

The fractional ZMA-transform for solving first and second order linear fractional differential equations

Zainab Mohammed Alwan

Department of Mathematics

College of Science

Mustansiriyah University

Baghdad 10052

Iraq

Abstract

For the work on paper, prepare new techniques by constructing a new definition called the fractional ZMA-transform (FZMAT). These techniques are used to make a solution to a fractional differential equation (FDE) easier by employing fractional analysis through the Riemann-Liouville (R-L) derivative. Also proposed attractive details and studied the relationship between the ZMA-transform (ZMAT) and the Laplace transform (LT). Moreover, present some applications for solving first and second order (1st and 2nd) linear fractional differential Equations (LFDEs).

Subject Classification: Primary 26A33, Secondary 44A10.

Keywords: Fractional differential equations, Riemann-Liouville, ZMA-transform, Mittag-leffler function, Fractional analysis, Inverse ZMA-transform.

1. Introduction

Many researchers in the past have used integral transform to solve various problems in different branches of engineering, chemistry, physics, mathematics, cryptography, and fractional. Fractional calculus, which deals with derivatives of non-integer order, is a significant branch of mathematics with wide applications in science and real-world problems. The fractional derivative (FD) has been defined by many literatures, and the commonly used definition is (R-L) in [1-5]. Several transforms (T), such as Laplace (LT) [6], [7], Khan and Khan [8], [9], [10], and ZMAT [12, 14, 15] exist. This work introduces FZMAT, a modified fractional LT [16], built on ZMAT concepts. It explores applications,

This article has been corrected with minor changes. These changes do not impact the academic content of the article.

E-mail: lionwight_2009@uomustansiriyah.edu.iq

enhances understanding, and presents fractional calculus facts using the Mittag-Leffler function (M-L) in [1] as:

$$E_{\alpha}(\tau) = \sum_{\ell=0}^{\infty} \frac{\tau^{\ell}}{\Gamma(\alpha\ell+1)}, \tau \text{ is complex number, } \Re(\alpha) > 0 \quad (1)$$

In [5], [13] using the fractional difference to realize a fractional derivative (FD):

$$\Delta^{\alpha} f(\tau) = (\mathcal{FW} - h)^{\alpha} f(\tau) = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f(\tau + (\alpha - \ell)h) \quad (2)$$

Where (τ) a function was continuous, the differentiable is not needful, a fractional difference for $f(\tau)$ and $\mathcal{FW}(h)$ is a forward operator appointed.

$$\mathcal{FW}(h)f(\tau) = f(\tau + h), h \in \mathbb{R}^+, \text{ constant discretization span} \quad (3)$$

$$D^{\alpha} f(\tau) = \lim_{h \rightarrow 0} \frac{\Delta^{\alpha} f(\tau)}{h^{\alpha}}, \alpha - \text{derivative} \quad (4)$$

The modified fractional (R-L) derivative of $f(\tau)$ write in (2) and presented in [1] as:

i) If a is a constant and $f(\tau) = a$, then it's α - derivative get by:

$$D^{\alpha} f(\tau) = \begin{cases} \frac{c}{\Gamma(1-\alpha)\tau^{\alpha}}, \alpha \leq 0 \\ 0, \text{ else} \end{cases} \quad (5)$$

ii) If $(\tau) \neq a$, $f(\tau)$ is not permanent and the (FD) offered as:

$$f(\tau) = f(0) + (f(\tau) - f(0)) \quad (6)$$

$$f^{\alpha}(\tau) = \mathcal{D}_t^{\alpha} f(0) + \mathcal{D}_t^{\alpha} (f(\tau) - f(0)) \quad (7)$$

$$D_t^{\alpha} (f(\tau) - f(0)) = D_t^{\alpha} f(\tau) = D_{\alpha} (f^{\alpha-1}(\tau)), \alpha > 0 \quad (8)$$

$$f^{\alpha}(\tau) = (f^{\alpha-\mu}(\tau))^{\alpha}, \mu \leq \alpha \leq \mu + 1, \mu \geq 1, < \alpha < \mu + 1 \quad (9)$$

