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Sharpened bounds for the Caratheodory function associated with the generating function of the Lucas-balancing polynomials

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Abstract

The lower bounds of $|\mathcal{T}'(a)|$ were estimated for an analytic function $\mathcal{T}$ with an angular limit at a boundary point on the unit disk $\mathfrak{A} = \{z \in \mathbb{C}: |z| < 1\}$, where $\mathcal{T}$ has the property $\mathcal{T}(\mathfrak{A}) = \mathfrak{A}$. We gave results about inequalities involving the module for $|\mathcal{T}'(a)|$ in case the zero of the function $(\mathcal{T}(z) - 1)$ are different from zero in the disk A, We relied on those results on the first non-zero the taylor's coefficients d_1 and d_1.

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1. Introduction

The performance of a function $\mathcal{T}$ on the unit disk $\mathfrak{A} = \{z \in \mathbb{C}: |z| < 1\}$ boundaries have been studied, and its results are presented in the well-known "Schwarz lemma", where $\mathcal{T}$ is an analytic such that $\mathcal{T}(\mathfrak{A}) = \mathfrak{A}$ and associates the origin with itself; thereafter, as research progressed and the generalization of choice regarding the lemma emerged, consensus was reached to replace the origin with alternative points within the disk.

The Schwarz lemma [7, 3, 4] is a powerful tool for investigating several domains of complex analysis.

Nearly all conclusions in geometric function theory are based on the Schwarz lemma as presented in references [2] and [5].

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Furthermore, more diversified and intriguing outcomes have been obtained [7], [8], [10], and [11]. In order to get the results we planned for, we will need the following facts:

Lemma 1.1: [7] (The boundary Schwarz lemma). *If $f(z)$ be a function, which is analytic and $(0) = 0$, If there exists a boundary point $\mathfrak{a}$ in $\mathfrak{U}$ such that f extends continuously to $\mathfrak{a}$, $f'(\mathfrak{a})$ exists and $|f(z)| = 1$, then*

$$\frac{2}{1+|f'(0)|} \leq |f'(\mathfrak{a})| \,. \tag{1}$$

This pertains to the maximal principle.

Conclude that in accordance with Lemma 1 and for all $z \in \mathfrak{U}$,

$$|z| \geq |f(z)|, \tag{2}$$

and

$$1 \leq |f'(0)|. \tag{3}$$

Thus, for an analytic function that satisfies the boundary conditions of the Schwarz lemma, we have

$$|f'(\mathfrak{a})| \geq 1, \tag{4}$$

and

$$|f'(\mathfrak{a})| > 1 \text{ unless } \mathcal{F}(z) = e^{i\alpha} z, \, (\, \alpha \in \mathbb{R}) \tag{5}$$

Lemma 1.2: [1] ("Julia Wolff's lemma"). *So, let f be an analytic function in, $f(0) = 0$ and $f(\mathfrak{U}) \subset \mathfrak{U}$. If at the boundary point $\mathfrak{a}$ in $\mathcal{D}$ the function f has an angular limit $f(\mathfrak{a})$, and $|f(\mathfrak{a})| = 1$, then $1 \leq |f'(\mathfrak{a})| \leq \infty$ and the angular derivative $f'(\mathfrak{a})$ exist. Where $\mathcal{D} = \partial\mathfrak{U}$.*

Corollary 1.3: [1]. *For the analytic function, at $\mathfrak{a} \in \mathcal{D}$ f' has the finite angular limit $f'(\mathfrak{a})$ if and only if f has a finite angular derivative $f'(\mathfrak{a})$., the following relation*

$$\mathcal{L}_i(y) = 6y\mathcal{L}_{i-1}(y) - \mathcal{L}_{i-2}(y), \, (\, i \geq 2, \, y \in \mathbb{C}\,), \tag{6}$$

Called the Lucas-Balancing polynomials [6], where $\mathcal{L}_0(y) = 1$, $\mathcal{L}_1(y) = 3y$, $\mathcal{L}_2(y) = 18y^2 - 1$ and $\mathcal{L}_3(y) = 108y^3 - 9y$, and the generating function [6] for Balancing polynomials is given by

$$\mathcal{B}(y,) = \sum_{i=0}^{\infty} \mathcal{L}_i(y)\xi^i = \frac{1-3yz}{1-6yz+z^2}\,, \tag{7}$$

Where $x \in$ [-1, 1], and z lies in.

