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On $\mathcal{L}$- exterior points and some of its properties

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Abstract

The purpose of this paper is to introduce the exterior concept in the $\mathcal{L}$-space where this space satisfies the axioms for a topological space. Some important properties for this novel concept are investigated. Furthermore, the relationship among some concepts, such as exterior, closure, dense, and interior, in the $\mathcal{L}$-space is proven.

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1. Introduction

The authors in [1, 2] studied a family of sets that plays a role in many areas of mathematics, particularly in measure theory and topology.

The notion of the exterior point is a generalization of an idea from Euclidean geometry, where the exterior of a circle is the set of points farther from the center than from the radius. In a topological space, the exterior point is defined using a more abstract notion of openness. We are studying an exterior point in a topological space because it helps us understand the structure of topological spaces by partitioning the space into three regions: the interior, the boundary, and the exterior of the set [3]. The aim of studying the notion of dense set-in topology is to provide a way to approximate elements in larger spaces, which is basic to some areas of mathematics [4]. A study of nearly open sets, such as pre-open sets, aims to generalize the notion of an open set [5].

2. Preliminaries and Basic Notions.

Definition 2.1: [6]. Suppose we have the topology $\Im$ on the nonempty set M. A subset T of M be named $\mathcal{L}$ −open if $\mathrm{T} \subseteq \mathrm{H} \cup Int(\mathrm{T})$ $\forall \mathrm{H} \in \Im$ & $\mathrm{H} \neq \Phi$.

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We shall denote A family for every all $\mathcal{L}-$open sets by $\mathcal{L}o_{\mathrm{M}}$ and hence $(\mathrm{M}, \mathcal{L}o_{\mathrm{M}})$ is said to be the $\mathcal{L}-$space.

Definition 2.2: [6]. In the $\mathcal{L}-$space $(\mathrm{M}, \mathcal{L}o_{\mathrm{M}})$. The $\mathcal{L}-$closure for a subset $\mathcal{T}$ of M is an intersection for every $\mathcal{L}-$closed sets that include $\mathcal{T}$ and it's denoted by $cl_{\mathcal{L}}(\mathcal{T})$.

Theorem 2.3: [6]. *For any $\mathcal{L}-$space $(\mathrm{M}, \mathcal{L}o_{\mathrm{M}})$, we have $cl_{\mathcal{L}}(\mathcal{T})$ is the smallest $\mathcal{L}-$closed set including $\mathcal{T}$ for any subsets $\mathcal{T}$ of M.*

Corollary 2.4: [6]. *In the $\mathcal{L}-$space $(\mathrm{M}, \mathcal{L}o_{\mathrm{M}})$. The subset $\mathcal{T}$ of M is $\mathcal{L}-$closed iff $\mathcal{T} = cl_{\mathcal{L}}(\mathcal{T})$.*

Theorem 2.5: [6]. *Assume $(\mathrm{M}, \mathcal{L}o_{\mathrm{M}})$ is $\mathcal{L}-$space and $\mathcal{T}_1, \mathcal{T}_2 \subseteq \mathrm{M}$. If $\mathcal{T}_1 \subseteq \mathcal{T}_2$, then $cl_{\mathcal{L}}(\mathcal{T}_1) \subseteq cl_{\mathcal{L}}(\mathcal{T}_2)$.*

Definition 2.6: [6]. Assume $(\mathrm{M}, \mathcal{L}o_{\mathrm{M}})$ is a $\mathcal{L}-$ space. The subset $\mathcal{T}$ of M is called $\mathcal{L}-$dense set if $cl_{\mathcal{L}}(\mathcal{T}) = \mathrm{M}$.

Definition 2.7: [7]. In the $\mathcal{L}-$space $(\mathrm{M}, \mathcal{L}o_{\mathrm{M}})$. the point h in the subset $\mathcal{T}$ is said to be $\mathcal{L}-$interior point for the $\mathcal{T}$ if there is $\mathcal{R} \in \mathcal{L}o_{\mathrm{M}}$ such that $h \in \mathcal{R} \subseteq \mathcal{T}$, and $lnt_{\mathcal{L}}(\mathcal{F})$ denote to the set for each $\mathcal{L}-$interior points for the set $\mathcal{F}$.

