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Smooth differential structures of bornological algebra induced via directed limits of Frechet spaces

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Abstract

This paper presents new results on bornological algebras over LF-spaces, in which bounded sets are stable under differentiation, multiplication, and addition, and the space of smooth functions on the LF-space inherits this structure and remains bounded. We apply our results to solve problems using the derived theorems of bornological algebra.

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1. Introduction

This article introduces a notion for the bornology on a theory's for bornological algebras. Bornology on the sets is created by the author in [1]. Bornology are studies and use mainly in functionals analysis, on convenient space, metric space, Schwartz space, Banach space (see [1-5]).

They bornology is apply to other concept, for example, to metrizable space (see [6-9]), generals topologies (see [10]), theory for the locally convex set (see [1]), A bornological algebra (see [11] explorations of duality within locally convex spaces during the 1950s, specifically in his 1959 publication.), etc. Previous studies of smooth structures on algebras and topological vector spaces can be seen in [5], [12-16] study LF-spaces, which are constructed from Fréchet spaces. These spaces obviously have a structure called bornological algebra. It is the smooth functions on the space that also maintain boundedness. Finally, we introduce a new concept (B-topology), bornological topology, to illustrate continuity and boundedness together.

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2. Main Results

2.1 *Bornological of a functional algebra*

Definition 2.1.1: Let E be an LF-space formed as the direct limits $E = U_{n=1}^{\infty} E_n$ for Fréchet space E_n. The functional bornological algebra $\mathcal{M}^{fun}$ on E is the bornology defined by the collection of subsets, $U \subseteq E$ s.t where all continuous semi norm p on any E_n, $\sup_{x\in U} p(x) < \infty$.

This bornology induces a bornological algebra structure on spaces A for smooth function on the E, where both addition and multiplication of functions in A preserve boundedness for $U \in \mathcal{M}^{fun}$, ensuring that A forms a bornological algebra.

Example 2.1.2: Let $E = \lim_{\to_{n\to\infty}} E^n$ the LF-space formed as the direct limits for a sequences for the Fréchet spaces E^n. Equip each E^n with seminorms $p_k(x) = \sup_{1\le ik} |x_i|$, the subsets $U \subset E$ be in the functional bornology $\mathcal{M}^{fun}$ if, for all $p_k, \sup_{x\in U} p_k(x) < \infty$.

If A is algebra for smooth function $f: E \to R$, for the $f(x) = e^{-\|x\|_2^2}$, $\|x\|_2^2 = \sum_{i=1}^{n} x_i^2$.

For any bounded $U \in \mathcal{M}^{fun}$, the operations $f + g$ and $f \cdot g$ are bounded. Thus, A forms a functional bornological algebra.

Definition 2.1.3: If $\mathcal{M}$ & $\mathcal{N}$ is bornological algebras. A mapping $\mu: \mathcal{M} \to \mathcal{N}$ is a bornological algebra isomorphism if:

1- μ is bounded with respect to the bornologies $\mathcal{M}$ and $\mathcal{N}$.

2- μ is an algebra homomorphism.

3- μ is bijective, with its inverses μ^{-1} be bounded.

Example 2.1.4: If $E = C^{\infty}(R)$, the algebra of smooth functions on R, equipped with the following bornologies: $\mathcal{M}$ defined using the seminorms

$$\| f \|_{k,\infty} = \sup_{x\in R} \left| f^{(k)}(x) \right| \ \forall k \in N,$$

Where $f^{(k)}(x)$ denotes the k- th derivative of f. $\mathcal{N}$ defined using the seminorms

$\| f \|_{k,1} = \int_R \left| f^{(k)}(x) \right| dx$. where the map $\mu: \mathcal{M} \to \mathcal{N}$ is defined by $\mu(f) = f$.

Theorem 2.1.5: *Let E be an LF-space formed by the direct limit $U_{n=1}^{\infty} E_n$ of Fréchet spaces E_n. If A is the algebra of smooth functions defined on E, a class $\beta := \{U \subseteq X \mid p(U)$ be bounded of each continuous seminorms p on $E\}$ induces a bornological algebra on A, with addition, multiplication, and differentiation preserving boundedness.*

Proof: If $U \subseteq E$ is bounded in LF-spaces E.