Recall that the Caratheodory function [9] is an analytic function in $\mathfrak{U}$ and has the form

$$\mathcal{F}(z) = 1 + d_1 z + d_2 z^2 + d_3 z^3 + \cdots, \tag{8}$$

At $z \in \mathfrak{U}$ this function satisfies the property $Re\big(\mathcal{F}(z)\big) > 0$, where $\mathcal{F}(z) \in \mathcal{P}$.

The convolution of the Caratheodory function in (8) with the generating function of the Lacus-Balancing polynomials in (7) is defined as follows:

$$\mathcal{T}(z) = \left(\mathcal{F} * \mathcal{B}(y)\right)(z), \mathcal{K}(z) = \frac{\mathcal{T}(z)-(\delta+\frac{1}{2})}{(\delta+\frac{1}{2})}, \tag{9}$$

With
$$w(z) = \frac{\mathcal{K}(z)-\mathcal{K}(0)}{1-\left[\overline{\mathcal{K}(0)\mathcal{K}(z)}\right]}, \tag{10}$$

Each one of this function satisfying the first three conditions of lemma 1 for $\in \mathfrak{A}$, we conclude that

$$|\mathcal{T}(z)| \leq \frac{\left(\delta+\frac{1}{2}\right)(1+|z|)}{\left(\delta+\frac{1}{2}\right)+\left(\frac{1}{2}-\delta\right)|z|}, \tag{11}$$

and

$$|d_1| \leq \frac{2\delta}{(\delta+\frac{1}{2})|\mathcal{L}_1(y)|}. \tag{12}$$

$$\mathcal{T}(z) = \frac{\left(\delta+\frac{1}{2}\right)(1+ze^{i\vartheta})}{\left(\delta+\frac{1}{2}\right)+\left[\left(\frac{1}{2}-\delta\right)ze^{i\vartheta}\right]}, \vartheta \in \mathbb{R}.$$

Assume that $\mathcal{T}(z) = 1 + \mathcal{L}_1(y)d_1 z + \mathcal{L}_2(y)d_2 z^2 +$
$$\mathcal{L}_3(y)d_3 z^3 + \cdots, \mathcal{L}_1(y) \neq 0, x \in (0,1] \tag{13}$$

$$\mathcal{K}(z) = \frac{1}{\left(\frac{1}{2}+\delta\right)}\left[1 + \mathcal{L}_1(y)d_1 z + \mathcal{L}_2(y)d_2 z^2 + \cdots - \left(\frac{1}{2}+\delta\right)\right]$$

Where $\mathcal{K}(0) = \frac{\frac{1}{2}-\delta}{\frac{1}{2}+\delta}$ and $w(z) = \frac{\left[(\delta+\frac{1}{2})\mathcal{L}_1(y)\right]}{2\delta} d_1 z$

$$+\left[\frac{\left[(\delta+\frac{1}{2})\mathcal{L}_2(y)\right]}{2\delta} d_2 + \frac{\left[(\delta+\frac{1}{2})(\frac{1}{2}-\delta)\mathcal{L}_1^2(y)\right]}{(2\delta)^2} d_1^2\right] z^2 + \ldots \tag{14}$$

is the analytic function.

2. Main Results

Let $\mathcal{T}(z)$ denote a function defined by the formula $\mathcal{T}(z) = 1 + \mathcal{L}_1(y)d_1 z + \mathcal{L}_2(y)d_2 z^2 + \mathcal{L}_3(y)d_3 z^3 + \cdots$, if we know the coefficient d_2, we searched to find inequalities of Schwarz inequality at $\partial\mathfrak{A}$ are infer by using d_2 and the zeros of $(\mathcal{T}(z) - 1)$.

Theorem 2.1: *If $\mathcal{T}(z)$ According to Theorem 1, the function given in equation (13) is analytic throughout the unit disk, ($\mathcal{T}(z) - 1$) ossesses no zeros in $\mathfrak{A}$ other than z = 0 & $\left|\mathcal{T}(z) - (\delta + \frac{1}{2})\right| < \delta + \frac{1}{2}$ for $|z| < 1$, where $\delta \in \mathbb{R}$ and $0 < \delta \leq \frac{1}{2}$. Consider that for $\in \partial\mathfrak{A}$, $\mathcal{T}$ possesses an angular limit $\mathcal{T}(z)|_{\mathfrak{a}}$, $\mathcal{T}'(\mathfrak{a}) = 2\delta + 1$. Then Subsequently*

