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Structural novel properties of generalized quasi-fuzzy soft paranormal operators in Hilbert spaces domain

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Abstract

This article introduces a new and extended concept of quasi-fuzzy soft paranormal operators of great importance in pure mathematics, called $(\Bbbk - \Omega)$ – quasi-fuzzy soft paranormal operators) in brief "$(\Bbbk - \Omega)$ – QFSPO", which is a generalization of the quasi-fuzzy soft paranormal operators. It also mentions some of its properties and proves the relevant theorems. To make this more robust, it is compared with other types, and the results are recorded. We seek to expand the theoretical framework in ways that differ from traditional methods, while providing a basic foundation for similar future work.

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1. Introduction

Fuzzy theory (FT) is essential and important in applied and pure mathematics, as functional analysis is one of the crucial fields in mathematics due to its many connections [1-6]. In 1965, the author presented his theory for the first time (fuzzy theory), which contributed significantly to dealing with uncertainty and analyzing uncertain data [7]. After that, many researchers applied fuzzy theory to solve complex problems across different fields. See: [8-10]. Then, in 1999, another author introduced his theory (soft theory) to improve decision-making on fuzzy theory [11]. In the same field, the Authors in 2020 presented some important definitions of the soft fuzzy set within the theory of operators [12]. On the other_hand, in 2023, another author presented a new work focusing on the definition of supernatural soft fuzzy operators and on proving

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algebraic theorems and properties, such as the addition and multiplication of two operators [13]. Authors [14] in 2020 developed the fixed-points theorems of partials rectangular metric-like spaces using a comparison function. To make the soft fuzzy set broader and more comprehensive, researchers introduced concepts of the soft fuzzy point (FSP) and the fuzzy soft normed space (FSNS).

Recently, the authors [15] In 2001, they introduced the soft fuzzy inner product space (FSIN-space) and the soft fuzzy Hilbert space (FSHS). In [16] Another author in 2015 also introduced the Fuzzy Soft Operator Theory (FS-OT) by studying certain fuzzy soft linear operators on FS-H, including fuzzy soft right operators & left shift operators, and discussing a variety of related merits of Fuzzy Soft Spectral Theory (FS-ST). In addition, several scholars proposed various applications of FSS theory [17]. Soft fuzzy theory combines fuzzy logic and soft sets, giving it a broader scope for applications in artificial intelligence, disease diagnosis, and business management. There is a lot of research in this regard [18-20] . In the same field in 2022, the author presented a new generalization and proved several important theorems, demonstrating that addition and multiplication hold under certain conditions. This was in the context of quasi - normals operator in fuzzy soft Hilbert space [21]. Despite the great progress in research and technology worldwide, the study of operators in the fuzzy-soft environment is still in its infancy. It has not yet reached the level of generalization commensurate with the importance of this environment, which is unique in its kind, as it participates in solving differential equations and quantum physics. On this basis, this work seeks to reduce the gap by presenting our new generalization and addressing the most important structural characteristics that distinguish it from other research [22, 23].

2. Preliminaries:

This part of the research paper presents the most important and necessary definitions for the next section.

Definition 2.1: [15] Let U be a non-empty set with L is close interval, $L = [0,1]$ fuzzy sets $\tilde{A}$ onthe U be functions $\tilde{A}: U \to L$ and can be given by $\tilde{A} = \{(x, \mu_{\tilde{A}}(x): x \text{ belongs } to\ U\}$ and $\mu_{\tilde{A}}(x)$ it's indicated memberships for the x in a fuzzy $\tilde{A}$ where x belongs to U.

Definition 2.2: [11] Assume that H is function $H : \tilde{A} \to P(U)$, for each $\tilde{A} \subseteq D$, D is a set of attributes where U is universe set and a soft set $H_{\tilde{A}}$ on U given by $H_{\tilde{A}} = \{H(e) \in P(U): e \in \tilde{A}\}$.

Definition 2.3: [12] Assume that $H_{\tilde{A}}$ be soft sets & it said fuzzy soft sets on U. Whene H is mapping from $\widetilde{A}$ to I^U, I^U is indicated as the family of every fuzzy subset from U, symbolize as FSS.