Theorem 2.8: [7]. *Assume we have the $\mathcal{L}-$space $(\mathrm{M}, \mathcal{L}o_{\mathrm{M}})$, then*

$$lnt_{\mathcal{L}}(\mathcal{T}) = \bigcup_{\sigma \in \mathrm{I}} \{\mathcal{R}_\sigma : \mathcal{R}_\sigma \text{ is } \mathcal{L}-open,\ \mathcal{R}_\sigma \subseteq \mathcal{T}\ \forall \sigma \in \mathrm{I}\}.$$

Theorem 2.9: [7]. *Assume $(\mathrm{M}, \mathcal{L}o_{\mathrm{M}})$ is $\mathcal{L}-$ space & $\mathcal{T} \subseteq \mathrm{M}$, then $lnt_{\mathcal{L}}(\mathcal{T})$ is the largest $\mathcal{L}-$ open set contained in $\mathcal{T}$.*

Theorem 2.10: [7]. *Suppose $(\mathrm{M}, \mathcal{L}o_{\mathrm{M}})$ is $\mathcal{L}-$space & $\mathcal{T}, \mathcal{F} \subseteq \mathrm{M}$. Then:*

1. If $\mathcal{T} \subseteq \mathcal{F}$, then $lnt_{\mathcal{L}}(\mathcal{T}) \subseteq lnt_{\mathcal{L}}(\mathcal{F})$.
2. $lnt_{\mathcal{L}}(\mathcal{T} \cap \mathcal{F}) = lnt_{\mathcal{L}}(\mathcal{T}) \cap lnt_{\mathcal{L}}(\mathcal{F})$.
3. $lnt_{\mathcal{L}}\big(lnt_{\mathcal{L}}(\mathcal{T})\big) = lnt_{\mathcal{L}}(\mathcal{T})$.
4. $lnt_{\mathcal{L}}(\Phi) = \Phi$ and $lnt_{\mathcal{L}}(\mathrm{M}) = \mathrm{M}$.

Theorem 2.11: [7]. *Assume $(\mathrm{M}, \mathcal{L}o_{\mathrm{M}})$ is $\mathcal{L}-$space & $\mathcal{T} \subseteq \mathcal{X}$, then $\mathcal{T}$ is $\mathcal{L}-$ open iff $lnt_{\mathcal{L}}(\mathcal{T}) = \mathcal{T}$.*

3. The Main Results.

Definition 3.1: Suppose $(\mathrm{M}, \mathcal{L}o_{\mathrm{M}})$ is the $\mathcal{L}-$space & $\mathcal{T} \subseteq \mathrm{M}$. the point $h \in \mathrm{M}$ be named an $\mathcal{L}-$ exterior points for $\mathcal{T}$ where there is $\mathcal{P} \in \mathcal{L}o_{\mathrm{M}}$ such that $h \in \mathcal{P} \subseteq \mathcal{T}^c$.

Definition 3.2: If $(\mathrm{M}, \mathcal{L}o_{\mathrm{M}})$ be the $\mathcal{L}-$space & $\mathcal{T} \subseteq \mathrm{M}$. The set of all $\mathcal{L}-$ exterior points of $\mathcal{T}$ is known as the $\mathcal{L}-$ exterior of $\mathcal{T}$ and is denoted by $ext_{\mathcal{L}}(\mathcal{T})$.

Theorem 3.3: *If* $(\mathrm{M}, \mathcal{L}o_{\mathrm{M}})$ *is* $\mathcal{L}-space$ *&* $\mathcal{T} \subseteq \mathrm{M}$. *Then* $ext_{\mathcal{L}}(\mathcal{T}) = lnt_{\mathcal{L}}(\mathcal{T}^c)$.