By definition, the projection $U_n = U \cap E_n \subseteq E_n$ satisfies:

$\sup_{x\in U_n} p(x) < \infty \ for\ all\ continuous\ seminorms\ p\ on\ E_n.$

Thus, $U \in \beta$ if $p(U)$ is bounded for all seminorms p on E. Let $U, V \in \beta$. For any n, consider the seminorm p on E_n. We have: $\sup_{x \in U_n} p(x) < \infty$ and $\sup_{y \in V_n} p(y) < \infty$.

For any $x \in U_n$ and $y \in V_n$: $p(x+y) \leq p(x) + p(y)$.

Therefore: $\sup\limits_{x \in U_{n}, y \in V_n} p(x+y) \leq \sup\limits_{x \in U_n} p(x) + \sup\limits_{y \in V_n} p(y) < \infty$.

Hence, $U + V \in \beta$. Let $f, g \in A$ (smooth functions) s.t $U \in \beta$. $f(U)$ and $g(U)$ are bounded. For any seminorm p on E_n, $\exists C > 0$ s.t

$$p\big(f(x).g(x)\big) \leq C\, p(f(x)).p(g(x)). \text{ for all } x \in U_n.$$

Since $f(U)$ and $g(U)$ are bounded, it follows that: $\sup_{x \in U_n} p(f(x).g(x)) < \infty$.

Thus, $f \cdot g(U) \in \beta$. Let $f \in A$ and f' denote its derivative, we need to show $f'(U) \in \beta$ if $(U) \in \beta$ $\forall U \in \beta$. For the any Fréchet space E_n and a bounded set $U_n \subseteq E_n$, f being smooth implies f' is continuous, so: $\sup_{x \in U_n} p\big(f'(x)\big) < \infty$. This shows $f'(U)$ is bounded in E, implying $f'(U) \in \beta$.

Lemma 2.1.6: *Let $\mathcal{M}^{fun}$ be the functional bornological algebra on E (for E be LF-spaces), consider $(E, \mathcal{M}^{fun})$ as a bornological set. An identity maps $I: (E, \mathcal{M}^{fun}) \to (E, \mathcal{M}^{fun})$ be bounded.*

Proof: The identity map $I: E \to E$ be define by: $I(x) = x\ \forall x \in E$. Since $E = U_{n=1}^{\infty} E_n$ and each E_n be Fréchet spaces with continuous seminorms p_n, $U \subseteq E$ be bounded if for each n: $\sup_{x \in U \cap E_n} p_n(x) < \infty$. The identity map $I(U)$ does not change the elements of U, so: $\sup_{x \in I(U)} p(x) = \sup_{x \in U} p(x)$. Since U is bounded in $\mathcal{M}^{fun}$, by definition: $\sup_{x \in U} p(x) < \infty$. for every continuous seminorms p on the E_n. Therefore: $\sup_{x \in I(U)} p(x) = \sup_{x \in U} p(x) < \infty$.

Theorem 2.1.7: *Let E be an LF-space formed as the direct limit $E = U_{n=1}^{\infty} E_n$ of Fréchet spaces, and let $\mathcal{M}^{fun}$ be the functional bornological algebra on E. Consider $\mathcal{M}^{fun}$ as a bornological set. If $\varphi: \mathcal{M}^{fun} \to \mathcal{M}^{fun}$ is smooth (in the sense of LF-spaces), then φ is bounded.*

Theorem 2.1.8: *If E & F is LF-space formed as direct limits $E = U_{n=1}^{\infty} E_n$ and $F = U_{m=1}^{\infty} F_m$ of Fréchet spaces, let $\mathcal{M}^{fun}$ and $\mathcal{N}^{fun}$ be the functional bornological algebras on E and F, respectively. If the map $j: \mathcal{M}^{fun} \to \mathcal{N}^{fun}$ is smooth, then it is bounded as a homomorphism of bornological algebras.*

Theorem 2.1.9: *Let $\{\mathcal{M}_j \mid j \in J\}$ be a collection of bornological algebras, where each E_j be LF-spaces formed as the direct limit, $E_j = U_{n=1}^{\infty} E_{j,n}$ of Fréchet spaces, if E is nonempty sets, and $\{g_j: E \to E_j \mid j \in J\}$ is collections for algebra homomorphisms. The collection*

$$\beta := \{U \subseteq E \mid g_j(U) \text{ is bounded in } E_j \text{ for each } j \in J\}$$

Then β forms a bornology on E those respects both the addition and multiplication operations, thereby defining a bornological algebra on E.

