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Solutions of fractional partial differential equations by combined new technique NTADM

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Abstract

An application of fractional differential equations is on the rise across physics, biology, and engineering. This is especially true when standard integer-order models fail to adequately describe a given phenomenon. A large number of practical systems rely on these equations, which enable the accounting for memory and nonlocal effects. But the mathematical complexity of high-order fractional models makes them difficult to address. To approximate solutions to these kinds of problems, this research employs the Adomian Decomposition Method in combination with the Natural Transform. The fractional derivatives are transformed to algebraic expressions using the Natural Transform, and nonlinear terms are systematically treated using the Adomian technique. This method is validated for fractional differential equations by the numerical results obtained for fractional orders between 4.7 and 4.9, which show that stable solutions with absolute errors less than one are produced.

Subject Classification: *35R11, 35R60.*

Keywords: *Natural transform, Analytical technique, Fractional order, High-order fractional models.*

1. Introduction

As a useful tool for modeling systems with memory, hereditary behavior, and nonlocal interactions, fractional partial differential equations (FPDEs) have evolved as a natural expansion of classical partial differential equations (PDEs). These systems include anomalous diffusion, viscoelastic fluids, and detailed biological processes, among others. Fractional differentiation has its roots in the

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17th-century writings of Leibniz and Euler; nonetheless, it was formally established by Riemann, Liouville, and Caputo, whose formulations provide the basis of contemporary fractional calculus. Many analytical and numerical methods were developed to solve first-order partial differential equations (FPDEs), including the Laplace and Sumudu transforms, finite difference schemes, spectral methods, and hybrid semi-analytical methods. For high-order nonlinear fractional models, a successful approach is to combine Natural Transforms (NT) with Adomian's Decomposition Methods (ADM). In contrast to ADM's methodical breakdown of nonlinear components into quickly convergent Adomian polynomials, the Natural Transform reduces fractional derivatives to manageable algebraic formulae. For solving complex fractional PDEs across a wide range of scientific and engineering contexts, our hybrid NT + ADM framework has proven an excellent tool due to its high precision, numerical stability, and extremely small absolute errors [1-11].

2. Properties of Caputo Fractional Derivative

Let ${}^{C}\boldsymbol{D}_{t}^{\alpha}(\boldsymbol{af}(\boldsymbol{t})$ denote a Caputo fractional derivative for the orders a > 0. Its main properties are:

1. Linearity ${}^{C}\boldsymbol{D}_{t}^{\alpha}\left(\boldsymbol{af}(\boldsymbol{t})+\boldsymbol{bg}(\boldsymbol{t})\right)=\boldsymbol{a}\,{}^{C}\boldsymbol{D}_{t}^{\alpha}\boldsymbol{f}(\boldsymbol{t})+\boldsymbol{b}\,{}^{C}\boldsymbol{D}_{t}^{\alpha}\boldsymbol{g}(\boldsymbol{t})$.
2. Derivative of Constants ${}^{C}\boldsymbol{D}_{t}^{\alpha}(\boldsymbol{C})=\boldsymbol{0}$ which makes Caputo suitable for initial-value problems in physics.
3. Compatibility with Classical Derivatives

   If $\mathbf{n}-\mathbf{1}<\boldsymbol{\alpha}<\mathbf{n}$ **then**: ${}^{C}\boldsymbol{D}_{t}^{\alpha}\boldsymbol{f}(\boldsymbol{t})=\boldsymbol{0}$ for all polynomials of degree < n.
4. Nonlocal (Memory) Property: The derivative depends on the full history of the function:

$$ {}^{C}\boldsymbol{D}_{t}^{\alpha}\boldsymbol{f}(\boldsymbol{t})=\frac{\mathbf{1}}{\Gamma(\boldsymbol{n}-\boldsymbol{\alpha})}\int_{\mathbf{0}}^{\boldsymbol{t}}\frac{\boldsymbol{f}^{(\boldsymbol{n})}(\xi)}{(\boldsymbol{t}-\xi)^{\boldsymbol{\alpha}-\boldsymbol{n}+\mathbf{1}}}\,\boldsymbol{d\xi} $$

   Natural Transform Properties for Fractional Derivatives.

Let $\mathcal{N}\{\mathbf{f}(\mathbf{t})\}=\mathbf{F}(\mathbf{s},\mathbf{u})$ denote the Natural Transform.

