Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 29 (2026), No. 5-B, pp. 1497–1502
DOI : 10.47974/JIM-2538

Solving the liquid tank system by generalization FAT

Azhar A. Abbas *
Department of Information Technology
College of Computer Science & Information Technology
University of Kerbala
Karbala 56001
Iraq

Masar F. Jabbar †
Mohammed A. Wannas §
Department of Mathematics
College of Education for Pure Sciences
Wasit University
Wasit 52001
Iraq

Abstract

In this work, we presented a generalization of FAT and proved some of the properties of (FAT). Eng. We also derive the FAT of fuzzy derivatives up to the third order and further generalize the said formula for the nth order fuzzy derivative by using the strongly generalized H-differentiability concept. The obtained formulas are then employed to solve the FIVP of a first-order liquid tank system.

Subject Classification: *Primary 93A30, Secondary 49K15.*

Keywords: *A-transform, FAT, Generalization of FAT, Liquid tank system.*

1. Introduction

The fuzzy integral transform is used to obtain solutions to various realistic mathematical problems. Due to their practical significance, many researchers have employed integrals transform, like as Laplace, Sumudu, and Fourier transforms, to solve differential equations. In recent years, some researchers have transformed some standard transforms into fuzzy transforms to solve fuzzy differential equations in their respective articles, such as FLT [1-6] and FST [7-

* *E-mail:* azhar.ali@uokerbala.edu.iq (Corresponding Author)
† *E-mail:* majabbar@uwasit.edu.iq
§ *E-mail:* mwannas@uowasit.edu.iq

© Taru Publications

13]. Now we have a new transform, named A. Hameed, that has been employed to aid in solving ordinary and partial differential equations in the time domain [14-18]. Thus, in the present work, we seek fuzzy derivatives of FAT as well as extended formulas for fuzzy n-th order derivatives and establish some properties for this transformation.

2. The main results.

Definition 1: Assume that $\mathcal{Y}(\upsilon)$ be an integrable function defined for $\upsilon \geq 0$, assume that $J(\csc(\rho)) \neq 0$ and $\mathcal{Y}(\csc(\rho)) \neq 0$ be positive functions S.T $J(\csc(\rho))\csc(\rho)\left(\sqrt[2\mathfrak{t}+1]{\csc(\rho)}\right)$ and $\mathcal{Y}(CSC(\rho)) = (\sqrt[2\mathfrak{t}+1]{CSC(\rho)})$, we define the integral transform A $S(CSC(\rho))$ of $\mathcal{Y}(\upsilon)$ by the formula:

$$S(CSC(\rho)) = A\{\mathcal{Y}(\upsilon), CSC(\rho)\} = CSC(\rho)^2 \int_0^\infty \mathcal{Y}(\upsilon) e^{-\sqrt[2\mathfrak{t}+1]{CSC(\rho)}\upsilon} d\upsilon$$

The variable $CSC(\rho)$ in these transforms, be used to the factor's variables υ in arguments for a function $\mathcal{Y}$.

Definition 2: Assume that $\mathcal{Y}: A \to F(A)$and it is represented by $\left[\underline{\mathcal{Y}}_\alpha(\upsilon), \overline{\mathcal{Y}}_\alpha(\upsilon)\right]$. For any fixed $\alpha \in (0,1]$ assume $\underline{\mathcal{Y}}_\alpha(\upsilon)$ and $\overline{\mathcal{Y}}_\alpha(\upsilon)$ are Riemann-integrals on [a, b] $\forall b \geq \mathfrak{t}$, if exist two positive $\underline{Z}_\alpha$ & $\overline{Z}_\alpha$ S.T $\int_{\mathfrak{t}}^b \left|\underline{\mathcal{Y}}_\alpha(\upsilon)\right| d\upsilon \leq \underline{Z}_\alpha$ & $\overline{Z_\alpha}$ $\forall b \geq \mathfrak{t}$. so, $F(A)$ be the improper fuzzy Riemann integral on $[\mathfrak{t}, \infty)$ &.