Proof: Suppose that $h \epsilon ext_{\mathcal{L}}(\mathcal{T})$, then h is an $\mathcal{L}-$ exterior point of $\mathcal{T}$, so there is $\mathcal{P} \epsilon \mathcal{L}o_{\mathrm{M}}$ such that $h \epsilon \mathcal{P} \subseteq \mathcal{T}^c$, that is h is an $\mathcal{L}-$interior point of $\mathcal{T}^c$, hence $h \epsilon lnt_{\mathcal{L}}(\mathcal{T}^c)$, thus $ext_{\mathcal{L}}(\mathcal{T}) \subseteq lnt_{\mathcal{L}}(\mathcal{T}^c)$.

Assume $h \epsilon lnt_{\mathcal{L}}(\mathcal{T}^c)$, then h is an $\mathcal{L}-$interior point of $\mathcal{T}^c$, hence there is $\mathcal{P} \epsilon \mathcal{L}o_{\mathrm{M}}$ such that $h \epsilon \mathcal{P} \subseteq \mathcal{T}^c$, thus by the definition of $\mathcal{L}-$ exterior we have h is a $\mathcal{L}-$ exterior point of $\mathcal{T}$, hence $h \epsilon ext_{\mathcal{L}}(\mathcal{T})$.

Therefore, $ext_{\mathcal{L}}(\mathcal{T}) \supseteq lnt_{\mathcal{L}}(\mathcal{T}^c)$.

Hence, $ext_{\mathcal{L}}(\mathcal{T}) = lnt_{\mathcal{L}}(\mathcal{T}^c)$.

Theorem 3.4: *If* $(\mathrm{M}, \mathcal{L}o_{\mathrm{M}})$ *is* $\mathcal{L}-space$ *&* $\mathcal{T} \subseteq \mathrm{M}$. *Then* $ext_{\mathcal{L}}(\mathcal{T}^c) = lnt_{\mathcal{L}}(\mathcal{T})$.

Proof: Assume that $h \epsilon ext_{\mathcal{L}}(\mathcal{T}^c)$, then h is an $\mathcal{L}-$ exterior point of $\mathcal{T}^c$, so there is $\mathcal{P} \epsilon \mathcal{L}o_{\mathrm{M}}$ such that $h \epsilon \mathcal{P} \subseteq (\mathcal{T}^c)^c$, but $\mathcal{T} = (\mathcal{T}^c)^c$ which implies that h is an $\mathcal{L}-$interior point of $\mathcal{T}$, hence $h \epsilon lnt_{\mathcal{L}}(\mathcal{T})$, thus $ext_{\mathcal{L}}(\mathcal{T}^c) \subseteq lnt_{\mathcal{L}}(\mathcal{T})$.

Let $h \epsilon lnt_{\mathcal{L}}(\mathcal{T})$, then h is an $\mathcal{L}-$interior point of $\mathcal{T}$, hence there is $\mathcal{P} \epsilon \mathcal{L}o_{\mathrm{M}}$ such that $h \epsilon \mathcal{P} \subseteq \mathcal{T}$.

Now, $\mathcal{T} = (\mathcal{T}^c)^c$, then$h \epsilon \mathcal{P} \subseteq (\mathcal{T}^c)^c$ where $\mathcal{P} \epsilon \mathcal{L}o_{\mathrm{M}}$, thus by the definition of $\mathcal{L}-$ exterior we have$h$ is a $\mathcal{L}-$ exterior point of $\mathcal{T}^c$, thus $h \epsilon ext_{\mathcal{L}}(\mathcal{T}^c)$. Which implies that

$ext_{\mathcal{L}}(\mathcal{T}^c) \supseteq lnt_{\mathcal{L}}(\mathcal{T})$. Therefore, $ext_{\mathcal{L}}(\mathcal{T}^c) = lnt_{\mathcal{L}}(\mathcal{T})$.