1. Linearity $\mathcal{N}\{\boldsymbol{af}(\boldsymbol{t})+\boldsymbol{bg}(\boldsymbol{t})\}=\boldsymbol{aF}_{\boldsymbol{f}}(\boldsymbol{s},\boldsymbol{u})+\boldsymbol{bF}_{\boldsymbol{g}}(\boldsymbol{s},\boldsymbol{u})$
2. Transform of Caputo Fractional Derivative

   For $\mathbf{n}-\mathbf{1}<\boldsymbol{\alpha}<\mathbf{n}$, $\mathcal{N}\{{}^{C}\boldsymbol{D}_{t}^{\alpha}\boldsymbol{f}(\boldsymbol{t})\}=\boldsymbol{u}^{\alpha}\boldsymbol{s}^{\alpha}\mathbf{F}(\mathbf{s},\mathbf{u})-\sum_{k=0}^{n-1}\boldsymbol{u}^{\alpha}\boldsymbol{s}^{\alpha-k-1}\boldsymbol{f}^{(k)}(\mathbf{0})$

   This property is the reason NT simplifies fractional PDES.
3. Transforms Derivatives to Simple Algebraic Expressions

   NT converts fractional derivatives into polynomial factors of s: ${}^{C}\boldsymbol{D}_{t}^{\alpha}(\boldsymbol{C})\longrightarrow\boldsymbol{s}^{\alpha}$.
4. Suitable for PDEs of High Fractional Order

   Because NT handles higher-order derivatives without complexity:

$$ \mathcal{N}\left\{\boldsymbol{f}^{(n)}(\boldsymbol{t})\right\}=\boldsymbol{u}^{n}\boldsymbol{s}^{n}\mathbf{F}(\mathbf{s},\mathbf{u})-\sum_{k=0}^{n-1}\boldsymbol{u}^{n}\boldsymbol{s}^{n-k-1}\boldsymbol{f}^{(k)}(\mathbf{0}) $$

3. Application and results

Example 3.1: put a **time-fractionals KdV-type equations**

$$^{C}D_t^{\alpha}u(\varkappa,t)+c\,\frac{\partial u}{\partial \varkappa}(\varkappa,t)+d\,u(\varkappa,t)\,\frac{\partial u}{\partial \varkappa}(\varkappa,t)+e\,\frac{\partial^3 u}{\partial \varkappa^3}(\varkappa,t)=0,\varkappa\in\mathbb{R},\ t>0,$$

where c,d,e are real constants. With

$$u(\varkappa,0)=\rho_0(\varkappa),u_t(\varkappa,0)=\rho_1(\varkappa),\quad u_{tt}(\varkappa,0)=\rho_2(\varkappa),\quad u_{ttt}(\varkappa,0)=\rho_3(\varkappa),\quad u_{tttt}(\varkappa,0)=\rho_4(\varkappa).$$

Let $\rho_1(\varkappa)=\rho_2(\varkappa)=\rho_3(\varkappa)=\rho_4(\varkappa)=0,$ and first conditions $u(\varkappa,0)=\rho_0(\varkappa)$ is nonzero.

Solution: Let the **Natural Transforms** for the $u(\varkappa,t)$ with respect to t be $\mathcal{N}\{u(\varkappa,t)\}(\varkappa,s)=U(\varkappa,s)$. Natural Transform has a forms

$$\mathcal{N}\{^{C}D_t^{\alpha}u(\varkappa,t)\}(\varkappa,s)=s^{\alpha}U(\varkappa,s)-\sum_{k=0}^{4}s^{\alpha-1-k}\,\frac{\partial^k u(\varkappa,0)}{\partial t^k}.$$

With our choice $\rho_1=\rho_2=\rho_3=\rho_4=0$, this reduces to

$$\mathcal{N}\{^{C}D_t^{\alpha}u\}=s^{\alpha}U(\varkappa,s)-s^{\alpha-1}\rho_0(\varkappa).$$

$$s^{\alpha}U(\varkappa,s)-s^{\alpha-1}\rho_0(\varkappa)+c\,\mathcal{N}\{u_{\varkappa}\}+d\,\mathcal{N}\{uu_{\varkappa}\}+e\,\mathcal{N}\{u_{\varkappa\varkappa\varkappa}\}=0.$$

Since the Natural Transform acts only on t, spatial derivatives pass through: $\mathcal{N}\{u_{\varkappa}\}=\frac{\partial U}{\partial \varkappa},\quad \mathcal{N}\{u_{\varkappa\varkappa\varkappa}\}=\frac{\partial^3 U}{\partial \varkappa^3}.$

So

$$s^{\alpha}U(\varkappa,s)-s^{\alpha-1}\rho_0(\varkappa)+c\,U_{\varkappa}(\varkappa,s)+d\,\mathcal{N}\{uu_{\varkappa}\}(\varkappa,s)+e\,U_{\varkappa\varkappa\varkappa}(\varkappa,s)=0.$$

Solve for $U(\varkappa,s):U(\varkappa,s)=\frac{\rho_0(x)}{s}-\frac{1}{s^{\alpha}}[c\,U_{\varkappa}(\varkappa,s)+d\,\mathcal{N}\{uu_{\varkappa}\}(\varkappa,s)+e\,U_{\varkappa\varkappa\varkappa}(\varkappa,s)]$. Using the known property

$$\mathcal{N}^{-1}\left\{\frac{1}{s^{\alpha}}\mathcal{N}\{f(t)\}\right\}=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f(\tau)\,d\tau,$$

$$u(\varkappa,t)=\rho_0(\varkappa)-\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}[c\,u_{\varkappa}(\varkappa,\tau)+d\,u(\varkappa,\tau)\,u_{\varkappa}(\varkappa,\tau)+e\,u_{\varkappa\varkappa\varkappa}(\varkappa,\tau)]\,d\tau.$$

We write solution as a **series**: $u(\varkappa\,,t)=\sum_{n=0}^{\infty}u_n\,(\varkappa,t)$.

$$N(u)=uu_{\varkappa}=\sum_{n=0}^{\infty}A_n\,(\varkappa,t),$$

where A_n are defined in terms of $u_0,u_1,\dots,u_n$.