$$A[\mathcal{Y}(\upsilon)] = S(\csc(\rho)) = CSC(\rho)\left(\sqrt[2\mathfrak{t}+1]{\csc(\rho)}\right)\int_0^\infty e^{-\sqrt[2\mathfrak{t}+1]{\csc(\rho)}\upsilon}\mathcal{Y}(\upsilon)d\upsilon,$$

$$CSC(\mathfrak{t})\left(\sqrt[2\mathfrak{t}+1]{CSC(\mathfrak{t})}\right)\int_0^\infty e^{-\sqrt[2\mathfrak{t}+1]{CSC(\mathfrak{t})}\upsilon}\mathcal{Y}(\upsilon)d\upsilon =$$

$$CSC(\mathfrak{t})\left(\sqrt[2\mathfrak{t}+1]{CSC(\mathfrak{t})}\right)\int_0^\infty e^{-\left(\sqrt[2\mathfrak{t}+1]{CSC(\mathfrak{t})}\right)\upsilon}\underline{\mathcal{Y}}(\upsilon;\vartheta)d\upsilon,$$

$$CSC(\rho)\left(\sqrt[2\mathfrak{t}+1]{CSC(\rho)}\right)\int_0^\infty e^{-\left(\sqrt[2\mathfrak{t}+1]{CSC(\rho)}\right)\upsilon}\overline{\mathcal{Y}}(\upsilon;\vartheta)d\upsilon$$

By using the definition of A transform, we get

$$A\left[\underline{\mathcal{Y}}(\upsilon;\vartheta)\right] = CSC(\rho)\left(\sqrt[2\mathfrak{t}+1]{CSC(\rho)}\right)\int_0^\infty e^{-\left(\sqrt[2\mathfrak{t}+1]{CSC(\rho)}\right)\upsilon}\underline{\mathcal{Y}}(\upsilon;\vartheta)d\upsilon,$$

$$A[\overline{\mathcal{Y}}(\upsilon;\vartheta)] = CSC(\rho)\left(\sqrt[2\mathfrak{t}+1]{CSC(\rho)}\right)\int_0^\infty e^{-\left(\sqrt[2\mathfrak{t}+1]{CSC(\rho)}\right)\upsilon}\overline{\mathcal{Y}}(\upsilon;\vartheta)d\upsilon$$

Then: $A[\mathcal{Y}(\upsilon;\vartheta)] = A\left[\underline{\mathcal{Y}}(\upsilon;\vartheta)\right], Z[\overline{\mathcal{Y}}(\upsilon;\vartheta)]$

were, $\mathcal{Y}(\upsilon)$ continuous fuzzy – valued function of the variable $CSC(\rho)$.

Theorem 1: *Assume that* $\mathcal{Y}(\upsilon)$ *be a fuzzy function and assume that* $CSC(\rho)^2$ *and* $\sqrt[2\mathfrak{t}+1]{CSC(\rho)}$ *are positive real functions, then:*

1. $A\{\upsilon\mathcal{Y}(\upsilon)\} = -\frac{v^2}{\sqrt[2\mathfrak{t}+1]{CSC(\rho)}}\left(\frac{A(CSC(\rho))}{CSC(\rho)^2}\right)'$
2. $A\{\upsilon^2\mathcal{Y}(\upsilon)\} = (-1)^2\frac{v^2}{\sqrt[2\mathfrak{t}+1]{CSC(\rho)}}\left(\frac{1}{\sqrt[2\mathfrak{t}+1]{CSC(\rho)}}\left(\frac{A(CSC(\rho))}{CSC(\rho)^2}\right)'\right)'$