Theorem 3.5: Assume that $(\mathrm{M}, \mathcal{L}o_{\mathrm{M}})$ is a $\mathcal{L}-space$ and $\mathcal{T} \subseteq \mathrm{M}$. Then

$$ext_{\mathcal{L}}(\mathcal{T}) = (cl_{\mathcal{L}}(\mathcal{T}))^c.$$

Proof: $cl_{\mathcal{L}}(\mathcal{T}) = \bigcap_{\sigma \epsilon \Delta} \{\mathcal{P}_\sigma : \mathcal{P}_\sigma \text{ is } \mathcal{L}-\text{closed sets } \mathcal{T} \subseteq \mathcal{P}_\sigma\}$, then

$$(cl_{\mathcal{L}}(\mathcal{T}))^c = (\bigcap_{\sigma \epsilon \Delta} \{\mathcal{P}_\sigma : \mathcal{P}_\sigma \text{ is } \mathcal{L}-\text{ closed sets } \mathcal{T} \subseteq \mathcal{P}_\sigma\})^c.$$

Now, $\mathcal{P}_\sigma$ be the $\mathcal{L}-$closed and $\mathcal{T} \subseteq \mathcal{P}_\sigma \; \forall \, \sigma \epsilon \Delta$, then ${\mathcal{P}_\sigma}^c$ is a $\mathcal{L}-open$ and ${\mathcal{P}_\sigma}^c \subseteq \mathcal{T}^c \; \forall \, \sigma \epsilon \Delta$. Hence by De-Morgan Laws we have:

$(cl_{\mathcal{L}}(\mathcal{T}))^c = \bigcup_{\sigma \epsilon \Delta} \{{\mathcal{P}_\sigma}^c : {\mathcal{P}_\sigma}^c$ be the $\mathcal{L}-$open sets, and ${\mathcal{P}_\sigma}^c \subseteq \mathcal{T}^c \; \forall \sigma \epsilon \Delta\}$. That is, $\bigcup_{\sigma \epsilon \Delta} \{{\mathcal{P}_\sigma}^c : {\mathcal{P}_\sigma}^c$ be the $\mathcal{L}-$open and ${\mathcal{P}_\sigma}^c \subseteq \mathcal{T}^c \; \forall \sigma \epsilon \Delta\}$ is a $\mathcal{L}-$interior of $\mathcal{T}^c$, thus $(cl_{\mathcal{L}}(\mathcal{T}))^c = lnt_{\mathcal{L}}(\mathcal{T}^c)$. But from Theorem (3.3), we have:

$$ext_{\mathcal{L}}(\mathcal{T}) = lnt_{\mathcal{L}}(\mathcal{T}^c).$$

Hence $ext_{\mathcal{L}}(\mathcal{T}) = (cl_{\mathcal{L}}(\mathcal{T}))^c$.

Corollary 3.6: *Suppose that* $(\mathrm{M}, \mathcal{L}o_{\mathrm{M}})$ *is a* $\mathcal{L}-space$ *and* $\mathcal{T} \subseteq \mathrm{M}$. *Then*

$$cl_{\mathcal{L}}(\mathcal{T}) = (ext_{\mathcal{L}}(\mathcal{T}))^c.$$

Proof: The result follows by Theorem (3.5).

Proposition 3.7: Assume that $(\mathrm{M}, \mathcal{L}o_{\mathrm{M}})$ is a $\mathcal{L}-space$ and $\mathcal{T} \subseteq \mathrm{M}$. Then

$$lnt_{\mathcal{L}}(\mathcal{T}^c) = (cl_{\mathcal{L}}(\mathcal{T}))^c.$$

Proof: $ext_{\mathcal{L}}(\mathcal{T}) = lnt_{\mathcal{L}}(\mathcal{T}^c)$ from Theorem (3.3) and $ext_{\mathcal{L}}(\mathcal{T}) = (cl_{\mathcal{L}}(\mathcal{T}))^c$ by Theorem (3.5). Therefore, $lnt_{\mathcal{L}}(\mathcal{T}^c) = (cl_{\mathcal{L}}(\mathcal{T}))^c$.