Jumarie's suggestion (R-L) of the fractional derivative in [1] and the (FDE) and the solution produced in [13], [16] as:

$$\mathbf{d}y = g(\tau)(\mathbf{d}\tau)^{\alpha}, y(0) = 0, \tau \geq 0 \quad (10)$$

$$y(\tau) = \int_0^{\tau} g(\zeta) (d\zeta)^{\alpha}, y(0) = 0 \quad (11)$$

$$y(\tau) = \alpha \int_0^{\tau} (\tau - \zeta)^{\alpha-1} g(\zeta) d\zeta, 0 < \alpha \leq 1 \quad (12)$$

2. Fundamentals Concepts of (FZMAT)

The (ZMAT) of the assignment $f(\tau)$ offered [12, 14, 15] as:

$$\mathbb{Z}_{\mathbf{MA}}(\mathbf{v}; \mathbf{s}) = \Lambda\{f(\tau)\} = \frac{1}{\mathbf{s}} \int_0^{\infty} f(\mathbf{v}\tau) \mathbf{e}^{-\frac{\tau}{\mathbf{s}}} \mathbf{d}\tau, \mathbf{0} \leq \tau \quad (13)$$

The (FZMAT) of the assignment $f(\tau)$ presented by:

$$\mathbb{Z}_{\mathbf{MA}}^{\alpha}(\mathbf{v}; \mathbf{s}) = \Lambda_{\alpha}\{f(\tau)\} = \frac{1}{\mathbf{s}^{\alpha}} \int_0^{\infty} \mathbf{E}_{\alpha} \left(-\left(\frac{\tau}{\mathbf{s}}\right)^{\alpha} \right) f(\mathbf{v}\tau) (\mathbf{d}\tau)^{\alpha}, \mathbf{0} \leq \tau^{\alpha} \quad (14)$$

$$\mathbb{Z}_{\mathbf{MA}}^{\alpha}(\mathbf{v}; \mathbf{s}) = \Lambda_{\alpha}\{f(\tau)\} = \frac{1}{(\mathbf{s}\mathbf{v})^{\alpha}} \int_0^{\infty} \mathbf{E}_{\alpha} \left(-\left(\frac{\tau}{\mathbf{s}\mathbf{v}}\right)^{\alpha} \right) f(\tau) (\mathbf{d}\tau)^{\alpha}, \mathbf{0} \leq \tau^{\alpha} \quad (15)$$

In the following, some properties of (FZMAT) as:

Theorem 1: Give a, b was fixed and $f(t), g(t)$ are assignment, then:

a- Linearity Property

$$\Lambda_\alpha [p_1 f(t) + p_2 g(t)] = p_1 \Lambda_\alpha [f(t)] + p_2 \Lambda_\alpha [g(t)] \quad (16)$$

b- Changing Scale

$$\Lambda_\alpha \{f(at)\} = \frac{1}{(a)^\alpha} \mathbb{Z}_{MA}^\alpha(av; s) \quad (17)$$

c- Multiplication by (t^α)

$$\Lambda_\alpha \{t^\alpha f(t)\} = \frac{-1}{(sv)^\alpha} D_s^\alpha ((sv)^\alpha \mathbb{Z}_{MA}^\alpha(v; s)) \quad (18)$$

d- Shifting Property

$$\Lambda_\alpha \{f(t-a)\} = \left(E_\alpha \left(-\left(\frac{a}{sv}\right)^\alpha \right) \right) \Lambda_\alpha [f(u)] \quad (19)$$

e- Convolution of $(\alpha - \text{order})$ of $f(t)$ and $g(t)$ write:

$$(f(t) *_\alpha g(t)) = \int_0^t f(t-u) \cdot g(u) (du)^\alpha \quad (20)$$

f- Inversion Equation of (FZMAT)