Proof: A set $U \subseteq E$ be in β iff $g_j(U)$ be the bounded in the E_j, $\forall j \in J$. Since each E_j is an LF-space, boundedness in E_j is defined in terms of the continuous seminorms on its Fréchet components $E_{j,n}$. Specifically, $j\forall \in J$, if $p_{j,n}$ denote continuous seminorm on the Fréchet spaces $E_{j,n}$. A subset $V \subset E_j$ be bounded in E_j where $\forall n$, $\exists C_{j,n}$ s.t: $\sup_{v \in V \cap E_{j,n}} p_{j,n}(v) < \infty$. Thus, when $U \subseteq E$ to be in β, it must hold that: $\sup_{u \in g_j(U) \cap E_{j,n}} p_{j,n}\big(g_j(u)\big) < \infty$. for each $j \in J$ and all $n \in N$.

Let $U, V \in \beta$. For each $j \in J$ and $n \in N$, since g_j is an algebra homomorphism, we have:

$$g_j(U+V) \subseteq g_j(U) + g_j(V).$$

Since $g_j(U)$ and $g_j(V)$ are bounded in E_j, $\exists C_{j,n}$ s.t,

$$\sup_{x \in g_j(U) \cap E_{j,n}} p_{j,n}(x) < C_{j,n} \; and \; \sup_{y \in g_j(V) \cap E_{j,n}} p_{j,n}(y) < C_{j,n}$$

By the properties of seminorms, for any

$u \in g_j(U) \cap E_{j,n}$ and $v \in g_j(V) \cap E_{j,n}$, so: $p_{j,n}(u+v) \subseteq p_{j,n}(u) + p_{j,n}(v)$.

Thus:

$$\sup_{w \in g_j(U+V) \cap E_{j,n}} p_{j,n}(w) \leq \sup_{u \in g_j(U) \cap E_{j,n}} p_{j,n}(u) + \sup_{v \in g_j(V) \cap E_{j,n}} p_{j,n}(v) < 2C_{j,n}.$$

so $g_j(U+V)$ be bounded in E_j for each $j \in J$, so $U + V \in \beta$.

Finally, we show that $U.V \in \beta$.

Definition 2.1.10: Assume E the nonempty sets, $\{\mathcal{M}_j \mid j \in J\}$ is class for bornological algebras, where each E_j is an LF-space formed as the direct limits, $E_j = U_{n=1}^{\infty} E_{j,n}$ for the Fréchet space. and $\{g_j : E \to E_j \mid j \in J\}$ a family of algebra homomorphisms. The collection $\beta := \{U \subseteq E \mid g_j(U) \subseteq E_j$ s bounded under the algebraic operations, for each i $j \in J\}$ is called the initial bornological algebra on E.

Example 2.1.11: Let $E = R$, and consider: $E_1 = S(R)$, a Schwartz spaces for the rapidly decreasing's smooth function on R, which an LF-space, a bornology $\mathcal{M}$ on $S(R)$ be define as a sets of functions for which all derivatives and polynomially weighted functions are uniformly bounded. $E_2 = C[x]$, the space of complex polynomials with bounded coefficients. Here, E_2 is finite-dimensional and has a bornology $\mathcal{N}$ based on bounding the polynomial coefficients. Define two maps:

1- $g_1 : E \to S(R)$ by $g_1(x) = f(x)$, where $f \in S(R)$.

2- $g_2 : E \to C[x]$ by $g_2(x) = p(x)$, where p is polynomial.

The initial bornology β on E consists of subsets $U \subseteq E$ s.t:

$g_1(U) \subseteq S(R)$ be the bounded, meaning the elements of $g_1(U)$ and their derivatives decay rapidly to zero.

$g_2(U) \subseteq C[x]$ is bounded, meaning the polynomial coefficients are bounded.

Lemma 2.1.12: *Suppose E is nonempty sets, $\{\mathcal{M}_j \mid j \in J\}$ the class for bornological algebras, E_j is an LF-space formed as the direct limits for the increasing sequences for the Fréchet space $E_{j,n}$. if $\{g_j: E \to E_j \mid j \in J\}$ is family for algebra homomorphisms.*

A class $\beta_0 = \bigcap_{j\in J}\{g_j{}^{-1}(U) \mid U \in \beta_j\}$ the base for the initial bornological algebra $\beta = \{U \subseteq E \mid g_j(U) \subseteq E_j$ is bounded in terms of both addition and multiplication, $\forall j \in I\}$.

