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On $\backsim_{\varepsilon}$-volterra and weakly $\backsim_{\varepsilon}$-volterra via $\backsim$-topological space and applications

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Abstract

In this paper, we will study some of the properties of the operator Ɛ(Ḩ) with respect to $\backsim$- a topological space defined by a cluster system, some of which are unconditionally true and others are conditionally true. We will also discuss some important theorems about operators $\backsim$-interior and $\backsim$-closure, and use these concepts to define operators $\backsim_{\varepsilon}$-volterra and weakly $\backsim_{\varepsilon}$-Volterra. We will also review the relationship between these two concepts. In addition, we will present some basic theorems about these two concepts.

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1. Introduction

Classifying sets in mathematics is a basic concept, and many scientists have drawn on these ideas to develop the foundational concepts in topology. In 1966, Kuratowski [1] introduced an ideal space, and in 1947, Choquet [2] defined grill space. Letting Ɛ be a cluster system on Ꞩ, a family Ŧ of subsets of Ɛ, that satisfies the conditions: 1) the empty set ,Ꞩ element of Ŧ, the set is an inverted lazy s of subsets of Ɛ power set and Ɛ is called Π̧-network in Ꞩ iff Ɛ(Ꞩ) = Ꞩ, and Ɛ is a Π̧ −network in Ŗ if for any Ŗ, Ş ⊆ Ꞩ if ∀∅ ≠ Ş ∈ Ŧ, ∀∅ ≠ Ŗ ∈ Ŧ ∃ Ģ ∈ Ɛ ∋ Ģ ⊆ Ş ⊆ Ŗ. The authors in [3] introduced the relation $\backsim$ on Ꞩ defined by: Ḩ $\backsim$ Ķ if and only if ∀ Ģ ∈ Ɛ ∋ Ģ ⊈ Ḩ − Ķ. Through this bilateral relationship, they learned a new topological space by let Ɛ be a cluster system on

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Ꞩ is a family $Ŧ_{ꝏ}$ of subsets of Ɛ, that satisfies the conditions: 1) $\emptyset, Ꞩ \in Ŧ_{ꝏ}$. 2) For any $Ⱨ, Ⱪ \in Ŧ_{ꝏ}$, there exists a proper subset $Ɱ \in Ŧ_{ꝏ}$ such that $Ⱨ \cap Ⱪ$ ꝏ $Ɱ$. 3) For any index $ʌ, Ⱨ_{ƛ} \in Ŧ_{ꝏ}, \forall ƛ \in ʌ$, there exists a proper subset $Ɱ \in Ŧ_{ꝏ}$ such that $\bigcup_{ƛ \in ʌ} Ⱨ_{ƛ}$ ꝏ $Ɱ$. 4) Ɛ is a Ƞ −network in Ꞩ. Then $Ꞩ_{Ɛ}^{Ŧ_{ꝏ}}$ is called ꝏ −topological space. The members of $Ŧ_{ꝏ}$ is called ꝏ-open set and the complement of ꝏ-open set is called ꝏ-closed set and the intersection and union of two sets of ꝏ-open do not have to be ꝏ-open. They also provided a definition$ꝏ_{Ɨnŧ}(Ⱨ) = \cup\{Ꞧ \in Ŧ_{ꝏ} : Ꞧ \subseteq Ⱨ\}$, $ꝏ_{\mathcal{C}\ell}(Ⱨ) = \cap\{ƒ : ƒ \text{is ꝏ −closed}, Ⱨ \subseteq ƒ\}$ and $ꝏ_{Ɨnŧ}(Ⱨ)$, $ꝏ_{\mathcal{C}\ell}(Ⱨ)$is not ꝏ-open, ꝏ-closed respectively. Let Ꞡ be any a nonempty collection of subsets of Ꞩ is satisfy: I) The I-property iff $\forall\, Ⱨ, Ⱪ \in Ꞡ$,then $Ⱨ \cap Ⱪ \in Ꞡ$. II) The ƛ -additive property iff for any index $ʌ, Ⱨ_{ƛ} \in Ꞡ, \forall ƛ \in ʌ$ such that $\bigcup_{ƛ \in ʌ} Ⱨ_{ƛ} \in Ꞡ$. In 1881, Volterra [4] introduced the concept of Volterra and weakly Volterra, which is named after him. In 1993, Gauld and Piotrowski [5] reformulated the concept of Volterra into continuous functions and studied some of its properties. Jeyanthi, Geetha, and Al-Omari [6] and others presented a generalization of Volterra -space using the g-space. Many authors studied classes of sets see [3], [7], [8], [9].