Presented $\delta_\alpha(t)$ for order $\alpha, \alpha \in (0,1)$ inv [16]:

$$\int_{\mathbb{R}} f(t) \delta_\alpha(t-\varphi) (dt)^\alpha = \alpha f(\varphi) \quad (21)$$

The (FZMAT) of $\delta_\alpha(t-\varphi)$ applied as:

$$\begin{aligned} \Lambda_\alpha \{\delta_\alpha(t-\varphi)\} &= \frac{1}{(sv)^\alpha} \int_0^\infty E_\alpha \left(-\left(\frac{t}{sv}\right)^\alpha \right) \delta_\alpha(t-\varphi) (dt)^\alpha \\ &= \frac{\alpha}{(sv)^\alpha} E_\alpha \left(-\left(\frac{t}{sv}\right)^\alpha \right) \Lambda_\alpha \{\delta_\alpha(t)\} = \frac{\alpha}{(sv)^\alpha}. \end{aligned}$$

Lemma 2: The formula holds connection of $\delta_\alpha(t)$ and $E_\alpha(t^\alpha)$ in [16]:

$$\frac{\alpha}{(\mu_\alpha)^\alpha} \int_{\mathbb{R}} E_\alpha \left(-i \left(\frac{t}{sv}\right)^\alpha \right) (ds)^\alpha = \delta_\alpha(t) \quad (22)$$

$E_\alpha(i(\mu_\alpha)^\alpha) = 1, \mu_\alpha$ is the period of the complex - valued (M-L) (23)

$$\int_{\mathbb{R}} E_\alpha \left(i \left(\frac{t}{\xi u}\right)^\alpha \right) \delta_\alpha(t) (dt)^\alpha = \alpha \quad (24)$$

Theorem 2: (Inversion Theorem) The inverse of (FZMAT) of (14) present:

$$f(t) = \frac{1}{(\mu_\alpha)^\alpha} \int_{-i\infty}^{i\infty} (sv)^\alpha E_\alpha \left(\left(\frac{t}{sv}\right)^\alpha \right) \mathbb{Z}_{MA}^\alpha(v; s) (ds)^\alpha \quad (25)$$

3. Inversion Equation of (FZMAT)

The fractional (LT) of $f(t)$ define in [16] like:

$$\begin{aligned} \mathcal{L}_\alpha \{f(t)\} &= \mathcal{F}_\alpha(s) = \int_0^\infty E_\alpha(-s t)^\alpha f(t) (dt)^\alpha = \\ &= \lim_{m \rightarrow \infty} \int_0^m E_\alpha(-s t)^\alpha f(t) (dt)^\alpha \end{aligned} \quad (26)$$

Theorem 3: The (FZMAT) of $f(t)$ give $\Lambda_\alpha \{f(t)\} = \mathbb{Z}_{MA}^\alpha(v; s)$:

$$\Lambda_{\alpha}\{\mathcal{F}(t)\} = \frac{1}{(sv)^{\alpha}} \mathcal{L}_{\alpha}\left\{\frac{1}{sv}\right\} \quad (27)$$