$$|\mathcal{T}'(\mathfrak{a})| \geq$$
$$2\delta\left(\delta+\frac{1}{2}\right)\left(1 - \frac{2\left(\delta+\frac{1}{2}\right)^2|\mathcal{L}_1(y)||d_1|\left(\ln\left(\frac{\delta+\frac{1}{2}}{2\delta}\right)\mathcal{L}_1(y)d_1\right)^2}{\left(\delta+\frac{1}{2}\right)^2|\mathcal{L}_1(y)||d_1|\left|2\ln\left(\frac{\delta+\frac{1}{2}}{2\delta}\right)\mathcal{L}_1(y)d_1\right|-\left(\delta+\frac{1}{2}\right)^2|\mathcal{L}_2(y)||d_2|+2\delta(\frac{1}{2}-\delta)}\right), \tag{15}$$

and

$$|d_2| \leq 2\left[\frac{\left(2\delta\left(\delta+\frac{1}{2}\right)^2|\mathcal{L}_1(kl)||d_1|ln\left(\frac{\delta+\frac{1}{2}}{2\delta}\right)\mathcal{L}_1 kld_1\right)+\delta(\frac{1}{2}-\delta)}{(\delta+\frac{1}{2})|\mathcal{L}_2(y)|}\right]. \tag{16}$$

$$\mathcal{T}(z) = \frac{\left(\delta+\frac{1}{2}\right)(1+z)}{\left(\delta+\frac{1}{2}\right)+\left(\frac{1}{2}-\delta\right)z}, (0 < \delta \leq \frac{1}{2}).$$

(16) holds if and only if

$$\mathcal{T}(z) = \frac{\left(\delta+\frac{1}{2}\right)(1+ze^{\Phi})}{\left(\delta+\frac{1}{2}\right)+\left(\frac{1}{2}-\delta\right)ze^{\Phi}}, (0 < d_1 \leq 1),$$

$$ln\left(\frac{\delta+\frac{1}{2}}{2\delta}\right)\mathcal{L}_1(x)d_1 < 0, \phi(z) = \frac{1+ze^{i\Psi}}{1-ze^{i\Psi}} ln\left(\frac{\delta+\frac{1}{2}}{2\delta}\right)\mathcal{L}_1(x)d_1, \text{ and } \Psi \in \mathbb{R}.$$

Proof: Let $d_1 > 0$, $\pi_1(z)$, $w(z)$ and $\mathcal{M}(z)$ Functions possessing identical formulas in the proof of Theorem 1 indicate that the logarithmic norm of $ln\mathcal{M}(z)$ corresponds to the analytic branch under the specified condition

$$ln\mathcal{M}(0) = Ln\left[\left(\frac{\delta+\frac{1}{2}}{2\delta}\right)\mathcal{L}_1(y)d_1\right] < 0 .$$

Consider the following conducive analytic function in $\mathfrak{A}$

$$\mathcal{X}_1(z) = \frac{ln\mathcal{M}(z)-ln\mathcal{M}(0)}{ln\mathcal{M}(z)+ln\mathcal{M}(0)},$$

Satisfy the properties of lemma 1. Thus from (1), we obtain

$$\frac{2}{1+|\mathcal{X}_1{}'(0)|} \leq |\mathcal{X}_1{}'(\mathfrak{a})| = \frac{-2ln\mathcal{M}(0)}{|ln\mathcal{M}(z_0)+ln\mathcal{M}(0)|^2}\left|\frac{\mathcal{M}'(\mathfrak{a})}{\mathcal{M}(\mathfrak{a})}\right|$$

$$= \frac{-2ln\mathcal{M}(0)}{ln^2\mathcal{M}(0)+arg^2\mathcal{M}(\mathfrak{t})}\{|w'(\mathfrak{a})| - |\pi_1{}'(\mathfrak{a})|\}$$

$$= \frac{-2ln\mathcal{M}(0)}{ln^2\mathcal{M}(0)+arg^2\mathcal{M}(\mathfrak{a}}\left\{\frac{|\mathcal{T}'(\mathfrak{a})|}{\delta(2\delta+1)} - 1\right\}$$