Definition 2.4: [18] The fuzzy soft point (FSP) on U, It is a particular instance of FSS (FS-Point) on U, If $x \in U$ and e $\in A$, such that:

$$\mu_{H(e)}(x) = \begin{cases} \alpha \text{ if } x = x_o \text{ and } e = e_0 \in \tilde{A} \\ 0 \text{ if } x \in U - \{x_0\} \text{ or } e \in A - \{e_0\} \text{ where } \alpha \in (0,1]. \end{cases}$$

Definition 2.5: [18] let $\widetilde{U}$ FS-vector, and a function $\|\widetilde{.}\|: \widetilde{U} \to \tilde{R}^{+}(\tilde{A})$ is said FS-Norm of $\widetilde{U}$, if $\|\widetilde{.}\|$ fulfills the next conditions:

(a) $\|\widetilde{\tilde{x}_{\mu_{H(e)}}}\| \geq \tilde{0}\ \forall \tilde{x}_{\mu_{H(e)}} \in \widetilde{U}$, and $\|\widetilde{\tilde{x}_{\mu_{H(e)}}}\| = \tilde{0} \leftrightarrow \tilde{x}_{\mu_{H(e)}} = \tilde{\theta}$.

(b) $\|\widetilde{\tilde{\mu} \cdot \tilde{x}_{\mu_{H(e)}}}\| = |\tilde{\tilde{\mu}}| \|\widetilde{\tilde{x}_{\mu_{H(e)}}}\|, \forall \tilde{x}_{\mu_{H(e)}}$ belong $\widetilde{U}, \forall \tilde{\mu}$ belong $\tilde{R}^{+}(\tilde{A})$.

(c) $\left\|\widetilde{\tilde{x}_{\mu_{1H(e_1)}}}\right\| + \left\|\widetilde{\tilde{y}_{\mu_{2H(e_2)}}}\right\| \geq \|\widetilde{\tilde{x}_{\mu_{1H(e_1)}} + \tilde{y}_{\mu_{2H(e_2)}}}\|\ \forall \tilde{x}_{\mu_{1H(e_1)}}, \tilde{y}_{\mu_{2H(e_2)}}$ belong $\widetilde{U}$.

$\widetilde{U}$ & FSN $\|\widetilde{.}\|$ It named fuzzy soft Normed spaces (FSNS).

Definition 2.6: [18] A fuzzy soft vector spaces (real or complex) $\widetilde{H}$ with the inners product and be complete relative to criterion derived from the inner products, then $\widetilde{H}$ is as fuzzy soft Hilbert space it and symbolize as FSHS.

Definition 2.7: [18] Let $\widetilde{H}$ is FSHS and let the function $\tilde{\Lambda}: \widetilde{H} \to \widetilde{H}$ be fuzzy soft operators, thus $\tilde{\Lambda}$ it said fuzzy soft linear operators (FSLO) achieves:

a) $\tilde{\Lambda}(\widetilde{\tilde{x}_{\mu 1H(e1)}}) + \tilde{\Lambda}(\widetilde{\tilde{y}_{\mu 2H(e2)}}) = \tilde{\Lambda}(\widetilde{\tilde{x}_{\mu 1H(e1)} + \tilde{y}_{\mu 2H(e2)}})$,
for evry $\tilde{x}_{\mu 1H(e1)}, \tilde{y}_{\mu 2H(e2)}$ belongs to $\widetilde{H}$.

b) $\tilde{\Lambda}(\widetilde{\tilde{\mu}\tilde{x}_{\mu 1H_{(e1)}}}) = \tilde{\mu}\tilde{\Lambda}(\widetilde{\tilde{x}_{\mu 1H_{(e1)}}})$, for evry $\tilde{x}_{\mu 1H(e1)}$ belongs to $\widetilde{H}$ with $\tilde{\mu}$ belongs to $\tilde{R}^{+}(\tilde{A})$.