$u_0(\varkappa,t)=\rho_0(\varkappa)$ First correction u_1, from the recurrence:

$$u_1(\varkappa,t)=-\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\,[c\,(u_0)_{\varkappa}+d\,A_0+e\,(u_0)_{\varkappa\varkappa\varkappa}]d\tau,$$

Thus: $u_1(\varkappa,t)=-[c\,\rho_0{}'(\varkappa)+d\,\rho_0(\varkappa)\,\rho_0{}'(\varkappa)+e\,\rho_0{}'''(\varkappa)]\,\frac{t^{\alpha}}{\Gamma(\alpha+1)}.$

$$u_2(\varkappa,t)=-[c\,B_1{}'(\varkappa)+d\,C_1(\varkappa)+e\,B_1{}'''(\varkappa)]\,\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}.$$

Approximate solution up to the fifth term is

$$u^{(5)}(\varkappa,t)=\sum_{n=0}^{5}u_n(\varkappa,t)=\sum_{n=0}^{5}B_n(\varkappa)\,\frac{t^{n\alpha}}{\Gamma(n\alpha+1)},4<\alpha\leq 5.$$

The following Figures 1-2 express the results of approximate solutions and consecutive error at various values of t, equal to 0, 0.1, 0.2,... 1. Also, numerical simulations were performed to determine whether the technique described would lead to increased accuracy.

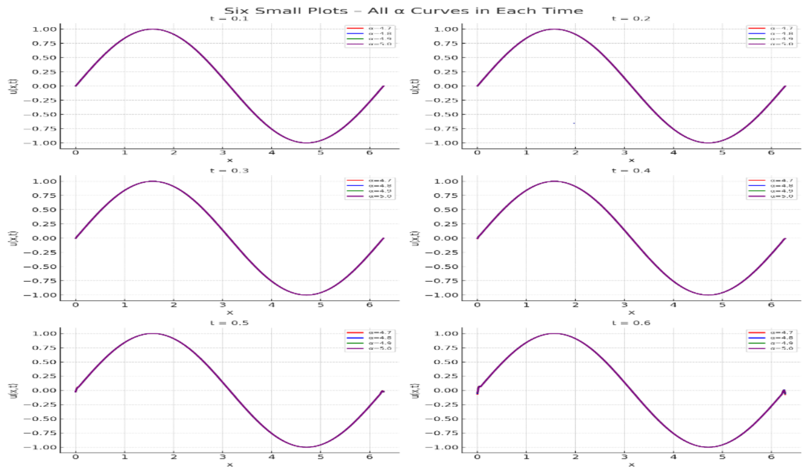

Figure 1
Show the approximate solutions using NT+ADM

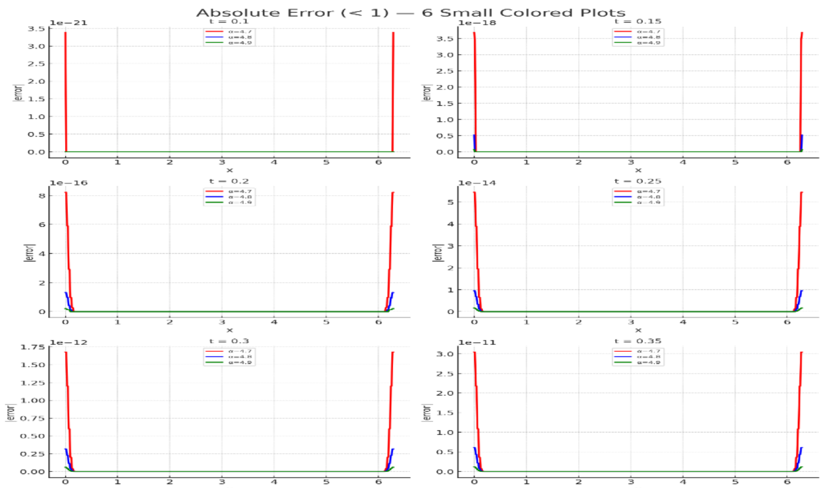

Figure 2
Show the consecutive errors between the current approximate solution and previous approximate solution

4. Conclusion

The application of Natural Transforms combined with Adomian Decompositions Methods (NT+ADM) proved highly effective in solving the high-order fractional PDE, providing stable and smooth approximate solutions. The Natural Transform greatly simplified the fractional derivatives, while ADM efficiently handled the nonlinear terms through rapidly convergent series. The obtained results across different fractional orders showed excellent accuracy, with the absolute error consistently remaining below 1. The method captured the sensitivity of the solution to variations in the fractional order, reflecting the realistic memory behavior of fractional systems. Overall, NT+ADM offers a robust and efficient semi-analytical framework for fractional modeling without requiring numerical discretization.

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