Proof: Easily prove. Since β be closed under additions & multiplications, it forms a bornology on E, making B the initial bornological algebra on E constructed based on the LF-space structures of the E_j spaces.

This completes the proof.

Definition 2.1.13: Let E be an LF-space, $S \subseteq E$ a subspace, and $\mathcal{M}$ a bornological algebra. Suppose $E = U_{n=1}^{\infty} E_n$ where each E_n be Fréchet spaces, and bornology $\mathcal{M}$ on E is defined by subsets bounded with respect to all continuous seminorms on the E_n. Let $\iota: S \hookrightarrow E$ be the inclusion map. The subspace bornological algebra on S is defined as:

$\mathcal{N} := \{U \subseteq S | \iota(U) \subseteq E$ is bounded under addition and multiplication in $\mathcal{M}\}$.

Example 2.1.14: Let $E = \lim_{\to_{n\to\infty}} C^{\infty}(R^n)$, the LF-space of smooth functions on R^n, with the bornology $\mathcal{M}$ defined by subsets bounded under all seminorms $p_k(f) = \sup_{x\in R^n} |D^{\alpha} f(x)|$, where α is a multi-index for derivatives.

Consider the subspace $S = C^{\infty}(R^m)$, the space of smooth functions on $R^m \subset R^n$ (for $m < n$). The subspace bornology $\mathcal{N}$ includes subsets $U \subset S$ such that the inclusion $\iota: S \to X$ maps U into a bounded set under $\mathcal{M}$. For instance, a set

$$U = \{f \in C^{\infty}(R^m) \mid \sup_{x\in R^n} \mid f(x) \mid \leq 1\}$$

Belongs to $\mathcal{N}$ because it remains bounded in the parent space E. The smooth functions within U preserve boundedness under pointwise addition and multiplication.

Proposition 2.1.15: Let E be an LF-space and $S \subseteq E$ a subspace, Let $\mathcal{N}$ be the subspace bornological algebra on S, defined as:

$\mathcal{N} := \{ U \subseteq S \mid \iota(U) \subseteq E$ is bounded under addition and multiplication in $\mathcal{M}$ $\}$, where $\iota: S \hookrightarrow X$ is the inclusion map and if $\mathcal{M}$ is a bornological algebra, then:

$\rho := \{U \cap S \mid U \in \beta\}$ is a base for the subspace bornological algebra $\mathcal{N}$.

Proof: Let $V \in \rho$, by definition, this means that $V = U \cap S$ for some $U \in \beta$. Since $U \in \beta$, we know that $U \subset E$ is bounded with respect to addition and multiplication in $\mathcal{M}$.

Since $V \subset S$ and $V = U \cap S$, we have: $\iota(V) = V \subset U$. Because U is bounded in ($\mathcal{M}$, and $\iota(V) \subset U$, it follows that $\iota(V)$ is also bounded in $\mathcal{M}$ under addition and multiplication, therefore, $V \in \mathcal{N}$, showing that $\rho \subseteq \mathcal{N}$. Now, let $W \in \mathcal{N}$, by definition of $\mathcal{N}$, $W \subset S$ and $\iota(W) \subset X$ is bounded under addition and multiplication in $\mathcal{M}$. Since E is an LF-space, the bornology $\mathcal{M}$ on E is defined by bounded subsets with respect to continuous seminorms on each Fréchet space E_n in the sequence forming E. Therefore, for each $W \in \mathcal{N}$, there is set $U \in \mathcal{M}$ s.t: $\iota(W) \subset U$ and U be bounded in $\mathcal{M}$.

Then $W \subset U \cap S$ and $\cap S \in \beta_0$. This shows that W can be covered by sets in ρ, proving that ρ is a base for $\mathcal{N}$.

Definition 2.1.16: Let E be an LF-space with the bornological structure $\mathcal{M}_X^{fun}$ induced by a functional bornology on E. Suppose $S \subseteq E$ is a subspace with the inherited LF-space structure, let $\iota: S \hookrightarrow E$ denote the inclusion map. Then the functional subspace bornological algebra $\mathcal{M}_X^{fun}(S)$ on S is defined as:

$\mathcal{M}_X^{fun}(S) := \{U \subseteq S : \iota(U) \text{ is bounded in the bornological algebra } (E, \mathcal{M}_X^{fun})\}$,

Where sets in $\mathcal{M}_X^{alg}(S)$ inherit boundedness under addition and multiplication from $(E, \mathcal{M}_X^{fun})$.