Proof 1:

$$A\{\gamma(\upsilon), CSC(\rho)\} = CSC(\rho)^2 \int_0^{\infty} \underline{\gamma}(\upsilon;\vartheta) e^{-\left(\sqrt[2\mathfrak{t}+1]{CSC(\rho)}\right)\upsilon} d\upsilon \,, CSC(\rho)^2 \int_0^{\infty} \overline{\gamma}(\upsilon;\vartheta) e^{-\left(\sqrt[2\mathfrak{t}+1]{CSC(\rho)}\right)\upsilon} d\upsilon$$

$$\frac{A\{\gamma(\upsilon),CSC(\rho)\}}{CSC(\rho)^2} = \int_0^{\infty} \underline{\gamma}(\upsilon;\vartheta) e^{-\left(\sqrt[2\mathfrak{t}+1]{CSC(\rho)}\right)\upsilon} d\upsilon \,, \int_0^{\infty} \overline{\gamma}(\upsilon;\vartheta) e^{-\left(\sqrt[2\mathfrak{t}+1]{CSC(\rho)}\right)\upsilon} d\upsilon$$

$$\left(\frac{A\{\gamma(\upsilon),CSC(\rho)\}}{CSC(\rho)^2}\right)' = \frac{d}{dCSC(\rho)}\left[\int_0^{\infty} \underline{\gamma}(\upsilon;\vartheta) e^{-\left(\sqrt[2\mathfrak{t}+1]{CSC(\rho)}\right)\upsilon} d\upsilon \,, \int_0^{\infty} \overline{\gamma}(\upsilon;\vartheta) e^{-\left(\sqrt[2\mathfrak{t}+1]{CSC(\rho)}\right)\upsilon} d\upsilon\right]$$

$$\left(\frac{A\{\gamma(\upsilon),CSC(\rho)\}}{CSC(\rho)^2}\right)' = -\left(\sqrt[2\mathfrak{t}+1]{CSC(\rho)}\right) \int_0^{\infty} \upsilon \underline{\gamma}(\upsilon;\vartheta) e^{-\left(\sqrt[2\mathfrak{t}+1]{CSC(\rho)}\right)\upsilon} d\upsilon,$$

$$-\left(\sqrt[2\mathfrak{t}+1]{CSC(\rho)}\right) \int_0^{\infty} \upsilon \overline{\gamma}(\upsilon;\vartheta) e^{-\left(\sqrt[2\mathfrak{t}+1]{CSC(\rho)}\right)\upsilon} d\upsilon = -\left(\sqrt[2\mathfrak{t}+1]{CSC(\rho)}\right) \frac{A\left\{\upsilon \underline{\gamma}(\upsilon;\vartheta), CSC(\rho)\right\}}{CSC(\rho)^2},$$

$$-\left(\sqrt[2\mathfrak{t}+1]{CSC(\rho)}\right) \frac{A\{\upsilon \overline{\gamma}(\upsilon;\vartheta), CSC(\rho)\}}{CSC(\rho)^2}$$

$$\left(\frac{A\{\gamma(\upsilon),CSC(\rho)\}}{CSC(\rho)^2}\right)' = -\left(\sqrt[2\mathfrak{t}+1]{CSC(\rho)}\right) \frac{A\{\upsilon\gamma(\upsilon),CSC(\rho)\}}{CSC(\rho)^2}$$

Then: $A\{\upsilon\gamma(\upsilon), CSC(\rho)\} = -\frac{CSC(\rho)^2}{\left(\sqrt[2\mathfrak{t}+1]{CSC(\rho)}\right)}\left(\frac{A\{\gamma(\upsilon),CSC(\rho)\}}{CSC(\rho)^2}\right)'$

