ω} - Δ-approach N-algebra.

Support that $\{A_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $(S_{\mathbb{C}}(L^0(\mu)), t^{\omega})$, $A_n, B \in 2^{S_{\mathbb{C}}(L^0(\mu))}$ and Support that $\wp(f) \in B$, This means,

for all $B \in \Gamma\left(2^{S_{\mathbb{C}}(L^0(\mu))}\right)$, $lim_{n\to\infty}\inf_{\wp_i(f)\in A_i, \wp(f)\in B, x\neq 0} t^{\omega}(A_n, B) = 0, i = 1, \dots, n \in Z^+$ $0 = lim_{n\to\infty}\inf_{\wp_i(f)\in A_i, \wp(f)\in B, x\neq 0} t^{\omega}(A_n, B) = lim_{n\to\infty}\inf_{\wp_j(f)\in A_j \subseteq B}\inf_{\wp_i(f)\in A_i}||\wp_i(f) - \wp_j(f)||_{\Delta}; j = 1, \dots, n \in Z^+$;

that is, $lim_{n\to\infty}\inf_{\wp_i(f)\in A_i, \wp(f)\in B}||\wp_i(f) - \wp(f)||_{\Delta}$

$$= lim_{n\to\infty}\inf_{\wp_i(f)\in A_i, \wp(f)\in B}\int \frac{|\wp_i(f)(x) - \wp(f)(x)|}{1 + |\wp_i(f)(x) - \wp(f)(x)|} \ d\mu(x)$$

$$\leq lim_{n\to\infty}\inf_{\wp_i(f)\in A_i, \wp(f)\in B}\int \frac{|\wp_i(f)(x)|}{1 + |\wp_i(f)(x)|} \ d\mu(x)$$

$$+ lim_{n\to\infty}\inf_{\wp_i(f)\in A_i, \wp(f)\in B}\int \frac{|\wp(f)(x)|}{1 + |\wp(f)(x)|} \ d\mu(x) = 0.$$

Then, for all $\wp_i(f), \wp(f) \in S_{\mathbb{C}}(L^0(\mu))$, $lim_{n\to\infty}\inf_{\wp_i(f)\in A_i, \wp(f)\in B}||\wp_i(f) - \wp(f)||_{\Delta} = 0$

Thus, $(S_{\mathbb{C}}(L^0(\mu)), \|.\|_{\Delta}, t^{\omega}{}_{\|.\|_{\Delta}})$ is complete t^{ω}- Δ-approach N-algebra.

Therefore, $S_{\mathbb{C}}(L^0(\mu))$ is t^{ω} - Δ-approach N-space, so, it is a symmetric t^{ω} - Δ- approach B-algebra.

4. Conclusions

In this paper, we introduced some important definitions and we proved that the t^{ω} - Δ - approach N-space $(C_0(\chi), \|.\|_{\Delta}, t^{\omega}{}_{\|.\|_{\Delta}})$, , $(L^0(\mu), \|.\|_{\Delta}, t^{\omega}{}_{\|.\|_{\Delta}})$, $(S_{\mathbb{C}}(L^0(\mu)), \|.\|_{\Delta}, t^{\omega}{}_{\|.\|_{\Delta}})$ are complete t^{ω}- Δ- approach N-algebra, also, they are symmetric t^{ω} - Δ- approach B-algebra. An example is given to explain that the space $(B(\chi, \gamma), \|.\|_{\Delta}, t^{\omega}{}_{\|.\|_{\Delta}})$ is also symmetric t^{ω} - Δ- approach B-algebra.

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