Proposition 1: Assume that $0 < \alpha \leq 1$,

- 1- $\Lambda_{\alpha}\{1\} = 1$
- 2- $\Lambda_{\alpha}\{t^n\} = \alpha! \Gamma(n+1) (sv)^{n\alpha}$
- 3- $\Lambda_{\alpha}\{E_{\alpha}(-(a^{\alpha} t^{\alpha}))\} = \frac{1}{(1-asv)^{\alpha}} \mathbb{Z}_{MA}^{\alpha}\left(\frac{1}{(1-asv)}, s\right)$
- 4- $\Lambda_{\alpha}\{E_{\alpha}(-(i a t)^{\alpha})\} = \frac{(1-iasv)^{\alpha}}{(1+(asv)^2)^{\alpha}}$
- 5- $\Lambda_{\alpha}\{(D_t^{\alpha})[\mathcal{F}(t)]\} = \frac{1}{(sv)^{\alpha}} \mathbb{Z}_{MA}^{\alpha}(v; s) - \frac{1}{(sv)^{\alpha}} \mathcal{F}_{\alpha}(0)$
- 6- $\Lambda_{\alpha}\{(D_t^{\alpha})^2[\mathcal{F}(t)]\} = \frac{1}{(sv)^{2\alpha}} \mathbb{Z}_{MA}^{\alpha}(v; s) - \frac{1}{(sv)^{2\alpha}} \mathcal{F}_{\alpha}(0) \frac{1}{(sv)^{\alpha}} (D_t^{\alpha})[\mathcal{F}_{\alpha}(t)](0)$
- 7- $\Lambda_{\alpha}\{(D_t^{\alpha})^n \mathcal{F}(t)\} = \frac{1}{(sv)^{n\alpha}} \mathbb{Z}_{MA}^{\alpha}(v; s) - \sum_{k=0}^{n-1} \frac{1}{(sv)^{(n-k)\alpha}} \mathcal{F}^{n-k}_{\alpha}(0)$
- 8- $\Lambda_{\alpha}\{t \mathcal{F}(t)\} = (sv)^{\alpha} \mathbb{Z}_{MA}^{\alpha}(v; s) + s^{\alpha} v^{2\alpha} \frac{d}{dv^{\alpha}} (\mathbb{Z}_{MA}^{\alpha}(v; s))$
- 9- $\Lambda_{\alpha}\{t (D_t^{\alpha})[\mathcal{F}(t)]\} = \Gamma(\alpha+1) s^{\alpha} v^{2\alpha} \frac{d}{dv^{\alpha}} \left(\frac{1}{(sv)^{\alpha}} \mathbb{Z}_{MA}^{\alpha}(v; s) \right) + \mathbb{Z}_{MA}^{\alpha}(v; s)$
- 10- $\Lambda_{\alpha}\{t (D_t^{\alpha})^2[\mathcal{F}(t)]\} = \frac{\Gamma(\alpha+2)}{s^{\alpha}} \frac{d}{dv^{\alpha}} \left(\mathbb{Z}_{MA}^{\alpha}(v; s) - \frac{1}{(sv)^{\alpha}} \mathbb{Z}_{MA}^{\alpha}(v; s) \right) + \frac{1}{(sv)^{\alpha}} \mathcal{F}_{\alpha}(0)$

4. Applications

Application 1: Permit the 1st order of (LFDEs):

$$D^{\alpha}[\mathcal{Y}_{\alpha}(t)] + \mathcal{Y}_{\alpha}(t) = \mathcal{F}_{\alpha}(t), 0 < \alpha \leq 1 \quad (28)$$

Corresponding premier provision

$$\mathcal{Y}_{\alpha}(0) = 0 \quad (29)$$

Solution: by (FZMAT) of (28) obtained:

$$\Lambda_{\alpha}\{D^{\alpha}[\mathcal{Y}_{\alpha}(t)] + \mathcal{Y}_{\alpha}(t)\} = \Lambda_{\alpha}\{\mathcal{F}_{\alpha}(t)\} \quad (30)$$

$$\frac{1}{(sv)^{\alpha}} \mathbb{Z}_{MA}^{\alpha}(v; s) - \frac{1}{(sv)^{\alpha}} \mathcal{Y}_{\alpha}(0) + \mathbb{Z}_{MA}^{\alpha}(v; s) = \mathcal{F}_{\alpha}(s) \quad (31)$$