Definition 2.8 [19] Assume that $\widetilde{H}$ is FSH-space and $\tilde{\Lambda}$ belongs to $\widetilde{\mathrm{B}}(\widetilde{H})$, thus $\tilde{\Lambda}^{*}$is FS $-$ adjoint operator and given as:

$\langle \widetilde{\tilde{x}_{\mu 1H_{(e1)}}, \tilde{\Lambda}^{*}\tilde{y}_{\mu 2H_{(e2)}}} \rangle = \langle \widetilde{\tilde{\Lambda}\tilde{x}_{\mu 1H_{(e1)}}, \tilde{y}_{\mu 2H_{(e2)}}} \rangle\ \forall \tilde{x}_{\mu 1H_{(e1)}}, \tilde{y}_{\mu 2H_{(e2)}}$ belongs to $\widetilde{H}$.

Definition 2.9: [19] Assume that $\widetilde{H}$ be FSHS and $\tilde{\Lambda}$ belongs to $\widetilde{\mathrm{B}}(\widetilde{H})$. if $\tilde{\Lambda}\tilde{\Lambda}^{*} \cong \tilde{\Lambda}^{*}\tilde{\Lambda}$ then $\tilde{\Lambda}$ is indicated as a fuzzy soft normal operator (FSN).

Definition 2.10: [19] Let $\tilde{\Lambda}$ FS-operator of $\widetilde{H}$ and is said FS-self-adjoint operator if $\tilde{\Lambda} \cong \tilde{\Lambda}^{*}$.

Definition 2.11: [21] Assume that $\widetilde{H}$ is FSHS and $\tilde{\Lambda}$ belongs to $\widetilde{\mathrm{B}}(\widetilde{H})$. thus $\tilde{\Lambda}$ named fuzzy soft- isometry operators where achieved $\langle \widetilde{\tilde{\Lambda}\,\tilde{x}_{\mu 1H(e1)}, \tilde{\Lambda}\,\tilde{y}_{\mu 2H(e2)}} \rangle \cong \langle \widetilde{\tilde{x}_{\mu 1H(e1)}, \tilde{y}_{\mu 2H(e2)}} \rangle\ \forall\ \tilde{x}_{\mu 1H(e1)}, \tilde{y}_{\mu 2H(e2)}\ \tilde{\in}\ \widetilde{H}$, where $\widetilde{H}$ It is a Hilbert Space.

Definition 2.12: [19] Assume that $\widetilde{H}$ is FSHS and $\tilde{\Lambda}$ belongs to $\widetilde{\mathrm{B}}(\widetilde{H})$, then $\tilde{\Lambda}$ is said fuzzy soft-unitary operator if achieve $\tilde{\Lambda}\tilde{\Lambda}^{*} = \tilde{\Lambda}^{*}\tilde{\Lambda} = I$.

Definition 2.13: [19] Assume that $\widetilde{H}$ be an FSHS and $\widetilde{\mathrm{T}}$ belongs to $\widetilde{\mathrm{B}}(\widetilde{H})$. thus $\tilde{\Lambda}$ is indicated the FS-paranormal operator if achieved $\|\Lambda^2\widetilde{\tilde{x}_{\mu H(e)}}\|\|\widetilde{\tilde{x}_{\mu H(e)}}\| \gtreqqless \|\tilde{\Lambda}\widetilde{\tilde{x}_{\mu H(e)}}\|^2 \forall\ \tilde{x}_{\mu H(e)}\ \widetilde{\in}\ \widetilde{H}$ and symbolizes by (FSPO).

3. Result discussions

Definition 3.1: Let $\widetilde{H}$ is FSHS with $\tilde{\Lambda}$ belongs in $\widetilde{\mathrm{B}}(\widetilde{H})$, then $\tilde{\Lambda}$ is said $(k,m)-$ quasi fuzzy soft paranormal operator, where Ω and $\Bbbk$ are non-negative integers, in brief $(\Bbbk-\Omega)-QFSPO$ if the condition below is met:

$$\left\|\widetilde{\tilde{\Lambda}^{2+\Omega}\tilde{x}_{\mu H(e)}}\right\|^2 \lesseqqgtr \Bbbk\|\widetilde{\tilde{\Lambda}^{3+\Omega}\tilde{x}_{\mu H(e)}}\|\|\widetilde{\tilde{\Lambda}^{1+\Omega}\tilde{x}_{\mu H(e)}}\| \text{ for all } \tilde{x}_{\mu H(e)}\ \widetilde{\in}\ \widetilde{H}\ .$$

Let $\widetilde{H}$ is (FSHS), & suppose that operator $\tilde{\Lambda}$ It works on the space elements in a way that preserves the fuzzy soft structure but slightly shifts it. When checking the condition in the definition, we find that $\tilde{\Lambda}$ achieves the inequality $(\Bbbk-\Omega)-QFSPO$. Thus, we can take any simple numerical example and see that the definition achieves an extension of the classical definitions.