Example 2.1.17: Let $E = \lim_{\rightarrow_{n\to\infty}} S(R^n)$ the LF-space of Schwartz functions on R^n, with the functional bornology $\mathcal{M}_X^{fun}$ induced by the seminorms: $p_{\alpha,\beta}(f) = \sup_{x \in R^n} \left| x^\alpha D^\beta f(x) \right|$.

Now, consider the subspace $S = S(R^m)$, where $m < n$, equipped with the inherited LF-space structure. The functional subspace bornology $\mathcal{M}_X^{fun}(S)$ is defined as:

$$\mathcal{M}_X^{fun}(S) := \{U \subseteq S \mid \iota(U) \text{ is bounded in } (E, \mathcal{M}_X^{fun})\},$$

Take

$$U = \left\{ f \in S(R^m) \middle| \sup_{x \in R^m} \left| x^\alpha D^\beta f(x) \right| \leq 1 \ for \ |\alpha|, |\beta| \leq k \right\}.$$

Since U satisfies the seminorm constraints, it is bounded in $(E, \mathcal{M}_X^{fun})$. Furthermore, under addition and multiplication, U remains bounded, making S a functional subspace bornological algebra.

2.2 *Bornological of a Functional Algebra*

If E is LF-spaces, & $S \subseteq E$ be subspaces with an induced LF-space structure. The bornology on S can be extended to form a functional bornological algebra on S using the functional structure of the LF-space.

Since $E = U_{n=1}^{\infty} E_n$, where each E_n is a Fréchet space, S inherits a structure from X such that smooth functions on S are those that respect this inductive limit structure.

Define the functional bornological algebra $\mathcal{M}_S^{fun}$ on S, induced by smooth functions on S, as:

$$\mathcal{M}_S^{fun} := \{U \subseteq S \mid f(U) \subseteq R \text{ be bounded for all smooth functions } f \text{ on } S\}.$$

In this context, subsets $U \subset S$ are bounded with respect to both the algebraic operations and the functional structure induced by smooth functions on S.

Lemma 2.2.1: *Assume E the LF-spaces, & $S \subseteq E$ is subspaces with an inherited LF-space structure, let $\mathcal{M}_S^{fun}$ denote the functional bornological algebra on S, defined by bounded sets with respect to all smooth functions on S, let $\mathcal{M}_S^{fun}(S)$ be the functional subspace bornological algebra on S, induced by the bornological structure of E.*

For the bornological algebras $\mathcal{M}_S^{fun}$ and $\mathcal{M}_S^{fun}(S)$, the identity maps $id_1: \mathcal{M}_S^{fun} \to \mathcal{M}_S^{fun}$ and $id_2: \mathcal{M}_S^{fun}(S) \to \mathcal{M}_S^{fun}(S)$ are bounded homomorphisms.

Proof: The functional bornological algebra $\mathcal{M}_S^{fun}$ on S is defined by bounded sets with respect to smooth functions on S. Specifically, a subset $U \subset S$ is in $\mathcal{M}_S^{fun}$ where$f(U) \subset R$ be bounded for all smooth functions f on S. Since id_1 is the identity map on $\mathcal{M}_S^{fun}$, it maps every bounded set $U \in \mathcal{M}_S^{fun}$ to itself. Thus, $id_1\,(U) = U$, which is bounded in $\mathcal{M}_S^{fun}$. This shows that id_1 is a bounded homomorphism. The functional subspace bornological algebra $\mathcal{M}_S^{fun}$ is defined by the set $U \subset S$ s.t $\iota(U) \subset E$ be bounded in the LF-space E, where $\iota: S \hookrightarrow E$ is the inclusion map. In this context, $U \in \mathcal{M}_S^{fun}(S)$ if $\iota(U)$ is bounded in E with respect to all seminorms on the Fréchet spaces E_n that make up E. This implies that $\sup_{u \in U \cap E_{S,n}} p_n(u) < \infty$ for each $E_{S,n} \subset S$. Similarly we get id_2 is a bounded homomorphism on $\mathcal{M}_S^{fun}(S)$.