2. Preliminaries

In this section, we will review the properties of Ɛ(Ⱨ) as well as some basic theorems about $ꝏ_{Ɨnŧ}$ and $ꝏ_{\mathcal{C}\ell}$. Some of these properties can be satisfied under certain conditions, while others can be satisfied unconditionally.

Theorem 2.1: [10, 11] *Let $Ꞩ_{Ɛ}^{Ŧ}$ be a cluster system topological space. Then the following are equivalent:*

1) *Ɛ is a Ƞ-network in Ꞧ, for any $Ꞧ \in Ŧ - \{\emptyset\}$.*
2) *For any $Ꞧ \in Ŧ - \{\emptyset\}$, then $Ꞧ \subseteq Ɛ(Ꞧ)$. 3) $Ɛ(Ꞩ) = Ꞩ$.*

Proposition 2.2: Let $Ꞩ_{Ɛ}^{Ŧ_{ꝏ}}$ be a ꝏ- topological space and $Ⱨ, Ⱪ \subseteq Ꞩ$, then:

1) If $Ⱨ \subseteq Ⱪ$,then $ꝏ_{Ɨnŧ}(Ⱨ) \subseteq ꝏ_{Ɨnŧ}(Ⱪ)$.
2) $ꝏ_{Ɨnŧ}(Ⱨ) \subseteq Ⱨ$.
3) $ꝏ_{Ɨnŧ}(ꝏ_{Ɨnŧ}(Ⱨ)) \subseteq ꝏ_{Ɨnŧ}(Ⱨ)$.
4) $ꝏ_{Ɨnŧ}(Ⱨ \cap Ⱪ) \subseteq ꝏ_{Ɨnŧ}(Ⱨ) \cap ꝏ_{Ɨnŧ}(Ⱪ)$.
5) If $Ⱨ \in Ŧ_{ꝏ}$, then $ꝏ_{Ɨnŧ}(Ⱨ) = Ⱨ$.

Proposition 2.3: Let $Ꞩ_{Ɛ}^{Ŧ_{ꝏ}}$ be a ꝏ- topological space and let $Ⱨ, Ⱪ \subseteq Ꞩ$, then:

1) $Ⱨ \subseteq ꝏ_{\mathcal{C}\ell}(Ⱨ)$.
2) $ꝏ_{\mathcal{C}\ell}(Ⱨ) \subseteq ꝏ_{\mathcal{C}\ell}(ꝏ_{\mathcal{C}\ell}(Ⱨ))$.
3) If $Ⱨ \subseteq Ⱪ$, then $ꝏ_{\mathcal{C}\ell}(Ⱨ) \subseteq ꝏ_{\mathcal{C}\ell}(Ⱪ)$.
4) $ꝏ_{\mathcal{C}\ell}(Ⱨ) \cup ꝏ_{\mathcal{C}\ell}(Ⱪ) \subseteq ꝏ_{\mathcal{C}\ell}(Ⱨ \cup Ⱪ)$.

5) If Ⱨ is ꝏ-closed set, then ꝏ$_{\mathcal{C}\ell}$ (Ⱨ) = Ⱨ.

Remark 2.4:

1) If Ȿ$_{\varepsilon}^{Ŧꝏ}$ is satisfy the ɉ-additive property, then:
   a) ꝏ$_{Ɨnƭ}$ (ꝏ$_{Ɨnƭ}$ (Ⱨ)) =ꝏ$_{Ɨnƭ}$ (Ⱨ).
   b) ꝏ$_{\mathcal{C}\ell}$ (ꝏ$_{\mathcal{C}\ell}$ (Ⱨ)) =ꝏ$_{\mathcal{C}\ell}$ (Ⱨ) .
   c) ꝏ$_{\mathcal{C}\ell}$ (Ⱨ) ∪ꝏ$_{\mathcal{C}\ell}$ (Ⱪ) =ꝏ$_{\mathcal{C}\ell}$ (Ⱨ ∪ Ⱪ).
   d) ꝏ$_{\mathcal{C}\ell}$ (Ⱨ) is ꝏ-closed set e) ꝏ$_{Ɨnƭ}$ (Ⱨ) is a ꝏ-open set.
2) If Ȿ$_{\varepsilon}^{Ŧꝏ}$ is satisfy the I-property, then ꝏ$_{Ɨnƭ}$ (Ⱨ ∩ Ⱪ) =ꝏ$_{Ɨnƭ}$ (Ⱨ) ∩ ꝏ$_{Ɨnƭ}$ (Ⱪ), ∀Ⱨ , Ⱪ ⊆ Ȿ.

