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Oscillation proportions for a second order linear neutral difference equation together with negative and positive coefficients

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Abstract

The framework for necessary and sufficient conditions was created to guarantee oscillatory behavior for each linear solution of a neutral delay difference equation. We have also examined the properties of oscillation. The results gained are illustrated with examples.

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1. Introduction

The study of the neutral delay difference equation with both negative and positive coefficients has been considered by several authors globally over the last several years. See [1], [2], [3], [4] They investigated cases with variable coefficients, which are both positive and negative; see [5], Ketab and Abdullah [1] investigated this concept in 2021. The oscillation solving for a second-order

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neutral difference equation with negative & positive coefficients is shown in our paper:

$$\Delta^2(\boldsymbol{g_n} - \boldsymbol{b_n g_{n-u}}) + \boldsymbol{a_n}(\boldsymbol{g_{n-e}}) - \boldsymbol{c_n g_{n-t}} = \boldsymbol{Z_n} \tag{1}$$

is studying, when Δ be difference operators forward, there are two infinite sequences for the real numbers: z_n , c_n, a_n, c_n, which are both non-negative.

In this study, we seek new sufficient and necessary criteria for all solutions to equation (1) to oscillate. The following suppositions are used:

(B_1) $\sum_{j=n_1}^{\infty}\sum_{i=g+e-t}^{g-1} c_i < \infty$

(B_2) $\sum_{j=n_1-e+t}^{\infty} a_i < \infty$

(B_3) There exists a equations $\{Z_n\}$ such that $\Delta^2 Z_n = Z_n and\ \lim_{n\to\infty} Z_n = 0$.

The obtained results may be applied in several fields such as real analysis, topology, and algebra, for example, see [6], [7], [8].

2. The main result

The following findings establish sufficient and necessary criteria for all bounding solutions of equation (1) to oscillate. For a simple set:

$$Z_n = g_n - b_n g_{n-u} \tag{2}$$

let the sequence V_n be defined as

$$V_n = g_n - b_n g_n + \sum_{j=n}^{\infty}\sum_{i=j+e-t}^{j-1} c_i g_{i-e} - Z_n, t > e \tag{3}$$

And the sequence V_n be defined as

$$V_n = g_n - b_n g_n - \sum_{j=n}^{\infty}\sum_{i=j+e-t}^{j-1} c_i g_{i-e} - Z_n\ e > t \tag{4}$$

Theorem 2.1: ([1], pp. 184) *Let t be the positive integer & $\{b_n\}$ be the non-negative sequence for the real integers. Assume that*

$$\lim_{i\to\infty} inf \sum_{i=n-t}^{n-1} b_i > \frac{t^{t+1}}{(t+1)^{t+1}}$$

then

i. *The difference inequality*

$$g_{n+1} - g_n + b_n g_{n-t} \le 0, n = 0,1,2,3, ...$$

does not have an eventual positive solution.

ii. *A difference inequality*

$g_{n+1} - g_n + b_n\, g_{n-t} \ge 0, n = 0,1,2,3\ ....$ *can't have unfavorable final results.*

Theorem 2.2: ([8] pp. 10) *Let $z, g\ \varepsilon\ R$ then*

i. $z < g + \varepsilon, \forall\, \varepsilon > 0$, iff $z \le g$

ii. $z > g - \varepsilon, \forall\, \varepsilon > 0$, iff $z \ge g$

Theorem :1 *Consider $b_n \geq 1$ is bounded ($a_{n+e-t} - c_n \leq 0$), let (B_1) and (B_3) are hold, in addition to*

$$\lim_{n\to\infty} inf \sum_{j=n(t-b-u)}^{n-1} \sum_{i=j}^{j+b} \frac{|c_{i+e-t}-a_i|}{b_{i-t+u}} > \frac{(t-b-u)^{t-b-u}}{(t-b-u+1)^{t-b-u+1}} \text{ , } t > b+u \tag{5}$$

Following that, every solution limited by equation (1) oscillates.