Since

$$\mathcal{X}_1{}'(z) = \frac{-2ln\mathcal{M}(0)}{(ln\mathcal{M}(0)+ln\mathcal{M}(\mathfrak{a}))^2}\frac{\mathcal{M}'(z)}{\mathcal{M}(z)},$$

so

$$|\mathcal{X}_1{}'(0)| = \frac{\left(\delta+\frac{1}{2}\right)^3|\mathcal{L}_2(y)||d_2|+2\delta(\frac{1}{4}-\delta^2)}{\left(\delta+\frac{1}{2}\right)^3|\mathcal{L}_1(y)||d_1|\left|2\ln\left(\frac{\delta+\frac{1}{2}}{2\delta}\right)\mathcal{L}_1(y)d_1\right|},$$

thus

$$\frac{2}{1+\frac{\left(\delta+\frac{1}{2}\right)^3|\mathcal{L}_2(y)||d_2|+2\delta(\frac{1}{4}-\delta^2)}{\left(\delta+\frac{1}{2}\right)^3|\mathcal{L}_1(y)||d_1|\left|2\ln\left(\frac{\delta+\frac{1}{2}}{2\delta}\right)\mathcal{L}_1(y)d_1\right|}} \leq \frac{-2ln\mathcal{M}(0)}{ln^2\mathcal{M}(0)+arg^2\mathcal{M}(\mathfrak{a})}\left\{\frac{|\mathcal{T}'(\mathfrak{a}|}{\delta(2\delta+1)} - 1\right\}.$$

Place $arg^2\tau(\mathfrak{a}) = 0$ in the above inequality, since $\ln\left(\frac{\delta+\frac{1}{2}}{2\delta}\right)\mathcal{L}_1(y)d_1 < 0$, we have

$$|\mathcal{T}'(\mathfrak{a})| \geq \delta(2\delta+1)\left(1 - \frac{2\left(\delta+\frac{1}{2}\right)^3|\mathcal{L}_1(y)||d_1|\left(\ln\left(\frac{\delta+\frac{1}{2}}{2\delta}\right)\mathcal{L}_1(y)d_1\right)^2}{\left(\delta+\frac{1}{2}\right)^3|\mathcal{L}_1(y)||d_1|\left|2\ln\left(\frac{\delta+\frac{1}{2}}{2\delta}\right)\mathcal{L}_1(y)d_1\right| - \left(\delta+\frac{1}{2}\right)^3|\mathcal{L}_2(y)||d_2| + 2\delta\left(\frac{1}{4}-\delta^2\right)}\right).$$

Consequently, we obtain an inequality (17). The analytical function $\mathcal{X}_1(z)$ adheres to the assumptions of Lemma 1, and observe that

$$1 \geq |\mathcal{X}_1{}'(0)| = \frac{|2ln\mathcal{M}(0)|}{|2ln\mathcal{M}(0)|^2}\left|\frac{\mathcal{M}'(\mathfrak{a})}{\mathcal{M}(\mathfrak{a})}\right|$$
$$= \frac{1}{\left|2\ln\left(\frac{\delta+\frac{1}{2}}{2\delta}\right)\mathcal{L}_1(y)d_1\right|}\left[\left(\frac{\left(\delta+\frac{1}{2}\right)|\mathcal{L}_2(y)|}{2\delta}\right)|d_2| + 1\right].$$

Therefor we obtain inequality (18).

As well as, from the last inequality if $|d_2| \leq \left[4\delta\frac{\left|ln\left(\frac{\delta+\frac{1}{2}}{2\delta}\right)\mathcal{L}_1(y)d_1+\frac{1}{2}\right|}{(\delta+\frac{1}{2})|\mathcal{L}_2(y)|}\right]$ and $|\mathcal{X}_1{}'(0)| = 1$, note that

$$ln\mathcal{M}(z) = \frac{ln\mathcal{M}(0)(1+ze^{i\psi})}{(1-ze^{i\psi})},$$

$$\mathcal{M}(z) = e^{\frac{ln\mathcal{M}(0)(1+ze^{i\psi})}{(1-ze^{i\psi})}} = e^{\frac{\left[ln\left(\frac{\delta+\frac{1}{2}}{2\delta}\right)\mathcal{L}_1(y)d_1\right](1+ze^{i\psi})}{(1-ze^{i\psi})}}.$$
$$= ze^{\Phi}.$$

Since from theorem 1,

$$\mathcal{M}(z) = \frac{w(z)}{\pi_1(z)},$$

hence using straightforward calculations, we ascertain that

$$\mathcal{K}(z) = \frac{\mathcal{K}(0)+ze^{\Phi}}{1+ze^{\Phi}\overline{\mathcal{K}(0)}} = \frac{\left(\frac{1}{2}+\delta\right)+z(\frac{1}{2}+\delta)e^{\Phi}}{\left(\frac{1}{2}+\delta\right)+z(\frac{1}{2}-\delta)e^{\Phi}},$$

that is

$$\mathcal{T}(z) = \frac{\left(\frac{1}{2}+\delta\right)+(1+ze^{\Phi})}{\left(\frac{1}{2}+\delta\right)+z\left(\frac{1}{2}-\delta\right)e^{\Phi}}.$$