Proposition 2.5: Let Ȿ$_{\varepsilon}^{Ŧꝏ}$ be a ꝏ- topological space, and Ⱨ ⊆ Ȿ, if ȿ ∈ ꝏ$_{\mathcal{C}\ell}$ (Ⱨ) then Ɽ ∩ Ⱨ ≠ ∅, for any ꝏ-open set Ɽ of Ȿ.

Proposition 2.6: Let Ȿ$_{\varepsilon}^{Ŧꝏ}$ be a ꝏ- topological space with ɉ-additive property and Ⱨ ⊆ Ȿ , if for any ꝏ-open set Ɽ of Ȿ , Ɽ ∩ Ⱨ ≠ ∅, then ȿ ∈ ꝏ$_{\mathcal{C}\ell}$ (Ⱨ).

Proof: Let ȿ ∉ꝏ$_{\mathcal{C}\ell}$ (Ⱨ), since Ŧ$_{\infty}$ with ɉ-additive property, then ꝏ$_{\mathcal{C}\ell}$ (Ⱨ) is ꝏ-closed set, let Ɽ = Ȿ −ꝏ$_{\mathcal{C}\ell}$ (Ⱨ) is ꝏ-open set containing ȿ, since Ⱨ ⊆ꝏ$_{\mathcal{C}\ell}$ (Ⱨ), then Ⱨ ∩ Ɽ = ∅.

Proposition 2.7: Let Ȿ$_{\varepsilon}^{Ŧꝏ}$ be a ꝏ- topological space with ɉ-additive property and I-property and Ⱨ ⊆ Ȿ, if for any ꝏ-open set Ɽ of Ȿ then Ɽ ∩ꝏ$_{\mathcal{C}\ell}$ (Ⱨ) ⊆ ꝏ$_{\mathcal{C}\ell}$ (Ɽ ∩ Ⱨ).

Proposition 2.8: Let Ȿ$_{\varepsilon}^{Ŧꝏ}$ be a ꝏ- topological space with ɉ-additive property and I-property and Ⱨ, Ⱪ ⊆ Ȿ, if Ⱨ ∈ Ŧ$_{\infty}$ or Ⱪ ∈ Ŧ$_{\infty}$then, ꝏ$_{Ɨnƭ}$ (ꝏ$_{\mathcal{C}\ell}$ (Ⱨ ∩ Ⱪ)) = ꝏ$_{Ɨnƭ}$ (ꝏ$_{\mathcal{C}\ell}$ (Ⱨ)) ∩ꝏ$_{Ɨnƭ}$ (ꝏ$_{\mathcal{C}\ell}$ (Ⱪ)).

Corollary 2.9: *Let Ȿ$_{\varepsilon}^{Ŧꝏ}$ be a ꝏ- topological space with ɉ-additive property and I-property and* Ⱨ, Ⱪ ⊆ Ȿ, *if* Ⱨ, Ⱪ ∈ Ŧ$_{\infty}$ *then,* ꝏ$_{Ɨnƭ}$ (ꝏ$_{\mathcal{C}\ell}$ (Ⱨ ∩ Ⱪ)) =ꝏ$_{Ɨnƭ}$ (ꝏ$_{\mathcal{C}\ell}$ (Ⱨ)) ∩ꝏ$_{Ɨnƭ}$ (ꝏ$_{\mathcal{C}\ell}$ (Ⱪ)).

Definition 2.10: Let Ȿ$_{\varepsilon}^{Ŧꝏ}$ be a ꝏ- topological space and Ⱨ, Ⱪ ⊆ Ȿ, then Ⱪ is called ꝏ-dense in Ⱨ iff Ⱨ ⊆ ꝏ$_{\mathcal{C}\ell}$ (Ⱪ).

Definition 2.11: Let Ȿ$_{\varepsilon}^{Ŧꝏ}$ be a ꝏ- topological space. For any Ⱨ ⊆ Ȿ, we define Ɛ(Ⱨ) = {ȿ ∈ Ȿ: ∀ Ɽ ∈ Ŧ$_{\infty(ȿ)}$ ∃Ģ ∈ Ɛ ∋ Ģ ⊆ Ɽ ∩ Ⱨ}, is called ꝏ-operator of Ɛ(Ⱨ).

Theorem 2.12: *Let Ȿ$_{\varepsilon}^{Ŧꝏ}$ be a ꝏ- topological space and* Ⱨ, Ⱪ ⊆ Ȿ, *then:*

1) $\mathcal{E}(\emptyset) = \emptyset$.
2) *If* $H \subseteq K$, *then* $\mathcal{E}(H) \subseteq \mathcal{E}(K)$.
3) $\mathcal{E}(\mathcal{E}(H)) \subseteq \mathcal{E}(H)$.