The following Theorem provides the equivalent definition of $(\Bbbk-\Omega)-QFSPO$.

Theorem 3.2: *An operator $\tilde{\Lambda}$ belongs in $\widetilde{\mathrm{B}}(\widetilde{H})$, is $(\Bbbk-\Omega)-QFSPO$ if and only if $k^2\tilde{\Lambda}^{*3+\Omega}\tilde{\Lambda}^{3+\Omega}-2\xi\tilde{\Lambda}^{*2+\Omega}\Lambda\tilde{\Lambda}^{2+\Omega}+\lambda^2\tilde{\Lambda}^{*1+\Omega}\tilde{\Lambda}^{1+\Omega}\gtreqqless 0$ for all non-negative ξ.*

Proof: for all $\tilde{x}_{\mu H(e)}\ \widetilde{\in}\ \widetilde{H}$**,** we have $\tilde{\Lambda}$ is a $(\Bbbk-\Omega)-QFSPO$, assume that $\Bbbk\in R^+$is fixed : Consider this condition $\left\|\widetilde{\tilde{\Lambda}^{2+\Omega}\tilde{x}_{\mu H(e)}}\right\|^2 \lesseqqgtr \Bbbk\|\widetilde{\tilde{\Lambda}^{3+\Omega}\tilde{x}_{\mu H(e)}}\|$ $\|\widetilde{\tilde{\Lambda}^{1+\Omega}\tilde{x}_{\mu H(e)}}\|$ Then we get the following

$$\left\|\widetilde{\tilde{\Lambda}^{2+\Omega}\tilde{x}_{\mu H(e)}}\right\|^2-\Bbbk\left\|\widetilde{\tilde{\Lambda}^{3+\Omega}\tilde{x}_{\mu H(e)}}\right\|\left\|\widetilde{\tilde{\Lambda}^{1+\Omega}\tilde{x}_{\mu H(e)}}\right\|\lesseqqgtr 0$$

$$\text{i.e. } 4\left(\left\|\widetilde{\tilde{\Lambda}^{2+\Omega}\tilde{x}_{\mu H(e)}}\right\|^2\right)^2-4\Bbbk^2\left\|\widetilde{\tilde{\Lambda}^{3+\Omega}\tilde{x}_{\mu H(e)}}\right\|^2\left\|\widetilde{\tilde{\Lambda}^{1+\Omega}\tilde{x}_{\mu H(e)}}\right\|^2\lesseqqgtr 0$$

$$\xi^2\|\widetilde{\tilde{\Lambda}^{1+\Omega}\tilde{x}_{\mu H(e)}}\|-2\xi\left\|\widetilde{\tilde{\Lambda}^{2+\Omega}\tilde{x}_{\mu H(e)}}\right\|^2+\Bbbk^2\left\|\widetilde{\tilde{\Lambda}^{3+\Omega}\tilde{x}_{\mu H(e)}}\right\|^2\gtreqqless\ 0\ ,\forall\xi>0.$$

$$\xi^2\langle\tilde{\Lambda}^{*1+\Omega}\widetilde{\tilde{\Lambda}^{1+\Omega}}\tilde{x}_{\mu H(e)},\tilde{x}_{\mu H(e)}\rangle-2\xi\langle\tilde{\Lambda}^{*2+\Omega}\tilde{\Lambda}^{2+\Omega}\tilde{x}_{\mu H(e)},\tilde{x}_{\mu H(e)}\rangle$$
$$+\Bbbk^2\langle\tilde{\Lambda}^{*3+\Omega}\tilde{x}_{\mu H(e)},\widetilde{\mathrm{T}}^{3+\Omega}\tilde{x}_{\mu H(e)}\rangle\gtreqqless 0,$$