After simple steps get:

$$\mathbb{Z}_{MA}^{\alpha}(v; s) \left(\frac{1+(sv)^{\alpha}}{(sv)^{\alpha}} \right) = \mathcal{F}_{\alpha}(s) \quad (32)$$

$$\mathcal{Y}_{\alpha}(t) = \frac{1}{(\mu_{\alpha})^{\alpha}} \int_{-i\infty}^{i\infty} (sv)^{\alpha} (1 + (sv)^{\alpha}) E_{\alpha}\left(\left(\frac{t}{sv}\right)^{\alpha}\right) \Lambda_{\alpha}(\mathcal{Y}_{\alpha}(s)) (ds)^{\alpha} \quad (33)$$

Application 2: Look for the 1st order of (LFDEs):

$$D_t^{\alpha}[\mathcal{Y}_{\alpha}(t)] = \mathcal{Y}_{\alpha}(t) + 1, 0 < \alpha \leq 1 \quad (34)$$

The premier provision:

$$y_\alpha(0) = 0 \quad (35)$$

Solution: by (FZMAT) of (34) find as:

$$\Lambda_\alpha \{ D_t^\alpha [y_\alpha(t)] \} = \Lambda_\alpha \{ y_\alpha(t) + 1 \} \quad (36)$$

$$\frac{1}{(\mathfrak{s}v)^\alpha} \mathbb{Z}_{\text{MA}}^\alpha(v; \mathfrak{s}) - \frac{1}{(\mathfrak{s}v)^\alpha} y_\alpha(0) = \mathbb{Z}_{\text{MA}}^\alpha(v; \mathfrak{s}) + 1 \quad (37)$$

$$\rightarrow \mathbb{Z}_{\text{MA}}^\alpha(v; \mathfrak{s}) = \frac{(\mathfrak{s}v)^\alpha}{1 + (\mathfrak{s}v)^\alpha} \quad (38)$$

$$y_\alpha(t) = t^\alpha E_\alpha(-t^\alpha) \quad (39)$$

Application 3: Estimate the 2nd order of (LFDEs):

$$(D_t^\alpha)^2 [y_\alpha(t)] - y_\alpha(t) = t^\alpha, 0 < \alpha \leq 1 \quad (40)$$

The premier provision:

$$y_\alpha(0) = 1, D_t^\alpha [y_\alpha(t)](0) = 1 \quad (41)$$

Solution: by (FZMAT) of (40) find as:

$$\Lambda_\alpha \{ (D_t^\alpha)^2 [y_\alpha(t)] - y_\alpha(t) \} = \Lambda_\alpha \{ t^\alpha \} \quad (42)$$

$$\frac{1}{(\mathfrak{s}v)^{2\alpha}} \mathbb{Z}_{\text{MA}}^\alpha(v; \mathfrak{s}) - \frac{1}{(\mathfrak{s}v)^{2\alpha}}$$

$$y_\alpha(0) - \frac{1}{(\mathfrak{s}v)^\alpha} D_t^\alpha [y_\alpha(t)](0) = \Gamma(1 + \alpha) (\mathfrak{s}v)^\alpha \quad (43)$$

$$\rightarrow \mathbb{Z}_{\text{MA}}^\alpha(v; \mathfrak{s}) [1 - (\mathfrak{s}v)^{2\alpha}] = 1 + (\mathfrak{s}v)^\alpha + \Gamma(1 + \alpha) (\mathfrak{s}v)^{3\alpha} \quad (44)$$

$$\rightarrow \mathbb{Z}_{\text{MA}}^\alpha(v; \mathfrak{s}) = \frac{1 + (\mathfrak{s}v)^\alpha}{(1 - (\mathfrak{s}v)^{2\alpha})} + \frac{\Gamma(1 + \alpha) (\mathfrak{s}v)^{3\alpha}}{(1 - (\mathfrak{s}v)^{2\alpha})} \quad (45)$$

$$y_\alpha(t) = E_\alpha(t^\alpha) + \frac{\Gamma(1 + \alpha)}{2} E_\alpha(t^\alpha) - \frac{\Gamma(1 + \alpha)}{2} E_\alpha(-t^\alpha) \quad (46)$$