2. $A\{\upsilon\gamma(\upsilon), CSC(\rho)\} = -\frac{CSC(\rho)^2}{\left(\sqrt[2\mathfrak{t}+1]{CSC(\rho)}\right)}\left(\frac{A\{\gamma(\upsilon),CSC(\rho)\}}{CSC(\rho)^2}\right)'$

$$-\left(\sqrt[2\mathfrak{t}+1]{CSC(\rho)}\right) \frac{1}{CSC(\rho)^2} \int_0^{\infty} \upsilon^2 \underline{\gamma}(\upsilon;\vartheta) e^{-\left(\sqrt[2\mathfrak{t}+1]{CSC(\rho)}\right)\upsilon} d\upsilon,$$

$$-\left(\sqrt[2\mathfrak{t}+1]{CSC(\rho)}\right) \int_0^{\infty} \upsilon^2 \overline{\gamma}(\upsilon;\vartheta) e^{-\left(\sqrt[2\mathfrak{t}+1]{CSC(\rho)}\right)\upsilon} d\upsilon = \left(-\frac{CSC(\rho)^2}{\left(\sqrt[2\mathfrak{t}+1]{CSC(\rho)}\right)}\left(\frac{A\{\gamma(\upsilon),CSC(\rho)\}}{CSC(\rho)^2}\right)'\right) -$$

$$\frac{\left(\sqrt[2\mathfrak{t}+1]{CSC(\rho)}\right)}{CSC(\rho)^2} A\left\{\upsilon^2 \underline{\gamma}(\upsilon;\vartheta)\right\}, -\frac{\left(\sqrt[2\mathfrak{t}+1]{CSC(\rho)}\right)}{CSC(\rho)^2} A\{\upsilon^2 \overline{\gamma}(\upsilon;\upsilon)\}$$

$$= \left(-\frac{CSC(\rho)^2}{\left(\sqrt[2\mathfrak{t}+1]{CSC(\rho)}\right)}\left(\frac{A\{\gamma(\upsilon),CSC(\rho)\}}{CSC(\rho)^2}\right)'\right)'$$

Example: The simple structure of the tank is evident in Figure 1 below. Assume that $\hbar = 0$ be the inflow interruption in the tank during which the liquid experiences friction at the initial zone, $\Im$, $\gamma = 1$ is a flow regulator that can have a value of the following partnership:

$$\gamma'(\upsilon) = \frac{\hbar}{\mathrm{J}'} - \frac{1}{\mathrm{J}\gamma}\gamma(\upsilon),$$

$$\gamma(0) = \left(\gamma(0,r), \gamma(r,0)\right)$$

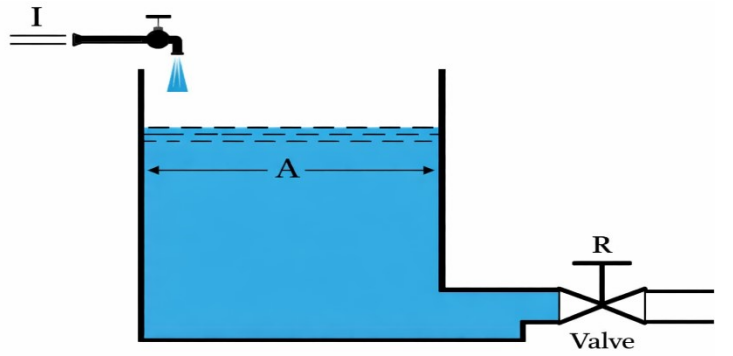


Figure 1
Schematic diagram of the studied system.

Case 1: When FAT is applied to both side for a given equation, it results in:

$$\left(\sqrt[2\mathfrak{t}+1]{CSC(\rho)}\right)A[\mathcal{Y}(\upsilon)].CSC(\rho)^2\mathcal{Y}(0) = -A[\mathcal{Y}(\upsilon)]$$

Equation becomes:

$$\left(\sqrt[2\mathfrak{t}+1]{CSC(\rho)}\right)A\left[\underline{\mathcal{Y}}(\upsilon)\right] - CSC(\rho)^2\underline{\mathcal{Y}}(0) = -A\left[\underline{\mathcal{Y}}(\upsilon)\right],$$

$$\left(\sqrt[2\mathfrak{t}+1]{CSC(\rho)}\right)A[\overline{\mathcal{Y}}(\upsilon)] - CSC(\rho)^2\overline{\mathcal{Y}}(0) = -A[\overline{\mathcal{Y}}(\upsilon)]$$