$$\langle\Bbbk^2\tilde{\Lambda}^{*3+\Omega}\tilde{\Lambda}^{3+\Omega}\tilde{x}_{\mu H(e)}-2\xi\ \tilde{\Lambda}^{*2+\Omega}\tilde{\Lambda}^{2+\Omega}\tilde{x}_{\mu H(e)}$$
$$+\lambda^2\tilde{\Lambda}^{*1+\Omega}\tilde{\Lambda}^{1+\Omega}\tilde{x}_{\mu H(e)},\tilde{x}_{\mu H(e)}\rangle\gtreqqless 0,\forall\xi>0.$$

Hence, $\Bbbk^2\tilde{\Lambda}^{*3+\Omega}\tilde{\Lambda}^{3+\Omega}-2\xi\tilde{\Lambda}^{*2+\Omega}\tilde{\Lambda}^{2+\Omega}+\lambda^2\tilde{\Lambda}^{*1+\Omega}\tilde{\Lambda}^{1+\Omega}\gtreqqless 0,\forall\xi>0.$

Regarding the reverse conclusion, you can go back to the previous steps.

Proposition 3.3: Assume $\tilde{\Lambda}$ belongs in $\widetilde{\mathrm{B}}(\widetilde{H})$ is $(\Bbbk-\Omega)-QFSPO$, if $\widetilde{\mathbb{S}}$ is unitary equivalents to the operators $\tilde{\Lambda}$, so $\widetilde{\mathbb{S}}$ be $(\Bbbk-\Omega)-QFSPO$.

Proof: If $\tilde{\Lambda}$ belongs in $\widetilde{B}(\widetilde{H})$ is $(\Bbbk - \Omega) - QFSPO$, because $\widetilde{\mathbb{S}}$ is unitary equivalent to operator $\tilde{\Lambda}$, then $\exists\ \mathbb{A}$ is a unitary operator so $\widetilde{\mathbb{S}} = \mathbb{A}^{*}\tilde{\Lambda}\,\mathbb{u}$ because $\tilde{\Lambda}$ is $(\Bbbk - \Omega) - QFSPO$ then from Theorem 3.2 we have:

$$\Bbbk^2 \tilde{\Lambda}^{*3+\Omega}\tilde{\Lambda}^{3+\Omega} - 2\xi\tilde{\Lambda}^{*2+\Omega}\tilde{\Lambda}^{2+\Omega} + \xi^2\tilde{\Lambda}^{*1+\Omega}\tilde{\Lambda}^{1+\Omega} \gtrapprox 0, \forall \xi > 0.$$

Hence, $\Bbbk^2 \widetilde{\mathbb{S}}^{*3+\Omega}\widetilde{\mathbb{S}}^{3+\Omega} - 2\xi\widetilde{\mathbb{S}}^{*2+\Omega}\widetilde{\mathbb{S}}^{2+\Omega} + \xi^2\widetilde{\mathbb{S}}^{*1+\Omega}\widetilde{\mathbb{S}}^{1+\Omega} \gtrapprox 0, \forall \xi > 0.$

$$= \Bbbk^2\left(\mathbb{A}^*\tilde{\Lambda}\,\mathbb{A}\right)^{*3+\Omega}\left(\mathbb{A}^*\tilde{\Lambda}\,\mathbb{A}\right)^{3+\Omega} - 2\xi\left(\mathbb{A}^*\tilde{\Lambda}\,\mathbb{A}\right)^{*2+\Omega}\left(\mathbb{A}^*\tilde{\Lambda}\,\mathbb{A}\right)^{2+\Omega} + \xi^2\left(\mathbb{A}^*\tilde{\Lambda}\,\mathbb{A}\right)^{*1+\Omega}\left(\mathbb{A}^*\tilde{\Lambda}\,\mathbb{A}\right)^{1+\Omega} \gtrapprox 0,$$

$$= \mathbb{A}^*\left(\tilde{\Lambda}^{*3+\Omega}\tilde{\Lambda}^{3+\Omega} - 2\xi\tilde{\Lambda}^{*2+\Omega}\tilde{\Lambda}^{2+\Omega} + \xi^2\tilde{\Lambda}^{*1+\Omega}\Lambda^{1+\Omega}\right)\mathbb{A} \gtrapprox 0, \forall \xi > 0.$$

So $\widetilde{\mathbb{S}}$ is $(\Bbbk - \Omega) - QFSPO$.