Proposition 3.8: If $(\mathrm{M}, \mathcal{L}\mathrm{o}_{\mathrm{M}})$ is a $\mathcal{L} - space$ and $\mathcal{T} \subseteq \mathrm{M}$. Then

$$(lnt_{\mathcal{L}}(\mathcal{T}))^c = cl_{\mathcal{L}}(\mathcal{T}^c).$$

Proof: Since $lnt_{\mathcal{L}}(\mathcal{T}) = \bigcup_{\sigma\in\Delta} \{\mathcal{P}_\sigma : \mathcal{P}_\sigma$ be the $\mathcal{L}$ −open set and $\mathcal{P}_\sigma \subseteq \mathcal{T}\ \forall\sigma\in\Delta\}$.

Then $(lnt_{\mathcal{L}}(\mathcal{T}))^c = (\bigcup_{\sigma\in\Delta} \{\mathcal{P}_\sigma : \mathcal{P}_\sigma$ be $\mathcal{L} -$ open , $\mathcal{P}_\sigma \subseteq \mathcal{T}\ \forall\sigma\in\Delta\})^c$.

Now, $\mathcal{P}_\sigma$ is $\mathcal{L} - open$ and $\mathcal{P}_\sigma \subseteq \mathcal{T}\ \forall\sigma\in\Delta$, then ${\mathcal{P}_\sigma}^c$ be the $\mathcal{L}$ −closed sets and $\mathcal{T}^c \subseteq {\mathcal{P}_\sigma}^c\ \forall\ \sigma\in\Delta$. Hence by De-Morgan Laws we get:

$(lnt_{\mathcal{L}}(\mathcal{T}))^c = \bigcap_{\sigma\in\Delta}\{\ {\mathcal{P}_\sigma}^c : {\mathcal{P}_\sigma}^c$ is $\mathcal{L} -$ closed and $\mathcal{T}^c \subseteq {\mathcal{P}_\sigma}^c\ \forall\ \sigma\in\Delta\}$.

Therefore, $(lnt_{\mathcal{L}}(\mathcal{T}))^c = cl_{\mathcal{L}}(\mathcal{T}^c)$.

Proposition 3.9: Assume that $(\mathrm{M}, \mathcal{L}\mathrm{o}_{\mathrm{M}})$ is a $\mathcal{L} - space$ and $\mathcal{T} \subseteq \mathrm{M}$. Then

$$(ext_{\mathcal{L}}(\mathcal{T}^c))^c = cl_{\mathcal{L}}(\mathcal{T}^c).$$

Proof: We obtain the prove by Theorem (3.4) and Proposition (3.8).

Proposition 3.10: If $(\mathrm{M}, \mathcal{L}\mathrm{o}_{\mathrm{M}})$ is a $\mathcal{L} - space$ and $\mathcal{T} \subseteq \mathrm{M}$. Then $\mathcal{T}$ is an $\mathcal{L} - open$ iff $ext_{\mathcal{L}}(\mathcal{T}^c) = \mathcal{T}$.

Proof: Since $\mathcal{T}$ is $\mathcal{L} -$ open, then $lnt_{\mathcal{L}}(\mathcal{T}) = \mathcal{T}$ by Theorem (2.11).

Thus we obtain the result by Theorem (3.4).

Proposition 3.11: If $(\mathrm{M}, \mathcal{L}\mathrm{o}_{\mathrm{M}})$ is a $\mathcal{L} - space$ and $\mathcal{T} \subseteq \mathrm{M}$. Then $\mathcal{T}$ is an $\mathcal{L} - closed$ iff $(ext_{\mathcal{L}}(\mathcal{T}))^c = \mathcal{T}$.

Proof: We obtains the prove from Corollary (2.4) and Corollary (3.6).