By solving equation 5, we get:

$$A\left[\underline{\mathcal{Y}}(\upsilon;\vartheta)\right] = \frac{CSC(\rho)^2}{\left(\sqrt[2\mathfrak{t}+1]{CSC(\rho)}\right)+1}\underline{\mathcal{Y}}(0;\upsilon),$$

Now we use inverse FAT for the above equation

$$\underline{\mathcal{Y}}(\upsilon;\vartheta) = e^{-\upsilon}\underline{\mathcal{Y}}(0;\upsilon), \overline{\mathcal{Y}}(\upsilon;\vartheta) = e^{-\upsilon}\overline{\mathcal{Y}}(0;\upsilon)$$

Case 2: When FAT is applied to both side for a given equation, it results in:

$$-CSC(\rho)^2\mathcal{Y}(0).\left(-\left(\sqrt[2\mathfrak{t}+1]{CSC(\rho)}\right)\right)A[\mathcal{Y}(\upsilon)] = -A[\mathcal{Y}(\upsilon)]$$

Equation becomes:

$$\left(\sqrt[2\mathfrak{t}+1]{CSC(\rho)}\right)A[\overline{\mathcal{Y}}(\upsilon)] - CSC(\rho)^2\overline{\mathcal{Y}}(0) = -A\left[\underline{\mathcal{Y}}(\upsilon)\right],$$

$$\left(\sqrt[2\mathfrak{t}+1]{CSC(\rho)}\right)A\left[\underline{\mathcal{Y}}(\upsilon)\right] - CSC(\rho)^2\underline{\mathcal{Y}}(0) = -A[\overline{\mathcal{Y}}(\upsilon)]$$

By solving equation 5, we get:

$$A\left[\underline{\mathcal{Y}}(\upsilon;\vartheta)\right] = \frac{CSC(\rho)^2\left(\sqrt[2\mathfrak{t}+1]{CSC(\rho)}\right)}{\left(\sqrt[2\mathfrak{t}+1]{CSC(\rho)}\right)^2-1}\overline{\mathcal{Y}}(0;\upsilon) - \frac{CSC(\rho)^2}{\left(\sqrt[2\mathfrak{t}+1]{CSC(\rho)}\right)^2-1}\underline{\mathcal{Y}}(0;\upsilon)$$

$$A[\overline{\mathcal{Y}}(\upsilon;\vartheta)] = \frac{CSC(\rho)^2\left(\sqrt[2\mathfrak{t}+1]{CSC(\rho)}\right)}{\left(\sqrt[2\mathfrak{t}+1]{CSC(\rho)}\right)^2-1}\underline{\mathcal{Y}}(0;\upsilon) - \frac{CSC(\rho)^2}{\left(\sqrt[2\mathfrak{t}+1]{CSC(\rho)}\right)^2-1}\overline{\mathcal{Y}}(0;\upsilon)$$

Now we use inverse FAT for the above equation

$$\underline{\mathcal{Y}}(\upsilon;\vartheta) = \cosh\upsilon\overline{\mathcal{Y}}(0;\upsilon) - \sinh\upsilon\underline{\mathcal{Y}}(0;\upsilon),$$
$$\overline{\mathcal{Y}}(\upsilon;\vartheta) = \cosh\upsilon\underline{\mathcal{Y}}(0;\upsilon) - \sinh\upsilon\overline{\mathcal{Y}}(0;\upsilon)$$

3. Conclusions

The obtained formulas are then employed to solve the FIVP of a first-order liquid tank system. In this work, we presented a generalization of FAT (FAT) and proved some of the properties of (FAT). Eng. We also derive the FAT of fuzzy derivatives up to the third order and further generalize the said formula for the nth order fuzzy derivative by using the strongly generalized H-differentiability concept.

References

[1] D. V. Widder, "A Simplified Approach To Cauchy'S Integral Theorem," *The American Mathematical Monthly*, vol. 53, no. 7, pp. 359–363 (1946).