Theorem 2.13: *Let* $S_{\mathcal{E}}^{T_\infty}$ *be a* ∞*- topological space,* $H \subseteq S$, *with* $\mathcal{E} = \{G: \emptyset \neq G \in T_\infty\}$, *then* $\infty_{Int}(H) = \emptyset$ *iff* $\mathcal{E}(H) = \emptyset$.

Theorem 2.14: *Let* $S_{\mathcal{E}}^{T_\infty}$ *be a* ∞*- topological space with* λ*-additive property and* $H \subseteq S$, *then* $\mathcal{E}(H) \subseteq \infty_{\mathcal{C}\ell}(H)$.

Theorem 2.15: *Let* $S_{\mathcal{E}}^{T_\infty}$ *be a* ∞*- topological space with* λ*-additive property and* $H \subseteq S$, *if* $\mathcal{E} = \{G: \emptyset \neq G \in T_\infty\}$, *then* $\mathcal{E}(H) \subseteq \infty_{\mathcal{C}\ell}\left(\infty_{Int}(H)\right)$.

Theorem 2.16: *Let* $S_{\mathcal{E}}^{T_\infty}$ *be a* ∞*- topological space with I-property and* $H \subseteq S$ *if* $\mathcal{E} = \{G: \emptyset \neq G \in T_\infty\}$, *then* $\infty_{\mathcal{C}\ell}\left(\infty_{Int}(H)\right) \subseteq \mathcal{E}(H)$.

Corollary 2.17: *Let* $S_{\mathcal{E}}^{T_\infty}$ *be a* ∞*- topological space with* λ*-additive property and I-property, if* $\mathcal{E} = \{G: \emptyset \neq G \in T_\infty\}$, *then* $\mathcal{E}(H) = \infty_{\mathcal{C}\ell}\left(\infty_{Int}(H)\right)$.

Theorem 2.18: *Let* $S_{\mathcal{E}}^{T_\infty}$ *be a* ∞*- topological space with* λ*-additive property and I-property and for any nonempty* ∞*-open set* R *in* S, *then* $\mathcal{E}(R) = \mathcal{E}\left(\infty_{\mathcal{C}\ell}(R)\right) = \infty_{\mathcal{C}\ell}(R)$.

Proof: From (theorem 2.14), we get $\mathcal{E}(R) \subseteq \infty_{\mathcal{C}\ell}(R)$ … (1). Now to prove $\infty_{\mathcal{C}\ell}(R) \subseteq \mathcal{E}(R)$, let $s \in \infty_{\mathcal{C}\ell}(R)$, then $\forall R_1 \in T_{\infty(s)}$ such that $R \cap R_1 \neq \emptyset$, since T_∞ with I-property, then $R \cap R_1$ is ∞-open set in S by (Theorem 2.1) ,we get $R \cap R_1 \subseteq \mathcal{E}(R \cap R_1)$ so $s \in \mathcal{E}(R \cap R_1)$, then $\forall V \in T_{\infty(s)}\ \exists G_s \in \mathcal{E}$ such that $G_s \subseteq V \cap (R \cap R_1) = (V \cap R_1) \cap R$, so $s \in \mathcal{E}(R)$, then $\infty_{\mathcal{C}\ell}(R) \subseteq \mathcal{E}(R)$ …(2), from (1) and (2), we get $\infty_{\mathcal{C}\ell}(R) = \mathcal{E}(R)$. Now, by (Proposition 2.3, part 1), we get $R \subseteq \infty_{\mathcal{C}\ell}(R)$ and by (Theorem 2.10, part2) then $\mathcal{E}(R) \subseteq \mathcal{E}(\infty_{\mathcal{C}\ell}(R))$ and by $(\mathcal{E}(R) = \infty_{\mathcal{C}\ell}(R))$, we get $\infty_{\mathcal{C}\ell}(R) \subseteq \mathcal{E}(\infty_{\mathcal{C}\ell}(R))$ … (1). Now, to prove $\mathcal{E}(\infty_{\mathcal{C}\ell}(R)) \subseteq \infty_{\mathcal{C}\ell}(R)$, by (Theorem 2.14), we get $\mathcal{E}(\infty_{\mathcal{C}\ell}(R)) \subseteq \infty_{\mathcal{C}\ell}(\infty_{\mathcal{C}\ell}(R))$, since T_∞ with λ-additive property, we get $\mathcal{E}\left(\infty_{\mathcal{C}\ell}(R)\right) \subseteq \infty_{\mathcal{C}\ell}\left(\infty_{\mathcal{C}\ell}(R)\right) = \infty_{\mathcal{C}\ell}(R)$ … (2), from (1) and (2), then $\mathcal{E}\left(\infty_{\mathcal{C}\ell}(R)\right) = \infty_{\mathcal{C}\ell}(R)$, therefore $\mathcal{E}(R) = \mathcal{E}\left(\infty_{\mathcal{C}\ell}(R)\right) = \infty_{\mathcal{C}\ell}(R)$.