Theorem 3.4: *Let* $\tilde{\Lambda}$ *belongs in* $\widetilde{B}(\widetilde{H})$ *is invertible* $(\Bbbk - \Omega) - QFSPO$ *it follows that* $\tilde{\Lambda}^{-1}$ *also* $(\Bbbk - \Omega) - QFSPO$.

Proof: Given that $\tilde{\Lambda}$ is a $(\Bbbk - \Omega) - QFSPO$, then $\|\widetilde{\tilde{\Lambda}^{2+\Omega}\tilde{x}_{\mu H(e)}}\|^2 \lessapprox \Bbbk \left\|\widetilde{\tilde{\Lambda}^{3+\Omega}\tilde{x}_{\mu H(e)}}\right\| \left\|\widetilde{\tilde{\Lambda}^{1+\Omega}\tilde{x}_{\mu H(e)}}\right\| \forall\ \tilde{x}_{\mu H(e)}$ belongs to $\widetilde{H}$

$$\frac{\|\widetilde{\tilde{\Lambda}^{2+\Omega}\tilde{x}_{\mu H(e)}}\|}{\|\widetilde{\tilde{\Lambda}^{3+\Omega}\tilde{x}_{\mu H(e)}}\|} \lessapprox \Bbbk \frac{\|\widetilde{\tilde{\Lambda}^{1+\Omega}\tilde{x}_{\mu H(e)}}\|}{\|\widetilde{\tilde{\Lambda}^{2+\Omega}\tilde{x}_{\mu H(e)}}\|}, \tilde{x}_{\mu H(e)} \neq 0$$

$\widetilde{\forall x_{\mu H(e)}}\ \widetilde{\in}\ \widetilde{H}$ by replace $\tilde{x}_{\mu H(e)}$ with $\tilde{\Lambda}^{-3-2\Omega}\,\tilde{y}$, we obtain$\tilde{y}\ \widetilde{\in}\ \widetilde{H}$

$\frac{\left\|\widetilde{\tilde{\Lambda}^{2+\Omega}\tilde{\Lambda}^{-3-2\Omega}\,\tilde{y}}\right\|}{\left\|\widetilde{\tilde{\Lambda}^{3+\Omega}\tilde{\Lambda}^{-3-2\Omega}\,\tilde{y}}\right\|} \lessapprox \Bbbk \frac{\left\|\widetilde{\tilde{\Lambda}^{1+\Omega}\tilde{\Lambda}^{-3-2\Omega}\,\tilde{y}}\right\|}{\left\|\widetilde{\tilde{\Lambda}^{2+\Omega}\tilde{\Lambda}^{-3-2\Omega}\,\tilde{y}}\right\|}$ so $\frac{\|\widetilde{\tilde{\Lambda}^{-1-\Omega}\,\tilde{y}}\|}{\|\widetilde{\tilde{\Lambda}^{-\Omega}\,\tilde{y}}\|} \lessapprox \frac{\Bbbk\|\widetilde{\tilde{\Lambda}^{-2-\Omega}\,\tilde{y}}\|}{\|\widetilde{\tilde{\Lambda}^{-1-\Omega}\tilde{y}}\|}$, assume that $\tilde{\Lambda}^{-1} = \tilde{\Gamma}$ we get $\frac{\|\widetilde{\tilde{\Gamma}^{1+\Omega}\,\tilde{y}}\|}{\|\widetilde{\tilde{\Gamma}^{\Omega}\,\tilde{y}}\|} \lessapprox \frac{\Bbbk\|\widetilde{\tilde{\Gamma}^{2+\Omega}\,\tilde{y}}\|}{\|\widetilde{\tilde{\Gamma}^{1+\Omega}\tilde{y}}\|}$ After rearranging, we get the definition of $(\Bbbk - \Omega) - QFSPO$ $\|\widetilde{\tilde{\Gamma}^{2+\Omega}\tilde{y}}\|^2 \lessapprox \Bbbk \left\|\widetilde{\tilde{\Gamma}^{3+\Omega}\tilde{y}}\right\| \left\|\widetilde{\tilde{\Gamma}^{1+\Omega}\tilde{y}}\right\|$

Sence $\tilde{\Lambda}^{-1} = \tilde{\Gamma}$, $\tilde{\Lambda}^{-1}$ also $(\Bbbk - \Omega) - QFSPO$.