Application 4: Calculate the 2nd order of (LFDEs):

$$(\mathcal{D}_t^\alpha)^2 [y_\alpha(t)] + 2 (\mathcal{D}_t^\alpha) [y_\alpha(t)] + 5 y_\alpha(t) = E_\alpha(-t^\alpha) \otimes \sin_\alpha(t^\alpha), 0 < \alpha \leq 1 \quad (47)$$

Identical premier provision:

$$y_\alpha(0) = 0, D_t^\alpha [y_\alpha(t)](0) = 1 \quad (48)$$

Solution: Advantage the (FZMAT) for (47) get like:

$$\Lambda_\alpha \{ (\mathcal{D}_t^\alpha)^2 [y_\alpha(t)] + 2 (D_t^\alpha) [y_\alpha(t)] + 5 y_\alpha(t) \} = \Lambda_\alpha \{ E_\alpha(-t^\alpha) \otimes \sin_\alpha(t^\alpha) \} \quad (49)$$

$$\left[\frac{1}{(\mathfrak{s}v)^{2\alpha}} \mathbb{Z}_{\text{MA}}^\alpha(v; \mathfrak{s}) - \frac{1}{(\mathfrak{s}v)^{2\alpha}} y_\alpha(0) - \frac{1}{(\mathfrak{s}v)^\alpha} D_t^\alpha [y_\alpha(t)](0) \right] + 2 \left[\frac{1}{(\mathfrak{s}v)^\alpha} \mathbb{Z}_{\text{MA}}^\alpha(v; \mathfrak{s}) - \frac{1}{(\mathfrak{s}v)^\alpha} y_\alpha(0) + \right] + 5 \mathbb{Z}_{\text{MA}}^\alpha(v; \mathfrak{s}) = \frac{1}{(1 + (\mathfrak{s}v))^\alpha} \frac{(\mathfrak{s}v)^\alpha}{(1 + (\mathfrak{s}v)^2)^\alpha} \quad (50)$$

$$\rightarrow \mathbb{Z}_{\text{MA}}^{\alpha}(v; s) \left[\frac{1}{(sv)^{2\alpha}} + \frac{2}{(sv)^{\alpha}} + 5 \right] = \frac{1}{(sv)^{\alpha}} + \frac{1}{(1+(sv))^{\alpha}} \frac{(sv)^{\alpha}}{1+(sv)^{2\alpha}} \quad (51)$$

$$\rightarrow \mathbb{Z}_{\text{MA}}^{\alpha}(v; s) = \left[\frac{1}{(sv)^{\alpha}} + \frac{1}{(1+(sv))^{\alpha}} \frac{(sv)^{\alpha}}{(1+(sv)^2)^{\alpha}} \right] \cdot \left[\frac{(sv)^{2\alpha}}{(1+2(sv)^{\alpha}+5(sv)^{2\alpha})} \right] \quad (52)$$

$$\rightarrow \mathbb{Z}_{\text{MA}}^{\alpha}(v; s) = \left[\frac{1}{(sv)^{\alpha}} + \frac{1}{(1+(sv))^{\alpha}} \frac{(sv)^{\alpha}}{(1+(sv)^2)^{\alpha}} \right] \cdot \left[\frac{(sv)^{2\alpha}}{((sv)^{\alpha}+1)^2+(2(sv)^{\alpha})^2} \right] \quad (53)$$

$$\rightarrow \mathcal{Y}_{\alpha}(\mathfrak{t}) = \mathbb{E}_{\alpha}(-\mathfrak{t}^{\alpha})[\sin_{\alpha}(2\mathfrak{t}^{\alpha}) + \mathfrak{t}^{\alpha}(\sin_{\alpha}(\mathfrak{t}^{\alpha}))] \quad (54)$$

5. Conclusion

A new technique, the Fractional ZMA Transform (FZMAT), is introduced to solve first- and second-order LFDEs. A new definition of FZMAT enhances efficiency, enabling the transformation of major functions. The study also examines the relationship between ZMAT and LT. Overall, this method proves highly accurate and effective for solving LFDEs.