3. Weakly $\infty_{\mathcal{E}}$-Volterra Space

Definition 3.1: Let $S_{\mathcal{E}}^{T_\infty}$ be a ∞- topological space. A subset H of S is called weakly $\infty_{\mathcal{E}}$-Volterra if for any $H_1, H_2 \subseteq S$, such that $\mathcal{E}(H) \subseteq \mathcal{E}(H_i), (i = 1,2)$, $H_1 \cap H_2 \neq \emptyset$.

$-A$ subset Ḩ of $Ꞩ$ is called not weakly ꝏ$_{Ꞓ}$-Volterra if there exists Ḩ$_1$, Ḩ$_2 \subseteq Ꞩ$ such that Ꞓ(Ḩ) $\subseteq$ Ꞓ(Ḩ$_i$), $(i = 1,2)$, Ḩ$_1 \cap$ Ḩ$_2 = \emptyset$.

Example 3.2: Let Ꞩ = {ḩ, ķ, ɱ, ʐ}, ₮$_{\infty}$ = {$\emptyset$, Ꞩ, {ɱ}, {ʐ}, {ɱ, ʐ}} and Ꞓ = {{ḩ}, {ɱ}, {ʐ}}. If Ḩ = {ḩ}, Ḩ$_1$ = {ʐ} and Ḩ$_2$ = {ķ, ʐ}, then Ꞓ(Ḩ) = {ḩ, ķ} and Ꞓ(Ḩ$_1$) = Ꞓ(Ḩ$_2$) = {ḩ, ķ, ʐ}, so Ꞓ(Ḩ) $\subseteq$ Ꞓ(Ḩ$_1$) and Ꞓ(Ḩ) $\subseteq$ Ꞓ(Ḩ$_2$), Ḩ$_1 \cap$ Ḩ$_2 \neq \emptyset$, then Ḩ is weakly ꝏ$_{Ꞓ}$-Volterra.

Proposition 3.3: Let Ꞩ$_{Ꞓ}^{₮_{\infty}}$ be a ꝏ- topological space and Ḩ $\subseteq Ꞩ$, if Ḩ is weakly ꝏ$_{Ꞓ}$-Volterra and Ḩ$_1$, Ḩ$_2$ are nonempty subset of $Ꞩ$, then Ꞓ(Ḩ) and Ḩ are nonempty of $Ꞩ$.

Proposition 3.4: Let Ꞩ$_{Ꞓ}^{₮_{\infty}}$ be a ꝏ- topological space and Ḩ, Ḩ$_1 \subseteq Ꞩ$, if Ḩ is weakly ꝏ$_{Ꞓ}$-Volterra and Ꞓ(Ḩ) $\subseteq$ Ꞓ(Ḩ$_1$), then Ḩ $\cap$ Ḩ$_1 \neq \emptyset$.

Proposition 3.5: Let Ꞩ$_{Ꞓ}^{₮_{\infty}}$ be a ꝏ- topological space and Ḩ, Ķ $\subseteq Ꞩ$, if Ḩ is weakly ꝏ$_{Ꞓ}$-Volterra and Ḩ $\subseteq$ Ķ, then Ķ is weakly ꝏ$_{Ꞓ}$-Volterra.

Proof: Since Ḩ is weakly ꝏ$_{Ꞓ}$-Volterra and Ḩ $\subseteq$ Ķ, let Ḩ$_1$, Ḩ$_2 \subseteq Ꞩ$ such that Ꞓ(Ķ) $\subseteq$ Ꞓ(Ḩ$_i$), $(i = 1,2)$, by (Theorem 2.12, part 2), since Ꞓ(Ḩ) $\subseteq$ Ꞓ(Ķ), by (Definition 3.1), so Ꞓ(Ḩ) $\subseteq$ Ꞓ(Ḩ$_i$), $(i = 1,2)$, since Ḩ is weakly ꝏ$_{Ꞓ}$-Volterra, we get Ḩ$_1 \cap$ Ḩ$_2 \neq \emptyset$, therefore Ķ is weakly ꝏ$_{Ꞓ}$-Volterra.

Proposition 3.6: Let Ꞩ$_{Ꞓ}^{₮_{\infty}}$ be a ꝏ- topological space and Ḩ, Ķ $\subseteq Ꞩ$, if Ḩ is not weakly ꝏ$_{Ꞓ}$-Volterra and Ķ $\subseteq$ Ḩ, then Ķ is not weakly ꝏ$_{Ꞓ}$-Volterra.