Proposition 3.5: Assume $\tilde{\Lambda}$ belongs in $\widetilde{B}(\widetilde{H})$ is $(\Bbbk - \Omega) - QFSPO$, if $\tilde{\Lambda}$ is isometric operator $\tilde{\mathcal{T}}$, then $\tilde{\Lambda}\tilde{\mathcal{T}}$ is $(\Bbbk - \Omega) - QFSPO$.

Proof: Let $\tilde{\mathcal{T}}$ is isometric operator, and assume that $D = \tilde{\mathrm{T}}\tilde{\mathcal{T}}$

We have the given$\tilde{\Lambda}$ commutes with the operator $\mathcal{T}$ so $\tilde{\Lambda}\tilde{\mathcal{T}} = \tilde{\mathcal{T}}\tilde{\mathrm{T}}$, $\tilde{\Lambda}\tilde{\mathcal{T}}^* = \tilde{\mathcal{T}}^*\tilde{\Lambda}$ and $\tilde{\mathcal{T}}^*\tilde{\mathcal{T}} = I$, So

$$\Bbbk^2 D^{*3+\Omega}D^{3+\Omega} - 2\lambda D^{*2+\Omega}D^{2+\Omega} + \lambda^2 D^{*1+\Omega}D^{1+\Omega} \gtrapprox 0,$$

$$= \Bbbk^2(\tilde{\Lambda}\tilde{\mathcal{T}})^{*3+\Omega}(\tilde{\Lambda}\tilde{\mathcal{T}})^{3+\Omega} - 2\xi(\tilde{\Lambda}\tilde{\mathcal{T}})^{*2+\Omega}(\tilde{\Lambda}\tilde{\mathcal{T}})^{2+\Omega} + \lambda^2(\tilde{\Lambda}\tilde{\mathcal{T}})^{*1+\Omega}(\tilde{\Lambda}\tilde{\mathcal{T}})^{1+\Omega} \gtrapprox 0,$$

$= \Bbbk^2\tilde{\Lambda}^{*3+\Omega}\tilde{\mathrm{T}}^{3+\Omega} - 2\xi\tilde{\Lambda}^{*2+\Omega}\tilde{\Lambda}^{2+\Omega} + \xi^2\tilde{\Lambda}^{*1+\Omega}\tilde{\Lambda}^{1+\Omega} \gtrapprox 0$, So $\tilde{\Lambda}\tilde{\mathcal{T}}$ is $(\Bbbk - \Omega) - QFSPO$.

Remark 3.6: Every $(1,0) - QFSPO$ is $QFSPO$.

Proof: We have $\tilde{\Lambda}$ is $(\Bbbk - \Omega) - QFSPO$ putting $\Bbbk = 1$ and $\Omega = 0$ we get $\left\|\widetilde{\tilde{\Lambda}^2\tilde{x}_{\mu H(e)}}\right\|^2 \lessapprox \|\widetilde{\tilde{\Lambda}^3\tilde{x}_{\mu H(e)}}\|\|\widetilde{\tilde{\Lambda}^1\tilde{x}_{\mu H(e)}}\|$ is $QFSPO$.

4. Discussion and Conclusion

In this research, the quasi-fuzzy soft paranormal operator was extended by providing a new definition "$(\mathbb{k} - \Omega) - QFSPO$", which is considered a new idea within the Hilbert space, especially since we discussed it in the soft fuzziness. The basic idea of this type was presented, mentioning the related properties such as the inverse and the equivalent theorem of the definition as well as the composition. With the expansion of the theoretical framework quasi-fuzzy soft paranormal operator, the structural properties and characteristics that support the validity of the definition and give it strength were identified, especially since our generalization relates to other types of operators. Finally, some related theorems were proven.

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