Theorem 3.12: *If* $(\mathrm{M}, \mathcal{L}\mathrm{o}_{\mathrm{M}})$ *is the* $\mathcal{L} - space$ *and* $\mathcal{T}, \mathcal{F} \subseteq \mathrm{M}$. *Then*

1. $ext_{\mathcal{L}}(\Phi) = \mathrm{M}$.
2. $ext_{\mathcal{L}}(\mathrm{M}) = \Phi$.
3. $ext_{\mathcal{L}}(\mathcal{T}) \subseteq \mathcal{T}^c$.
4. $ext_{\mathcal{L}}(\mathcal{T}) = ext_{\mathcal{L}}((ext_{\mathcal{L}}(\mathcal{T}))^c)$.
5. If $\mathcal{T} \subseteq \mathcal{F}$, then $ext_{\mathcal{L}}(\mathcal{F}) \subseteq ext_{\mathcal{L}}(\mathcal{T})$.
6. $lnt_{\mathcal{L}}(\mathcal{T}) \subseteq ext_{\mathcal{L}}\big(ext_{\mathcal{L}}(\mathcal{T})\big)$.
7. $ext_{\mathcal{L}}(\mathcal{T} \cup \mathcal{F}) = ext_{\mathcal{L}}(\mathcal{T}) \cap ext_{\mathcal{L}}(\mathcal{F})$.

Proof:

1. $ext_{\mathcal{L}}(\Phi) = lnt_{\mathcal{L}}(\Phi^c)$ by Theorem (3.3)
   $= lnt_{\mathcal{L}}(\mathrm{M})$
   $= \mathrm{M}$ by [Theorem (2.10) - 4].

2. $ext_{\mathcal{L}}(\mathrm{M}) = (cl_{\mathcal{L}}(\mathrm{M}))^c$ by Theorem (3.5)
   $= \mathrm{M}^c$ by Theorem (3.4)
   Hence, $ext_{\mathcal{L}}(\mathrm{M}) = \Phi$.
3. $ext_{\mathcal{L}}(\mathcal{T}) = lnt_{\mathcal{L}}(\mathcal{T}^c)$ by Theorem (3.3)
   $\subseteq \mathcal{T}^c$ by Theorem (2.9).
4. $ext_{\mathcal{L}}((ext_{\mathcal{L}}(\mathcal{T}))^c) = ext_{\mathcal{L}}((lnt_{\mathcal{L}}(\mathcal{T}^c))^c)$ by Theorem (3.3)
   $= (lnt_{\mathcal{L}}((lnt_{\mathcal{L}}(\mathcal{T}^c))^c))^c$
   $= lnt_{\mathcal{L}}(lnt_{\mathcal{L}}(\mathcal{T}^c))$
   $= lnt_{\mathcal{L}}(\mathcal{T}^c)$ by [Theorem (2.10) – 3]
   $= ext_{\mathcal{L}}(\mathcal{T})$ by Theorem (3.3).
5. Suppose that $\mathcal{T} \subseteq \mathcal{F}$. Then $\mathcal{F}^c \subseteq \mathcal{T}^c$, hence
   $lnt_{\mathcal{L}}(\mathcal{F}^c) \subseteq lnt_{\mathcal{L}}(\mathcal{T}^c)$ by [Theorem (2.10) – 1].
   From Theorem (3.27) we have,
   $ext_{\mathcal{L}}(\mathcal{T}) = lnt_{\mathcal{L}}(\mathcal{T}^c)$ and $ext_{\mathcal{L}}(\mathcal{F}) = lnt_{\mathcal{L}}(\mathcal{F}^c)$.
   Therefore, $ext_{\mathcal{L}}(\mathcal{F}) \subseteq ext_{\mathcal{L}}(\mathcal{T})$.
6. From Theorem (3.2) we have, $\mathcal{T} \subseteq cl_{\mathcal{L}}(\mathcal{T})$, hence
   $lnt_{\mathcal{L}}(\mathcal{T}) \subseteq lnt_{\mathcal{L}}\big(cl_{\mathcal{L}}(\mathcal{T})\big)$
   $= lnt_{\mathcal{L}}((ext_{\mathcal{L}}(\mathcal{T}))^c)$ by Corollary (3.6)
   $= ext_{\mathcal{L}}\big(ext_{\mathcal{L}}(\mathcal{T})\big)$ by Theorem (3.3).
7. $ext_{\mathcal{L}}(\mathcal{T}) \cap ext_{\mathcal{L}}(\mathcal{F}) = lnt_{\mathcal{L}}(\mathcal{T}^c) \cap lnt_{\mathcal{L}}(\mathcal{F}^c)$ by Theorem (3.3)
   $= lnt_{\mathcal{L}}(\mathcal{T}^c \cap \mathcal{F}^c)$ by [Theorem (2.10) – 2]
   $= lnt_{\mathcal{L}}((\mathcal{T} \cup \mathcal{F})^c)$ by De- Morgan Laws
   $= ext_{\mathcal{L}}(\mathcal{T} \cup \mathcal{F})$ by Theorem (3.3).