Proposition 3.7: Let Ꞩ$_{Ꞓ}^{₮_{\infty}}$ be a ꝏ- topological space and Ḩ $\subseteq Ꞩ$, if Ḩ is not weakly ꝏ$_{Ꞓ}$-Volterra, then Ꞓ(Ḩ) is not weakly ꝏ$_{Ꞓ}$-Volterra.

Proposition 3.8: Let Ꞩ$_{Ꞓ}^{₮_{\infty}}$ be a ꝏ- topological space with ʓ-additive property and I- property and $\emptyset \neq$ Ɽ $\in$ ₮$_{\infty}$, then Ɽ is weakly ꝏ$_{Ꞓ}$-Volterra iff ꝏ$_{\mathcal{C}\ell}$ (Ɽ) is weakly ꝏ$_{Ꞓ}$-Volterra.

Proof: $\Longrightarrow$) To prove ꝏ$_{\mathcal{C}\ell}$ (Ɽ) is weakly ꝏ$_{Ꞓ}$-Volterra, since Ɽ is weakly ꝏ$_{Ꞓ}$-Volterra and Ɽ $\subseteq$ꝏ$_{\mathcal{C}\ell}$ (Ɽ), by (Proposition 3.5), then ꝏ$_{\mathcal{C}\ell}$ (Ɽ) is weakly ꝏ$_{Ꞓ}$-Volterra.

$\Longleftarrow$) To prove Ɽ is weakly ꝏ$_{Ꞓ}$-Volterra, let Ḩ$_1$, Ḩ$_2 \subseteq Ꞩ$ such that Ꞓ(Ɽ) $\subseteq$ Ꞓ(Ḩ$_i$), $(i = 1,2)$, by (Theorem 2.18), we get Ꞓ(Ɽ) = Ꞓ(ꝏ$_{\mathcal{C}\ell}$ (Ɽ)), and since ꝏ$_{\mathcal{C}\ell}$ (Ɽ) is weakly ꝏ$_{Ꞓ}$-Volterra such that Ꞓ(ꝏ$_{\mathcal{C}\ell}$ (Ɽ)) $\subseteq$ Ꞓ(Ḩ$_i$), $(i = 1,2)$, so Ḩ$_1 \cap$ Ḩ$_2 \neq \emptyset$. Therefore Ɽ is weakly ꝏ$_{Ꞓ}$-Volterra.

Proposition 3.9: Let Ꞩ$_{Ꞓ}^{₮_{\infty}}$ be a ꝏ- topological space and Ḩ $\subseteq Ꞩ$, with Ꞓ = {Ģ: $\emptyset \neq$ Ģ $\in$ ₮$_{\infty}$}, if Ḩ is weakly ꝏ$_{Ꞓ}$-Volterra then ꝏ$_{Int}$ (Ḩ) $\neq \emptyset$.

Proof: Let H is weakly $\omega_{\mathcal{E}}$-Volterra, then for any $H_1, H_2 \subseteq S$, such that $\mathcal{E}(H) \subseteq \mathcal{E}(H_i), (i = 1,2)$, $H_1 \cap H_2 \neq \emptyset$. By (Proposition 3.3), then $\emptyset \neq \mathcal{E}(H)$. If possible $\omega_{Int}(H) = \emptyset$, from (Theorem 2.13), then $\mathcal{E}(H) = \emptyset$, which is contradiction with hypothesis, then $\omega_{Int}(H) \neq \emptyset$.

Proposition 3.10: Let $S_{\mathcal{E}}^{\mathcal{T}_\omega}$ be a ω- topological space with χ-additive property and I-property and $H \subseteq S$, with $\mathcal{E} = \{G : \emptyset \neq G \in \mathcal{T}_\omega\}$, if $\omega_{Int}(H) \neq \emptyset$ then H is weakly $\omega_{\mathcal{E}}$-Volterra.