Proof: From equations (1), (2), and (3), assuming that $\{g_n\}$ be positive solutions for the eq. (1)

$$for\ n \geq n_0 \geq 0, \text{ we obtain } \Delta^2 V_n = (c_{n+e-t} - a_n)(g_{n-t}) \leq 0 \tag{6}$$

Hence, $\Delta V_n, V_n$, are monotone sequences. We claim that thus, $\Delta V_n > 0$ for $n \geq n_1 \geq n_0$ otherwise, $\Delta V_n < 0, n \geq n_1, thus\ V_n < 0\ and\ V_n \to -\infty\ as$ $n \to \infty$. from (3) we get

$$V_n \geq -b_n g_{n-e} - Z_n$$
$$\geq -b g_{n-u} - Z_n, b_n \leq b$$

Then $g_n \to \infty\ as$ n $\to \infty$, which is a contradiction. Hence $\Delta V_n > 0$.

There are two cases for $n \geq n_2 \geq n_1$:

Case 1: $V_n > 0, \Delta V_n > 0, \Delta^2 V_n \leq 0$, then exists $\mu > 0$ s.t $V_n \geq \mu \geq 0$ where $n \geq n_2 \geq n_1$

Since g_n is bounded, let $\lim\inf_{n\to\infty} g_n = f_*, f_* \geq 0$,so there exists a subsequence $\{n_j\}\ of\ \{n\}$ such that $\lim_{j\to\infty} n_j = \infty$, $\lim_{j\to\infty} g_{nj} = f_*$ from (3) we get

$$g_{nj} = V_{nj} + b_{nj}\, g_{nj-u} - \sum_{r=nj}^{\infty} \sum_{i=r+e-t}^{r-1} c_i g_{i-e} + Z_{nj}$$
$$g_{nj} \geq \mu + b_{nj}\, g_{nj-u} - \alpha_2 \sum_{r=n_j}^{\infty} \sum_{i=r+e-t}^{r-1} c_i + Z_{nj}, (g_n) \leq \alpha_2$$
$$g_{nj} > \mu + g_{nj-u} - \varepsilon$$

Since ε is arbitrary, Theorem 2.2 states that for a sufficiently big j, we obtain

$$g_{nj} > \mu + g_{nj-u}$$

As $j \to \infty$, it follows that $f_* \geq \mu + f_*$ which is a contradiction.

Case 2: $V_n < 0, \Delta V_n > 0, \Delta^2 V_n \leq 0$.

By taking the summation of both sides of (6) from n to$n+b, 0 < b < t-u$ we get

$$\Delta V_{n+1} - \Delta V_n = \sum_{i=n}^{n+b} (c_{i+\mathrm{e}-\mathrm{t}} - a_i)(g_{i-t})$$
$$-\Delta V_n \leq \sum_{i=n}^{n+b} (c_{i+\mathrm{e}-\mathrm{t}} - a_i)(g_{i-t})$$
$$\Delta V_n \geq \sum_{i=n}^{n+b} |c_{i+\mathrm{e}-\mathrm{t}} - a_i| g_{i-t} \tag{7}$$

From (3) we get

$$V_n = g_n - b_n\, g_{n-u} + \sum_{r=n_j}^{\infty} \sum_{i=r+e-t}^{r-1} c_i\ (g_i - \mathrm{e}) - Z_n$$

$$\geq -b_n g_{n-u} - Z_n$$
$$V_n > -b_n g_{n-u} - \varepsilon, \varepsilon > 0$$
since ϵ is arbitrary, it follows that
$$V_n \geq -b_n g_{n-u}$$
$$g_{n-u} \geq -\frac{1}{b_n} V_n$$
$$g_{n-t} \geq -\frac{1}{b_{n-t+u}} V_{n-t+u} \tag{8}$$
Substituting (8) in (7) to obtain
$$\Delta V_n \geq -\sum_{i=n}^{n+b} \frac{|c_{i+e-t} - a_i|}{b_{i-t+u}} V_{i-t+u}$$
$$\Delta V_n + \sum_{i=n}^{n+b} \frac{|c_{i+e-t} - a_i|}{b_{i-t+u}} V_{n-(t-b-u)} \geq 0$$
A last inequality have not an ultimately nonpositive solutions, this contradiction, according to theorems (l-ii) and grace of (5).