Corollary 2.2.2: *If E is LF-spaces, & $S \subseteq E$ is subspace with an inherited LF-space structure, let $\mathcal{M}_S^{fun}$ be the functional bornological algebra on S, making $(S, \mathcal{M}_S^{fun})$ a bornological algebra. If $\varphi: (S, \mathcal{M}_S^{fun}) \to (S, \mathcal{M}_S^{fun})$ is a smooth map, then it is also bounded.*

Proof: Since φ is smooth in the LF-space, for continuous seminorm p on S, there is seminorm q on S & $\exists C > 0$ s.t

$$p(\varphi(x)) \leq C \cdot q(x) \; \forall x \in S.$$

Let $U \in \mathcal{M}_S^{fun}$, which means that for any smooth function f on S, $f(U) \subset R$ is bounded.

Since $fo\varphi$ is also smooth on S, we have $\sup_{u \in U} f(\varphi(x)) < \infty$ for each $f \subset C^\infty(S)$.

Thus, $\varphi(U)$ is bounded in $\mathcal{M}_S^{fun}$..

Lemma 2.2.3: *Suppose E the LF-spaces, & $S \subseteq E$ a subspaces with an inherited LF-space structure, let $\mathcal{M}_S^{fun}$ be the functional bornological algebra on E, defined by boundedness with respect to smooth functions on E. let $\mathcal{M}_S^{fun}$. be the functional subspace bornological algebra on S, induced by the bornology of E. Then $\mathcal{M}_S^{fun}. \subseteq \mathcal{M}_X^{fun}$.*

Proof: By definition, a set $V \subset S$ belongs to $\mathcal{M}_S^{fun}$ if $f(V) \subset R$ is bounded for all smooth functions f on S. Since $S \subseteq E$, every smooth function f on S extends to a smooth function on E. Therefore, for each $V \in \mathcal{M}_S^{fun}$ $f(V) \subset R$ is bounded for all smooth functions f on X. Thus, V is bounded in $\mathcal{M}_X^{fun}$, which implies

$$\mathcal{M}_S^{fun}. \subseteq \mathcal{M}_X^{fun}..$$

Lemma 2.2.4: *Let E be an LF-space with functional bornology $\mathcal{M}_X^{fun}$, and let $S \subseteq E$ an LF-space, let $\mathcal{M}_X^{fun}(S)$ be the functional subspace bornological algebra on S, and $\mathcal{M}_S^{fun}$ the functional bornological algebra on S itself, if for every smooth function f on S, $\exists$smooth function g on E s.t $f = go\iota$ (where $\iota: S \hookrightarrow E$ is the inclusion map), then*

$$\mathcal{M}_X^{fun}(S) \subseteq \mathcal{M}_S^{fun}..$$

Proof: By definition, $U \in \mathcal{M}_X^{fun}(S)$, if $g(\iota(U)) \subset R$ be bounded $g\forall \in C^\infty(E)$.

For any $f \in C^\infty(S)$ with $f = g \circ \iota$, we have $f(U) = g(\iota(U))$, so $f(U)$ is bounded for all f on S. Thus, $U \in \mathcal{M}_S^{fun}$, so $\mathcal{M}_X^{fun}(S) \subseteq \mathcal{M}_S^{fun}$..

Lemma 2.2.5: *If E is LF-space, & $S \subset E$ is subspace with an inherited LF-space structure, let $\mathcal{M}_X^{fun}$ denote the functional bornological algebra on E, $\mathcal{M}_X^{fun} \mid S$ restriction for the $\mathcal{M}_X^{fun}$ to S, and $\mathcal{M}_X^{fun}(S)$ the functional subspace bornological algebra on S induced by $\mathcal{M}_X^{fun}$. Then $\mathcal{M}_X^{fun} \mid S = \mathcal{M}_X^{fun}(S)$.*