Proof: Let $\omega_{Int}(H) \neq \emptyset$, since $\emptyset \neq \omega_{Int}(H) \subseteq H$,then $H \neq \emptyset$ by (Theorem 2.13), so $\mathcal{E}(H) \neq \emptyset$. Let $H_1, H_2 \subseteq S$, such that $\emptyset \neq \mathcal{E}(H) \subseteq \mathcal{E}(H_i), (i = 1,2)$, so $\mathcal{E}(H_i) \neq \emptyset$, if possible $H_1 = \emptyset$ or $H_2 = \emptyset$, then $\mathcal{E}(H_i) = \emptyset$ which is contradiction, so $H_1 \neq \emptyset$ and $H_2 \neq \emptyset$, since $\emptyset \neq \omega_{Int}(H_1) \subseteq H_1$, $\emptyset \neq \omega_{Int}(H_2) \subseteq H_2$ and $\mathcal{E}(H) \subseteq \mathcal{E}(H_i), (i = 1,2)$, by (Corollary 2.17),we get $\mathcal{E}(H) = \omega_{C\ell}(\omega_{Int}(H))$, so $\emptyset \neq \omega_{Int}(H) \subseteq \omega_{C\ell}(\omega_{Int}(H)) \subseteq \omega_{C\ell}(\omega_{Int}(H_1)) \cap \omega_{C\ell}(\omega_{Int}(H_2))$, from (Corollary 2.9) we get, $\emptyset \neq \omega_{Int}(\omega_{Int}(H)) \subseteq \omega_{Int}(\omega_{C\ell}(\omega_{Int}(H))) \subseteq \omega_{Int}(\omega_{C\ell}(\omega_{Int}(H_1)) \cap \omega_{C\ell}(\omega_{Int}(H_2))) = \omega_{Int}(\omega_{C\ell}(\omega_{Int}(H_1 \cap H_2)))$, therefore $\emptyset \neq \omega_{Int}(H) \subseteq \omega_{Int}\left(\omega_{C\ell}\left(\omega_{Int}(H_1 \cap H_2)\right)\right) \subseteq \omega_{C\ell}(\omega_{Int}(H_1 \cap H_2)) \subseteq \omega_{C\ell}\left((H_1 \cap H_2)\right)$, if possible $H_1 \cap H_2 = \emptyset$, then $\omega_{Int}(H) = \emptyset$, which is contradiction, then $H_1 \cap H_2 \neq \emptyset$, we have H is weakly $\omega_{\mathcal{E}}$-Volterra.

4. $\omega_{\mathcal{E}}$-Volterra Space

Definition 4.1: Let $S_{\mathcal{E}}^{\mathcal{T}_\omega}$ be a ω- topological space. A nonempty subset H of S is called $\omega_{\mathcal{E}}$-Volterra iff for any $H_1, H_2 \subseteq S$ such that $\mathcal{E}(H) \subseteq \mathcal{E}(H_i), (i = 1,2)$, $H_1 \cap H_2$ is ω-dense in H.

Example 4.2: $S = \{h, k, m, z\}$, $\mathcal{T}_\omega = \{\emptyset, S, \{m\}, \{z\}, \{m, z\}\}$ and $\mathcal{E} = \{\{h\}, \{m\}, \{z\}\}$. If $H = \{h\}, H_1 = \{h, m\}, H_2 = \{h, k, z\}$, then $\mathcal{E}(H) = \{h, k\}, \mathcal{E}(H_1) = \{h, k, m\}, \mathcal{E}(H_2) = \{h, k, z\}$, then $\mathcal{E}(H) \subseteq \mathcal{E}(H_i), (i = 1,2)$, and $\omega_{C\ell}(H_1 \cap H_2) = \omega_{C\ell}(\{h\}) = \{h, k\} \supseteq \{h\}$, so $H_1 \cap H_2$ is ω-dense in H.

Theorem 4.3: *Let $S_{\mathcal{E}}^{\mathcal{T}_\omega}$ be a ω- topological space and $H \subseteq S$, if H is $\omega_{\mathcal{E}}$-Volterra set then H is weakly $\omega_{\mathcal{E}}$-Volterra set.*

Proof: Since H is $\omega_{\mathcal{E}}$-Volterra, then for any $H_1, H_2 \subseteq S$ such that $\mathcal{E}(H) \subseteq \mathcal{E}(H_i), (i = 1,2), \emptyset \neq H \subseteq \omega_{C\ell}(H_1 \cap H_2)$, but $H_1 \cap H_2 \subseteq \omega_{C\ell}(H_1 \cap H_2) \neq \emptyset$. If possible that, $H_1 \cap H_2 = \emptyset$, we get that $\omega_{C\ell}(H_1 \cap H_2) = \emptyset$, which is contradiction, because $\omega_{C\ell}(H_1 \cap H_2) \neq \emptyset$, we hence $H_1 \cap H_2 \neq \emptyset$, therefore H is weakly $\omega_{\mathcal{E}}$-Volterra.

Remark 4.4: The converse of (Theorem 4.3) is not necessary true. See in the following example.