Consequently, from Theorem one and Theorem two, we derive the coefficients d_1 and d_2 from the aforementioned inequalities, as we utilize (1.1) in the proof of these theorems. In the present theorems, we derive a less strong inequality that includes an upper bound with out the coefficient d_2 by utilizing (4).

Theorem 2.3: *Let $\mathcal{T}(z)$ be an equation possessing the conditions outlined in Theorem 2; the result is*

$$|\mathcal{T}'(\mathfrak{a})| \geq \delta(2\delta+1)\left(1-\frac{1}{2}ln\left(\frac{\delta+\frac{1}{2}}{2\delta}\right)\mathcal{L}_1(y)d_1\right). \tag{17}$$

The relationship in (1.19) is valid if and only if

$$\mathcal{T}(z) = \frac{\left(\delta+\frac{1}{2}\right)(1+ze^{\phi})}{\left(\delta+\frac{1}{2}\right)+\left(\frac{1}{2}-\delta\right)ze^{\phi}},$$

$ln\left(\frac{\delta+\frac{1}{2}}{2\delta}\right)\mathcal{L}_1(y)d_1 < 0$, $d_1 \in (0,1]$ $\phi = \frac{1+ze^{i\Psi}}{1-ze^{i\Psi}} ln\left(\frac{\delta+\frac{1}{2}}{2\delta}\right)\mathcal{L}_1(y)d_1$, and $\Psi \in \mathbb{R}$.

Proof: By using (4)

$$|\mathcal{X}_1'(\mathfrak{a})| = \frac{-2ln\mathcal{M}(0)}{ln^2\mathcal{M}(0)}\left\{\frac{|\mathcal{T}'(\mathfrak{a})|}{\delta(2\delta+1)}-1\right\}.$$

So by the following inequality

$$\begin{aligned} 1 \leq |\mathcal{X}_1'(\mathfrak{a})| &= \frac{-2ln\mathcal{M}(0)}{ln^2\mathcal{M}(0)}\left\{\frac{|\mathcal{T}'(\mathfrak{a})|}{\delta(2\delta+1)}-1\right\} \\ &\leq \frac{-2}{ln\left(\frac{\delta+\frac{1}{2}}{2\delta}\right)\mathcal{L}_1(y)d_1}\left\{\left\{\frac{|\mathcal{T}'(\mathfrak{a})|}{\delta(2\delta+1)}\right\}-1\right\}, \end{aligned} \tag{18}$$

hence

$$|\mathcal{T}'(\mathfrak{a})| \leq \delta(2\delta+1)\left(1-\frac{1}{2}ln\left(\frac{\delta+\frac{1}{2}}{2\delta}\right)\mathcal{L}_1(x)d_1\right).$$

Consequently, we have the inequality (17).

From (18) if $|\mathcal{T}'(\mathfrak{a})| = \delta(2\delta+1)\left(1-\frac{1}{2}ln\left(\frac{\delta+\frac{1}{2}}{2\delta}\right)\mathcal{L}_1(y)d_1\right)$ and $|\mathcal{X}'(\mathfrak{a})| = 1$, we obtain

$$\mathcal{T}(z) = \frac{\left(\delta+\frac{1}{2}\right)(1+ze^{\phi})}{\left(\delta+\frac{1}{2}\right)+\left(\frac{1}{2}-\delta\right)ze^{\phi}}.$$

The subsequent theorem demonstrates that the connection (15) can be reinforced using an alternative way if $(\mathcal{T}(z)-1)$ possesses zeros separate from $z=0$, taking these zeros into consideration. Examine the subsequent analytic function within$\mathfrak{A}$ nd ensure it meets the criteria of the Schwarz lemma for $\in \partial\mathfrak{A}$,

$$\mathcal{X}_2(z) = \frac{\mathcal{N}(z)-\mathcal{N}(0)}{\left(1-\mathcal{N}(0)\overline{\overline{\mathcal{N}(z)}}\right)},$$