[2] S. H. Asaad, I. S. Ahmed, and H. H. Ebrahim, "On Finitely Null-additive and Finitely Weakly Null-additive Relative to the σ–ring," *Baghdad Sci. J*, vol. 19, no. 5, pp. 1148-1154 (2022).

[3] E. Christophe, D. Léger, and C. Mailhes, "Quality criteria benchmark for hyperspectral imagery," *IEEE Transactions on Geoscience and Remote Sensing*, vol. 43, no. 9, pp. 2103–2114 (2005).

[4] E. Silverberg, C. C. Boring, and T. S. Squires, "Cancer statistics, 1990," *CA: a cancer journal for clinicians*, vol. 40, no. 1, pp. 9–26 (1990).

[5] H. F. Abbas, H. H. Ebrahim, and A. Al-Fayadh, "Soft P*–field and some of its properties," *J. Disc. Math. Sci. Cryptogr*, vol. 26, no. 7, pp. 1875–1882 (2023).

[6] G. K. Watugala, "Sumudu transform: A new integral transform to solve differential equations and control engineering problems," *Applied Mathematics and Computation*, vol. 24, no. 1, pp. 35–43 (1993).

[7] H. Gündoğdu and Ö. F. Gözükızıl, "Solving nonlinear partial differential equations by using Adomian decomposition method, modified decomposition method and Laplace decomposition method," *MANAS Journal of Engineering*, vol. 5, no. 1, pp. 1–13 (2017).

[8] H.-J. Zimmermann, *Fuzzy set theory and its applications*, 4th ed. Berlin, Germany: Springer Science and Business Media (2001).

[9] W. N. Zhou, "Generalization of L-closure spaces," *Fuzzy Sets and Systems*, vol. 149, no. 3, pp. 415–432 (2005).

[10] H. M. Gloster and D. G. Brodland, "The epidemiology of skin cancer," *Dermatologic Surgery*, vol. 22, no. 3, pp. 217–226 (1996).

[11] H. Serrano, J. Scotto, G. Shornick, T. R. Fears, and E. R. Greenberg, "Incidence of nonmelanoma skin cancer in New Hampshire and Vermont," *Journal of the American Academy of Dermatology*, vol. 24, no. 4, pp. 574–579 (1991).

[12] J. Kaldor, D. Shugg, B. Young, T. Dwyer, and Y. G. Wang, "Non-melanoma skin cancer: ten years of cancer-registry-based surveillance," *International journal of cancer*, vol. 53, no. 6, pp. 886–891 (1993).

[13] J. Schiff, *The Laplace Transform: Theory and Application*. New York, USA: Springer (2013), doi: 10.1007/978-1-4614-4949-7.

[14] A. S. Mohammed, I. S. Ahmed, and S. H. Asaad, "Study of the rings in which each element expresses as the sum of an idempotent and pure," *Int. J. Nonlinear Anal. Appl,* vol. 12, no. 2, pp. 1719-1724 (2021), doi: 10.22075/IJNAA.2021.5311.

[15] S. M. Kadham and M. A. Mustafa, "Medical applications of the new-transform," *Journal of Interdisciplinary Mathematics*, vol. 26, no. 6, pp. 1341–1353 (2023).

[16] S. M. Kadham and M. A. Mustafa, "Fuzzy SHmath. Mbio-transform generalization and application to skin cancer imaging distributed diseases," *J. Interdiscip. Math,* vol. 26, no. 6, pp. 1031–1042 (2023).

[17] S. A. Al-Ameedee and A. J. Obaid, "New results and application of differential quasi subordinations for higher-order derivatives of meromorphic multivalent functions," *Journal of Interdisciplinary Mathematics*, vol. 26, no. 4, pp. 643–650 (2023), doi: 10.47974/JIM-1483.

[18] R. H. Hasan, M. A. Kadhim, and A. M. Darweesh, "Fuzzy partial T-transforms applied to a system of PDE," *J. Interdiscip. Math,* vol. 26, no. 6, pp. 1099–1107 (2023).

Received November, 2025