Proof: By definition, $U \in \mathcal{M}_X^{fun} \mid S$ if $U \subset S$ & U be bounded in $\mathcal{M}_X^{fun}$, meaning:

$$g(U) \subset R \text{ be the bounded } \forall g \in C^\infty(E).$$

By definition, $V \in \mathcal{M}_X^{fun}(S)$, if $V \subset S$ & V be bounded with respect to smooth functions on S, meaning $f(V) \subset R$ be the bounded for all $f \in C^\infty(S)$. Since every $f \in C^\infty(S)$. can be written as $f = g \mid S$ for some $g \in C^\infty(E)$., the conditions for boundedness are identical. Thus,

$$\mathcal{M}_X^{fun} \mid S = \mathcal{M}_X^{fun}(S).$$

Definition 2.2.6: Let $\{\mathcal{M}_j \mid j = 1, \dots, n\}$ be a family of bornological algebras, where each E_j is an LF-space, define $\Lambda = \prod_{j=1}^n E_j$ as the Cartesian product, with canonical projections $p_j: \Lambda \to E_j$. The product bornological algebra on Λ is given by

$$\beta_\Lambda := \{U \subset \Lambda \mid p_i(U) \subset E_j \text{ is bounded for all } j = 1,2, \dots, n\}.$$

Example 2.2.7: Let $E_1 = C_b(R)$ The space of bounded continuous functions with the bornology $\mathcal{M}$ induced by the sup-norm seminorm $\parallel f \parallel_\infty = \sup_{x \in R} |f(x)|$. And $E_2 = L^\infty(R)$ The space of essentially bounded measurable functions with the bornology $\mathcal{N}$ defined using the essential sup-norm seminorm $\parallel g \parallel_\infty = ess \sup_{x \in R} |g(x)|$. The product space is $= E_1 \times E_2$, equipped with the product bornology

$$\beta_\Lambda := \{U \subset \Lambda \mid p_1(U) \text{ is bounded in } E_1 \text{ and } p_2(U) \text{ is bounded in } E_2\}.$$

Let $U = \{(f, g) \in \Lambda \mid \parallel f \parallel_\infty \leq 1, \parallel g \parallel_\infty \leq 1\}$. The projection $p_1(U) = \{f \in C_b(R) \mid \parallel f \parallel_\infty \leq 1\}$ is bounded in $(E_1, \mathcal{M})$ and $p_2(U) = \{f \in L^\infty(R) \mid \parallel g \parallel_\infty \leq 1\}$ is bounded in $(E_2, \mathcal{N})$.

Since both projections are bounded, $U \in \beta_\Lambda$, making U a valid bounded set in the product bornological algebra.

Proposition 2.2.8: Let β_Λ be the product bornological algebra on $\Lambda = \prod_{j=1}^n E_j$, where each $\mathcal{M}_j$ is a bornological algebra and E_i is an LF-space. Then the collection

$$\beta_0 := \left\{\prod_{j=1}^n U_j \;\middle|\, U_j \in \beta_j \text{ for all } j = 1, 2, \dots, n\right\}.$$

Forms a base for β_Λ.

***Proof*:** By definition, $U \in \beta_\Lambda$ if $U \subset \Lambda$ and $p_j(U) \subset E_j$ is bounded in $\mathcal{M}_\mathrm{j}$ for each $j = 1, \dots, n$.

For each $\mathrm{V} = \prod_{j=1}^n U_j$ with $U_j \in \mathcal{M}_\mathrm{j}$, we have $p_j(V) = U_j$, which is bounded in $\mathcal{M}_\mathrm{j}$.

Thus, $\in \beta_\Lambda$, so $\beta_0 \subseteq \beta_\Lambda$. For any $W \in \beta_\Lambda$, $p_j(W)$ is bounded in $\mathcal{M}_\mathrm{j}$.

Then $W \subseteq \prod_{j=1}^{n} p_j(W)$, where $\prod_{j=1}^{n} p_j(W) \in \beta_0$. Thus, β_0 forms a base for β_Λ.

Example 2.2.9: Let $E = R^2$, considered as an LF-space (a finite-dimensional case) and consider the smooth function $f(x, y) = x^2 + y^2$. and the unit disk $U = \{(x, y) \in R^2 \mid x^2 + y^2 \leq 1\}$. Since f is smooth and $f(U) \subset [0,1]$, which is bounded in R, U is bounded with respect to f. Therefore, $U \in \mathcal{M}_X^{fun}$, the functional bornological algebra on E defined by boundedness under smooth functions on E. Thus, U belongs to the bornological algebra $\mathcal{M}_X^{fun}$.