Theorem 2: *Suppose that* $b_n \leq 1$, $(c_n - a_{n-e+t}) \geq 0$

(B_2) *and* (B_3) *are hold in addition to*
$$\lim_{n\to\infty} inf \sum_{j=n-(e-b)}^{n-1} \sum_{i=j}^{j+b} c_i - a_{i-e+t}) > \frac{(e-b)^{e-b+1}}{(e-b+1)^{e-b+1}}, e > b \tag{9}$$
Each of the equation (1) restricted solutions then oscillates.

Proof: If $\{y_n\}$ turns out to be eventual positive restricted solution for the eq. (1), then aske solution for contradiction may be obtained from equations (1), (2), and (3).
$$\Delta^2 V_n = (c_n - a_{n-e+t})(g_{n-e}) \geq 0 \tag{10}$$
Hence, $\Delta V_n, V_n$ are monotone sequences, we claim that $\Delta V_n < 0$ for
$$n \geq n_1 \geq n_1$$
otherwise $\Delta V_n > 0$ where $n \geq n_1$, hence, $V_n > 0$ & $V_n \to \infty$ when $n \to \infty$. Let
$$g_n \leq \propto_1$$
where $\propto_1$ is positive constants of (4) we get
$$V_n = g_n - b_n g_{n-u} - \sum_{j=n}^{\infty} \sum_{i=j-e+t}^{j-1} a_i(g_{i-t}) - Z_n$$
$$V_n \leq g_n - Z_n$$
Which implies that $g_n \to \infty\ as\ n \to \infty$, that is a contradiction. Thus $\Delta V_n < 0$,

The presence for no oscillatory solutions to eq. (1) when $n \geq n_2 \geq n_1$ must therefore be considered in two different scenarios.

Case 1: $V_n < 0, \Delta V_n < 0, \Delta^2 V_n \geq 0$. Then there exists $\mu < 0$ such that $V_n \leq \mu < 0\ for\ n \geq n_2$.

Since g_n is bounded, let $\limsup_{n\to\infty} g_n = f^*, f^* \geq 0$ so there exists a sequence $\{n_j\}$ such that

$\lim_{j\to\infty} n_j = \infty$, $\lim_{j\to\infty} g_{nj} = f^*$ from (4) we get

$$g_n = V_n + b_n g_{n-u} + \sum_{j=n}^{\infty}\sum_{i=j-e+t}^{j-1} a_i(g_{i-t}) + Z_n$$

$$g_n \leq \mu + b_n g_{n-u} + \propto_1 \sum_{j=n}^{\infty}\sum_{i=j-e+t}^{j-1} a_i + Z_n$$

$$g_n < \mu + g_{n-u} + \varepsilon\,, \varepsilon > 0$$

According to theorem (2.2), since ε is arbitrary, for sufficiently big j, it follows that:

$$g_{nj} \leq \mu + g_{nj-u}$$

As $j \to \infty$, from the previous inequality $f^* \leq \mu + f^*$ which is a contradiction.

Case 2: $V_n > 0, \Delta V_n < 0, \Delta^2 V_n \geq 0.$

By the summation of (10) from n to $n + b, b > 0$, it follows that

$$\Delta V_{n+1} - \Delta V_n = \sum_{i=n}^{n+b}(c_i - a_{i-e+t})(g_{n-e}) - \Delta V_n$$

$$\geq \sum_{i=n}^{n+b}(a_i - c_{i-e+t})(g_{i-e}) \tag{11}$$

From (4) we get

$$g_n = V_n + b_n g_{n-u} + \sum_{j=n}^{\infty}\sum_{i=j-e+t}^{j-1} a_i(g_{i-t}) + Z_n$$

$$\geq V_n + Z_n$$

$$g_n > V_n - \varepsilon\,, \varepsilon > 0$$

Since ϵ is arbitrary, it follows that

$$g_n \geq V_n$$

Substituting the last inequality in (11) we obtain

$$-\Delta V_n \geq \sum_{i=n}^{n+b}(c_i - a_{i-e+t})(V_{i-e})$$

$$-\Delta V_n \geq \left(\sum_{i=n}^{n+b}(c_i - a_{i-e+t})\right) V_{n+b-e}$$

$$\Delta V_n + \left(\sum_{i=n}^{n+b}(c_i - a_{i-e+t})\right) V_{n+(b-e)} \leq 0$$