Example 4.5: Let $\mathcal{S} = \{h, \kappa, м, z\}$ where $\mathcal{S}$ represents the number of types of mobile devices used by a segment of people, where h refers to Apple devices, κ refers to Samsung devices, $м$ refers to Nokia devices, and z refers to Honor devices, $\mathcal{T}_\infty = \{\emptyset, \mathcal{S}, \{h, z\}, \{\kappa, z\}, \{z\}\}$ and $\mathcal{E} = \{\{\kappa\}, \{м\}, \{z\}, \{h, z\}, \{\kappa, z\}\}$. If $H = \{h, м\}, H_1 = \{м, z\}$ and, $H_2 = \{h, \kappa, м\}$ then $\mathcal{E}(H) = \{м\}, \mathcal{E}(H_1) = \mathcal{S}$ and $\mathcal{E}(H_2) = \{\kappa, м\}$, then $\mathcal{E}(H) \subseteq \mathcal{E}(H_1)$ and $\mathcal{E}(H) \subseteq \mathcal{E}(H_2)$ and $H_1 \cap H_2 = \{м\} \neq \emptyset$ so, H is weakly ∞_ε-Volterra. But we note that $\infty_{\mathcal{C}\ell}\,(H_1 \cap H_2) = \infty_{\mathcal{C}\ell}\,(\{м\}) = (\{м\})$, where $H \nsubseteq \infty_{\mathcal{C}\ell}\,(H_1 \cap H_2)$, so H is not ∞_ε-Volterra.

Proposition 4.6: Let $\mathcal{S}_\varepsilon^{\mathcal{T}_\infty}$ be a ∞- topological space and $H, H_1 \subseteq \mathcal{S}$, if H is ∞_ε-Volterra and $\mathcal{E}(H) \subseteq \mathcal{E}(H_1)$, then $H_1 \cap H_2$ is dense in H.

Proof: Since H is ∞_ε-Volterra, and for any $H_1 \subseteq \mathcal{S}$ such that $\mathcal{E}(H) \subseteq \mathcal{E}(H_1)$ but $\mathcal{E}(H) \subseteq \mathcal{E}(H)$, by (Definition 4.1), we get $\emptyset \neq H \subseteq \infty_{\mathcal{C}\ell}\,(H \cap H_1)$, so $H \cap H_1$ is dense in H.

Proposition 4.7: Let $\mathcal{S}_\varepsilon^{\mathcal{T}_\infty}$ be a ∞- topological space with λ-additive property, and $H \subseteq \mathcal{S}$, if H is ∞_ε-Volterra, then $\infty_{\mathcal{C}\ell}\,(H)$ is ∞_ε-Volterra.

Proof: Since H is ∞_ε-Volterra, let $H_1, H_2 \subseteq \mathcal{S}$ such that $\mathcal{E}(\infty_{\mathcal{C}\ell}\,(H)) \subseteq \mathcal{E}(H_i), (i = 1{,}2)$, since $H \subseteq \infty_{\mathcal{C}\ell}\,(H)$, by (Theorem 2.12, part 2), we get $\mathcal{E}(H) \subseteq \mathcal{E}(\infty_{\mathcal{C}\ell}\,(H)) \subseteq \mathcal{E}(H_i), (i = 1{,}2)$, then $H \subseteq \infty_{\mathcal{C}\ell}\,(H_1 \cap H_2)$, we hence $\infty_{\mathcal{C}\ell}\,(H) \subseteq \infty_{\mathcal{C}\ell}\left(\infty_{\mathcal{C}\ell}\,(H_1 \cap H_2)\right) = \infty_{\mathcal{C}\ell}\,(H_1 \cap H_2)$, because T_∞ with λ-additive property, then $\infty_{\mathcal{C}\ell}\,(H) \subseteq \infty_{\mathcal{C}\ell}\,(H_1 \cap H_2)$ and $H_1 \cap H_2$ is dence in $\infty_{\mathcal{C}\ell}\,(H)$, therefore $\infty_{\mathcal{C}\ell}\,(H)$ is ∞_ε-Volterra.

Proposition 4.8: Let $\mathcal{S}_\varepsilon^{\mathcal{T}_\infty}$be a ∞- topological space with λ-additive property and *I*- property and $\emptyset \neq R \in \mathcal{T}_\infty$, if R is ∞_ε-Volterra iff $\infty_{\mathcal{C}\ell}\,(R)$ is ∞_ε-Volterra.

Proposition 4.9: Let $\mathcal{S}_\varepsilon^{\mathcal{T}_\infty}$ be a ∞- topological space with λ-additive property and I-property and $H \subseteq \mathcal{S}$, with $\mathcal{E} = \{G : \emptyset \neq G \in \mathcal{T}_\infty\}$, if H is any nonempty ∞-open set then H is ∞_ε-Volterra.

5. Conclusion

We proved some properties of operator $\mathcal{E}(H)$ using λ-additive property and I-property, which are indispensable. We also proved that every ∞_ε-Volterra is weakly ∞_ε-Volterra without any condition, but the converse is not true. These ideas can be generalized in [3], [7].

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