References

- [1] I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*, vol. 198, Mathematics in Science and Engineering, San Diego, CA, USA: Academic Press, (1999).
- [2] S. Das, *Functional Fractional Calculus*, 2nd ed. Berlin, Germany: Springer-Verlag (2011).
- [3] E. S. Bhaya, E. A. Hessen, and A. H. Ibrahim, "Dunkl weighted approximation of functions," *AIP Conf. Proc.*, vol. 2845, no. 1, Art. no. 050034 (Sept. 2023).
- [4] A. H. Ibrahim, "Inverse approximation theorem using Bernstein's Durrmeyer operator in terms of 2nd modulus of smoothness," *J. Interdiscip. Math.*, vol. 27, no. 4, pp. 815–819 (2024).
- [5] G. Jumarie, "Table of some basic fractional calculus formulae derived from a modified Riemann–Liouville derivative for non-differentiable functions," *Appl. Math. Lett.*, vol. 22, no. 3, pp. 378–385 (2009).
- [6] R. F. Khalaf and F. B. M. Belgacem, "Extraction of the Laplace, Fourier, and Mellin transforms from the Sumudu transform," *AIP Conf. Proc.*, vol. 1637, no. 1, pp. 1426–1432 (Dec. 2014).
- [7] A. K. Thakur and S. Panda, "Some properties of triple Laplace transform," *J. Math. Comput. Appl. Res.*, vol. 5, no. 2, pp. 2250–2408 (2015).
- [8] Z. H. Khan and W. A. Khan, "N-transform properties and applications," *NUST J. Eng. Sci.*, vol. 1, no. 1, pp. 127–133 (2008).

- [9] F. B. M. Belgacem and A. A. Karaballi, "Sumudu transform fundamental properties investigations and applications," *J. Appl. Math. Stochastic Anal.*, vol. 2006, Art. no. 91083, pp. 1–23 (2006), doi: 10.1155/JAMSA/2006/91083.
- [10] T. M. Elzaki, E. M. Hilal, and J. S. Arabia, "Solution of telegraph equation by modified double Sumudu transform," *Math. Theory Model.*, vol. 2, no. 4, pp. 95–103 (2012).
- [11] J. A. Jasim, M. R. Ali, and E. A. Kuffi, "Application of a new integral 'KAJ transform' technique in cryptography," *J. Discrete Math. Sci. Cryptogr.*, vol. 28, no. 2, pp. 311–319 (2025), doi: 10.47974/JDM-SC-1856.
- [12] Z. M. Alwan, "Solution homogeneous and non-homogeneous fractional ordinary differential equations by ZMA-transform," *J. Interdiscip. Math.*, vol. 27, no. 4, pp. 781–785 (2024).
- [13] K. B. Oldham and J. Spanier, *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order*, vol. 111, Mathematics in Science and Engineering. New York, NY, USA: Academic Press, (1974).
- [14] Z. M. Alwan, "ZMA-transform method," *J. Interdiscip. Math.*, vol. 24, no. 7, pp. 1841–1849 (2021), doi: 10.1080/09720502.2021.1963520.
- [15] Z. M. Alwan, "Solution of nonlinear partial differential equations by ZMA decomposition method," *WSEAS Trans. Math.*, vol. 20, pp. 712–716 (2021), doi: 10.37394/23206.2021.20.75.
- [16] G. Jumarie, "Laplace's transform of fractional order via the Mittag-Leffler function and modified Riemann–Liouville derivative," *Applied Mathematics Letters*, vol. 22, no. 11, pp. 1659–1664 (2009).

Received November, 2025