Theorem 3.13: *If we have* $\mathcal{L}$ *−space* $(\mathrm{M}, \mathcal{L}o_{\mathrm{M}})$ *&* $\mathcal{T}, \mathcal{F} \subseteq \mathrm{M}$. *Then the following statements are equivalent*

1. $\mathcal{T}$ *is* $\mathcal{L}$ *−dense set.*
2. *If* $\mathcal{F}$ *is* $\mathcal{L}$ *−closed &* $\mathcal{T} \subseteq \mathcal{F}$, *then* $\mathcal{F} = \mathrm{M}$.
3. $lnt_{\mathcal{L}}(\mathcal{T}^c) = \Phi$.

Proof: (1) ⇒ (2): Assume $\mathcal{T}$ is $\mathcal{L}$ −dense set and $\mathcal{F}$ is $\mathcal{L}$ −closed & $\mathcal{T} \subseteq \mathcal{F}$. To prove $\mathcal{F} = \mathrm{M}$ we must prove $\mathcal{F} \subseteq \mathrm{M}$ and $\mathrm{M} \subseteq \mathcal{F}$, clearly that $\mathcal{F} \subseteq \mathrm{M}$ by hypothesis, so it is only remains we prove $\mathrm{M} \subseteq \mathcal{F}$.

Since $\mathcal{T}$ is $\mathcal{L}$ −dense set, then $cl_{\mathcal{L}}(\mathcal{T}) = \mathrm{M}$.

Now, by hypothesis, we have, $\mathcal{F}$ is $\mathcal{L}$ −closed, then by Corollary (2.4) we get $\mathcal{F} = cl_{\mathcal{L}}(\mathcal{F})$. Also, from the hypothesis we have, $\mathcal{T} \subseteq \mathcal{F}$, then by Theorem (2.5) we get $cl_{\mathcal{L}}(\mathcal{T}) \subseteq cl_{\mathcal{L}}(\mathcal{F})$. So, we have $cl_{\mathcal{L}}(\mathcal{T}) \subseteq \mathcal{F}$. but $cl_{\mathcal{L}}(\mathcal{T}) = \mathrm{M}$, so $\mathrm{M} \subseteq \mathcal{F}$, hence $\mathcal{F} = \mathrm{M}$.

(2) ⇒ (3): we assume, $\mathcal{F} = \mathrm{M}$ for each $\mathcal{L}$ −closed $\mathcal{F}$ includes an arbitrary subset $\mathcal{T}$ of M. Now, from Theorem (2.3), we have $cl_{\mathcal{L}}(\mathcal{T})$ is $\mathcal{L}$ −closed set

including $\mathcal{T}$ for any subsets $\mathcal{T}$ of M. So by (2) we get $cl_{\mathcal{L}}(\mathcal{T}) = \mathrm{M}$. Now, by Proposition (3.7) we get $lnt_{\mathcal{L}}(\mathcal{T}^{c}) = (cl_{\mathcal{L}}(\mathcal{T}))^{c}$. So we get, $lnt_{\mathcal{L}}(\mathcal{T}^{c}) = (\mathrm{M})^{c} = \Phi$.

(3) ⇒ (1): Suppose $lnt_{\mathcal{L}}(\mathcal{T}^{c}) = \Phi$. Now, from Proposition (3.7) we have, $lnt_{\mathcal{L}}(\mathcal{T}^{c}) = (cl_{\mathcal{L}}(\mathcal{T}))^{c}$. Hence $(cl_{\mathcal{L}}(\mathcal{T}))^{c} = \Phi$, thus $cl_{\mathcal{L}}(\mathcal{T}) = \Phi^{c} = \mathrm{M}$. Therefore, $\mathcal{T}$ is $\mathcal{L}$ −dense set.