3. Application:

Example 3.1: Consider the following difference equations

$$\Delta^2(g_n - (3 + (-1)^n g_{n-1}) + \tfrac{3}{4} g_{n-2} - \tfrac{1}{4}(-1)^n g_{n-3} =$$

$$(1)^{3n}\left(\tfrac{-5}{2}(-1)^{n+2}\right) \quad (E_1)$$

Where u=1, t=3, e=2, b=1, b_n= 3+$(-1)^n$, $c_n = \frac{1}{4}(-1)^n$, $a_n = \frac{3}{4}$,

$$Z_n = (1)^{3n}\left(\tfrac{-5}{2}(-1)^{n+2}\right)$$

$$\sum_{j=n1}^{\infty}\sum_{i=j+e-t}^{j-1} c_i = \tfrac{1}{4}\sum_{j=n_1}^{\infty}(-1)^{i-1} = 1 < \infty$$

- $b_n = 3 + (-1)^n \geq 1$
- $c_{n+e-t} - a_n = \frac{1}{4}(-1)^{n-1} - \frac{3}{4} = \frac{-1}{2} < 0 \; n \geq 1$
- $\liminf_{n\to\infty} \sum_{j=n-1}^{n-1} \sum_{i=j}^{j+1} \frac{|c_{j+e-t}-a_i|}{b_{i-t+u}} = \lim_{n\to\infty} \sum_{n-1}^{n-1} \sum_{i=j}^{j+1} \frac{\frac{1}{4}(-1)^{n-1}-\frac{3}{4}}{3+(-1)^{i-2}} = \frac{1}{4}$

Theorem (3) indicates that each constrained solution $g_n = (1)^n$ be the such the solutions did solution of (E_1) oscillates, for instance.

Example 3.2: Put a difference equation

$$\Delta^2(g_n\text{-}\,(1 + \left(\frac{1}{3}\right)^n)(2)^{2n-1} + \frac{2}{3}(2)^{2n-2} - \left(\frac{1}{3}\right)^n (2)^{2n-3}$$

$$= (2)^{2n}\left(\frac{29}{24} - \frac{1}{4}\left(\frac{1}{3}\right)^{n+2}\right) \text{(E2)}$$

where t=3, u =1, e=2, b=1, $b_n = 1 + \left(\frac{1}{3}\right)^n$, $c_n = \left(\frac{1}{3}\right)^n$, $a_n = \frac{3}{2}$,

$$Z_n = (2)^{2n}\left(\frac{29}{24} - \frac{1}{4}\left(\frac{1}{3}\right)^{n+2}\right)$$

- $\sum_{j=n1}^{\infty} \sum_{i=j-e+t}^{j-1} a_i = \sum_{j=n_1}^{\infty} \sum_{i=j-1}^{j-1} \left(\frac{1}{3}\right)^j =$
  $\sum_{j=n_1}^{\infty} \left(\frac{1}{3}\right)^{j-1} = 3\sum_{j=n_1}^{\infty} \left(\frac{1}{3}\right)^j < \infty$
- $b_n = (\frac{1}{2})^n \leq 1$, n > 0
- $c_n - a_{n-e+t} = \frac{7}{3} - \left(\frac{1}{3}\right)^{n-1} = \frac{1}{3} > 0, n \geq 0$
- $\liminf_{n\to\infty} \sum_{j=n-(e-b)}^{n-1} \sum_{i=j}^{j+b} (c_i - a_{i-e+t}) = \lim_{\inf_{n\to\infty}} \sum_{j=n-1}^{n-1} \sum_{i=j}^{j+1} \frac{7}{3} -$
  $\left(\frac{1}{3}\right)^{n-1} > \frac{1}{4}$

By use theorems (5), each limit solution for (E2) oscillates, for example g_n=$(2)^{2\,n}$ be a such the solutions.

4. Conclusion

We study v_n, V_n are series and was established. Certain prerequisites must be met in order for any solution to a second-order neutral difference equation, together with negative and positive coefficients, to exhibit oscillation. In conditions (B_3) we can use $\lim_{n\to\infty} Z_n = 0$ and the results remain true.

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