Apply (1), we get

$$\frac{2}{1+|\mathcal{X}_2'(0)|} \leq |\mathcal{X}_2'(\mathfrak{a})| = \frac{1-|\mathcal{N}(0)|^2}{(1-|\mathcal{N}(0)|)^2}|\mathcal{N}'(\mathfrak{a})|$$

$$= \frac{(1+|\mathcal{N}(0)|)}{1-|\mathcal{N}(0)|}\left\{|w'(\mathfrak{a})| - |\pi_2'(\mathfrak{a})|\right\}.$$

$$= \frac{(2\delta\prod_{i=1}^{n}|q_i|+(\delta+\frac{1}{2})|\mathcal{L}_1(y)||d_1|)}{(2\delta\prod_{i=1}^{n}|q_i|-(\delta+\frac{1}{2})|\mathcal{L}_1(y)||d_1|)}\left\{\frac{|\mathcal{T}'(\mathfrak{a})|}{\delta(2\delta+1)} - \left[1+\sum_{i=1}^{n}\bar{\mu}\frac{(1-|q_i|^2)}{|\bar{\mu}z-q_i|}\right]\right\}. \quad (22)$$

In addition, since

$$|\mathcal{X}_2'(0)| = \left[\frac{1-|\mathcal{N}(0)|^2}{(1-|\mathcal{N}(0)|^2)^2}\right]|k_2'(0)| = \frac{\left(\delta+\frac{1}{2}\right)2\delta\prod_{i=1}^{n}|q_i|\left[\mathcal{L}_2(y)d_2+\frac{(\frac{1}{2}-\delta)\mathcal{L}_1^2(y)}{2\delta}d_1^2\right]}{(2\delta\prod_{i=1}^{n}|q_i|)^2-\left((\delta+\frac{1}{2})|\mathcal{L}_1(y)||d_1|\right)^2}.$$

Replace regarding $|\mathcal{X}_2'(0)|$in (22), we have

$$\frac{2\left((2\delta\prod_{i=1}^{n}|q_i|)^2-\left((\delta+\frac{1}{2})|\mathcal{L}_1(y)||d_1|\right)^2\right)}{(2\delta\prod_{i=1}^{n}|q_i|)^2-\left((\delta+\frac{1}{2})|\mathcal{L}_1(y)||d_1|\right)^2+\left(\delta+\frac{1}{2}\right)2\delta\prod_{i=1}^{n}|q_i|\left[\mathcal{L}_2(y)d_2+\frac{\left(\frac{1}{2}-\delta\right)\mathcal{L}_1^2(y)}{2\delta}d_1^2\right]}$$

$$\cdot\frac{(2\delta\prod_{i=1}^{n}|q_i| - (\delta+\frac{1}{2})|\mathcal{L}_1(y)||d_1|)}{(2\delta\prod_{i=1}^{n}|q_i| + (\delta+\frac{1}{2})|\mathcal{L}_1(y)||d_1|)}$$

$$\leq \frac{|\mathcal{T}'(\mathfrak{a})|}{\delta(2\delta+1)} - \left[1+\sum_{i=1}^{n}\bar{\mu}\frac{(1-|q_i|^2)}{|\bar{\mu}z-q_i|}\right]$$

$$\delta(2\delta+1)\left(1+\frac{2\left(2\delta\prod_{i=1}^{n}|q_i|-\left((\delta+\frac{1}{2})|\mathcal{L}_1(y)||d_1|\right)\right)^2}{(2\delta\prod_{i=1}^{n}|q_i|)^2-\left((\delta+\frac{1}{2})|\mathcal{L}_1(y)||d_1|\right)^2+\left(\delta+\frac{1}{2}\right)2\delta\prod_{i=1}^{n}|q_i|\left[\mathcal{L}_2(y)d_2+\frac{(\frac{1}{2}-\delta)\mathcal{L}_1^2(y)}{2\delta}d_1^2\right]} + \sum_{i=1}^{n}\bar{\mu}\frac{(1-|q_i|^2)}{|\bar{\mu}z-q_i|}\right) \leq |\mathcal{T}'(\mathfrak{a})|.$$

Thus, the proof is complete.

Conclusion

In this paper, lower bounds for $|T'(a)|$ were established for analytic functions on the unit disk with angular boundary limits and satisfying $T(A) = A$. Several inequalities involving $|T'(a)|$ were obtained when the zeros of $(T(z)-1)$ are nonzero in the disk. The results were derived using the first nonzero Taylor coefficients d_1 and d_2, providing useful contributions to analytic function theory.

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