Theorem 3.14: *If $\mathcal{L}$ −space $(\mathrm{M}, \mathcal{L}\mathrm{o}_{\mathrm{M}})$ is $\mathcal{L}$ −space & $\mathcal{T} \subseteq \mathrm{M}$. If $cl_{\mathcal{L}}(\mathcal{T}^{c}) = \Phi$, then $\mathcal{T}$ is $\mathcal{L}$ −dense set.*

Proof: Suppose $cl_{\mathcal{L}}(\mathcal{T}^{c}) = \Phi$. Now, from Proposition (3.8) we have $(lnt_{\mathcal{L}}(\mathcal{T}))^{c} = cl_{\mathcal{L}}(\mathcal{T}^{c})$. Thus $(lnt_{\mathcal{L}}(\mathcal{T}))^{c} = \Phi$, hence $lnt_{\mathcal{L}}(\mathcal{T}) = \mathrm{M}$. So from Theorem (2.9) we have $lnt_{\mathcal{L}}(\mathcal{T}) \subseteq \mathcal{T}$ and from Theorem (2.3) we have $\mathcal{T} \subseteq cl_{\mathcal{L}}(\mathcal{T})$, therefore $lnt_{\mathcal{L}}(\mathcal{T}) \subseteq cl_{\mathcal{L}}(\mathcal{T})$, but $lnt_{\mathcal{L}}(\mathcal{T}) = \mathrm{M}$, hence $cl_{\mathcal{L}}(\mathcal{T}) = \mathrm{M}$, this implies that $\mathcal{T}$ is $\mathcal{L}$ −dense set.

Theorem 3.15: *Suppose that $\mathcal{L}$ −space $(\mathrm{M}, \mathcal{L}\mathrm{o}_{\mathrm{M}})$ is an $\mathcal{L}$ −space & $\mathcal{T} \subseteq \mathrm{M}$. If $\mathcal{T}$ is $\mathcal{L}$ −dense set, then $ext_{\mathcal{L}}(\mathcal{T}) = \Phi$.*

Proof: Let $\mathcal{T}$ be the $\mathcal{L}$ −dense set, then $cl_{\mathcal{L}}(\mathcal{T}) = \mathrm{M}$.

Now, $ext_{\mathcal{L}}(\mathcal{T}) = (cl_{\mathcal{L}}(\mathcal{T}))^{c}$ by Theorem (3.5).
But $cl_{\mathcal{L}}(\mathcal{T}) = \mathrm{M}$, then $(cl_{\mathcal{L}}(\mathcal{T}))^{c} = \Phi$, hence $ext_{\mathcal{L}}(\mathcal{T}) = \Phi$.

4. Conclusion

The main results in this article are as follows:

1. If $(\mathrm{M}, \mathcal{L}\mathrm{o}_{\mathrm{M}})$ is $\mathcal{L} - space$ & $\mathcal{T} \subseteq \mathrm{M}$. Then $ext_{\mathcal{L}}(\mathcal{T}) = lnt_{\mathcal{L}}(\mathcal{T}^{c})$.
2. Assume $(\mathrm{M}, \mathcal{L}\mathrm{o}_{\mathrm{M}})$ is $\mathcal{L} - space$. Then $ext_{\mathcal{L}}(\mathcal{T}) = (cl_{\mathcal{L}}(\mathcal{T}))^{c}$.
3. If $(\mathrm{M}, \mathcal{L}\mathrm{o}_{\mathrm{M}})$ is a $\mathcal{L} - space$ and $\mathcal{T} \subseteq \mathrm{M}$. Then $\mathcal{T}$ is an $\mathcal{L} - open$ iff $ext_{\mathcal{L}}(\mathcal{T}^{c}) = \mathcal{T}$.

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