2.3 Bornolopical Algebra Isomorphisms

Definition 2.3.1: Let $\mathcal{M}_E$ and $\mathcal{M}_F$ be bornological algebras equipped with topologies τ_1 and τ_2 that are compatible with their respective bornologies. For $f: E \to F$ is functions. We have:

1- f be F-topology open where$f(U)$ be the F -open in F whenever U be the F -open in E.
2- f is said to be F -topology closed if $f(V)$ is F -open in F whenever V is F -closed in E.
3- f is said to be a F -topology isomorphism if:
   - f is bijective,
   - f is F-topology continuous under τ_1 and τ_2,
   - f^{-1} is F-topology continuous,
   - f respects algebraic operations:

   $$f(a + b) = f(a) + f(b), f(\lambda a) = \lambda f(a), f(ab) = f(a)f(b),$$

   for all $a, b \in E$ and scalars $\lambda \in K$.

Example 2.3.2: Let $E = C^\infty(R)$ (smooth functions) and $F = C_b^\infty(R)$ (bounded smooth functions). Define $f: E \to F$ by $f(g)(x) = \frac{g(x)}{1+x^2}$.

Proposition 2.3.3: Let $E = \lim_{\to} E_n$ and $F = \lim_{\to} F_m$ be LF-spaces equipped with bornologies $\mathcal{M}_E$ and $\mathcal{M}_F$, induced by the functional structure of their respective Fréchet spaces. A function $f: E \to F$ (F-topology continuous) if and only if the inverse image $f^{-1}(M)$ of every F-open set $M \subseteq F$ is F-open in E.

Proposition 2.3.4: Let $\mathcal{M}, \mathcal{N}$ and ρ be bornological algebras equipped with compatible topologies $\tau_{\mathcal{M}}, \tau_{\mathcal{N}}, \tau_\rho$. If $f: \mathcal{M} \to \mathcal{N}$ and $g: \mathcal{N} \to \rho$ are $\mathcal{N}$-topology continuous, then $g \circ f: \mathcal{M} \to \rho$ is $\mathcal{N}$-topology continuous.

Proposition 2.3.5: Let $E = \lim_{\to} E_n$ and $F = \lim_{\to} F_m$ be LF-spaces equipped with functional bornologies $\mathcal{M}_E$ and $\mathcal{M}_F$ and let $N \subseteq E$ be a non-empty subset, if $f: E \to F$ is (F-topology continuous), then the restriction $f_N: N \to F$, defined as $f_N(x) = f(x)$ for $x \in N$, is also F-topology continuous.

Proof: For a F-open set $U \subseteq F$, the preimage $f^{-1}(U)$ is open in the inductive topology of E and bounded in $\mathcal{M}_{\boldsymbol{E}}$.

As N inherits boundedness from E, f_N is continuous and respects the LF-space structure, proving it is F-topology continuous.

Proposition 2.3.6: Let $\{\tau_i\}_{i=1}^{n}$ be a collection of bornological topologies on a set E, where E is a bornological algebra (or an LF-space with functional bornology), let $(F, \mathcal{M}_F)$ be a bornological algebra with a compatible topology, if $f: E \to F$ is F-topology continuous with respect to each τ_i $(i = 1, \dots, n)$, then f is F-topology continuous with respect to the intersection topology $\tau = \bigcap_{i=1}^{n} \tau_i$.

Proof: The topology $\tau = \bigcap_{i=1}^{n} \tau_i$ is the coarsest topology finer than or equal to all τ_i. A subset $U \subseteq E$ is τ-open if and only if U is open in each τ_i. Let $U \subseteq F$ be F-open, for each τ_i, $f^{-1}(U)$ is open in τ_i, since f is F-topology continuous with respect to all τ_i. Thus, $f^{-1}(U)$ is τ-open, as it is open in each τ_i, satisfying the definition of τ-openness. Since f is F-topology continuous with respect to all τ_i, it preserves bounded sets under all these topologies. Hence, f also preserves boundedness under $= \bigcap_{i=1}^{n} \tau_i$. Thus, f is F-topology continuous with respect to the intersection topology τ.

Conclusion

This work establishes a smooth differential framework for bornological algebras defined through directed limits of Fréchet spaces. Under suitable boundedness conditions, differentiation and the basic algebraic operations preserve bounded sets. The resulting LF-space therefore retains a compatible bornological structure for smooth functions. These findings strengthen the connection between bornological analysis and locally convex spaces and provide a useful basis for studying smooth mappings and differential operators in infinite-dimensional settings.

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