Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 29 (2026), No. 5-B, pp. 1407–1415
DOI : 10.47974/JIM-2531

On a fuzzy statistically convergent in 2-norm (Fsc_n^2- Convergent) in a fuzzy 2-normed Riesz spaces and some properties

Shaimaa Abdul-Hussein Kadhim *
Shatha Abdul-Hussein Kadhim †
Mohammed Yahya Abed §
Department of Mathematics
College of Education for Pure Sciences
University of Kerbala
Karbala 56001
Iraq

Abstract

In this paper, we will be discussing fuzzy statistically convergent in 2-norm in a fuzzy 2-normed Riesz spaces. We also define the notion of a Fsc_n^2- Cauchy in fuzzy 2-normed Riesz spaces and establish some basic facts.

Subject Classification: *Primary 93A30, Secondary 49K15.*

Keywords: *Fuzzy statistical convergence in 2-norm, Fsc_n^2- Cauchy in 2-norm, 2-Banach fuzzy Riesz spaces, Fuzzy 2-normed Riesz spaces, Fuzzy sets, Normed spaces.*

1. Introduction

The concept of statistical convergence is studied by Fast [1] and Schoenberg [2]. The idea depends on a concept of density for subsets of N. A notion for 2-normed spaces is studied in [3], [4], [5], [6], [7], [8], [9], [10]. There are many authors who study this idea to obtain results; see [11], [12], [13]. Fuzzy Riesz spaces were studied in [14], which is an interesting and important generalization of properties, and subsequently, fuzzy statistical convergence in fuzzy Riesz spaces was discussed in [15], [16], [17]. In the past, we introduced fuzzy statistical convergence in the norm of a fuzzy Riesz space, and in this paper, we extend the work to fuzzy statistical convergence in the 2-norm of a fuzzy 2-normed Riesz space and establish some basic facts. Some properties for fuzzy statistical convergences are shown in norm fuzzy Riesz spaces, also hold for fuzzy 2-normed Riesz spaces.

* *E-mail:* **shaimaa.a@uokerbala.edu.iq (Corresponding Author)**
† *E-mail:* **shatha.abdulhussein@uokerbala.edu.iq**
§ *E-mail:* **mohammed.y@uokerbala.edu.iq**

2. Background, notations and preliminaries.

In this section, we recall some basic definitions and notions in 2-normed spaces, and some facts related to fuzzy Riesz spaces will help us in establishing our results.

Definition 2.1: The real linear space $\mathfrak{T}$ is a fuzzy ordered linear space if $\mathfrak{T}$ is foset & further $\mathfrak{T}$ satisfied:

i. For $ꞥ_1, ꞥ_2 \in \mathfrak{T}$ s.t $\dot{\mu}(ꞥ_1, ꞥ_2) > \frac{1}{2}$ so, $\dot{\mu}(ꞥ_1, ꞥ_2) \leq \dot{\mu}(ꞥ_1 + ʄ, ꞥ_2 + ʄ)$ for all $ʄ \in \mathfrak{T}$.

ii. If $ꞥ_1, ꞥ_2 \in \mathfrak{T}$ such that $\dot{\mu}(ꞥ_1, ꞥ_2) > \frac{1}{2}$ then $\dot{\mu}(ꞥ_1, ꞥ_2) \leq \dot{\mu}(\alpha ꞥ_1, \alpha ꞥ_2)$ for all $0 \leq \alpha \in \mathcal{R}$.

Fuzzy ordered linear spaces are also called the fuzzy Riesz spaces (for short: FRS).

Definition 2.2: Let $ꞥ$ be a member of $\mathfrak{T}$. So, a positive part for $ꞥ$ is $ꞥ_+ = ꞥ \vee 0$ and a negative part $ꞥ_- = (-ꞥ) \vee 0$, the absolute value of $ꞥ$ is $|ꞥ| = ꞥ \vee (-ꞥ)$. Means that we can write $ꞥ = ꞥ_+ - ꞥ_-$.

Theorem 2.3. [17] *In an FRS, the absolute value has the following properties:*

i. $\dot{\mu}(|ꞥ + ᵯ|, |ꞥ| + |ᵯ|) > \frac{1}{2}$,

ii. $|\alpha ꞥ| = |\alpha||ꞥ|$,

iii. $\dot{\mu}(||ꞥ| - |ᵯ||, |ꞥ - ᵯ|) > \frac{1}{2}$,

iv. $|ꞥ - ᵯ| = (ꞥ \vee ᵯ) - (ꞥ \wedge ᵯ)$.

3. The Main results.

In the following theorems, we present some properties of the fuzzy statistically convergent in 2-norm in a fuzzy 2-normed Riesz space (short F2-NRS) $(\mathfrak{T}, \|\bullet,\bullet\|)$, as well as proof that we have the F2-NRS, is a 2-Banach fuzzy Riesz space.

Definition 3.1: Let $\mathfrak{T}$ be a real FRS, equipped with 2-norm $\|\bullet,\bullet\|$. The 2-norm on $\mathfrak{T}$ is called a Riesz 2-norm if $ꞥ, ᵯ \in \mathfrak{T}$ then $\dot{\mu}(|ꞥ, ʄ|, |ᵯ, ʄ|) > \frac{1}{2}$ implies that $\dot{\mu}(\|ꞥ, ʄ\|, \|ᵯ, ʄ\|) > \frac{1}{2}$ for each nonzero $ʄ \in \mathfrak{T}$.

Any FRS equipped with Riesz 2-norm is named F2-NRS. An F2-NRS is also a Banach space.

Definition 3.2: The sequence $(ꞥ_ţ)_{ţ \in N}$ in the F2-NRS is a fuzzy statistically convergent in 2-norm to $\xi \in \mathfrak{T}$, when each $\varepsilon > 0$, & nonzero $ʄ \in \mathfrak{T}$, a set

$\left\{\mathfrak{t} \in \mathbb{N} : \dot{\mu}(\varepsilon, \| \mathfrak{n}_{\mathfrak{t}} - \xi, \mathfrak{f} \|) > \frac{1}{2}\right\}$ have the natural density, which is zero. So write $Fsc_n^2 - \lim_{\mathfrak{t}} \| \mathfrak{n}_{\mathfrak{t}} - \xi, \mathfrak{f} \| = 0$ (or) $Fsc_n^2 - \lim_{\mathfrak{t}} \| \mathfrak{n}_{\mathfrak{t}}, \mathfrak{f} \| = \|\xi, \mathfrak{f}\|$.

Definition 3.3: A sequence $(\mathfrak{n}_{\mathfrak{t}})_{\mathfrak{t} \in \mathbb{N}}$ in F2-NRS$(\mathfrak{X}, \|\bullet,\bullet\|)$ named Fsc_n^2- Cauchy seq. in $\mathfrak{X}$ for $\varepsilon > 0$ & nonzero $\mathfrak{f} \in \mathfrak{X}$, there is Ns.t for every $\mathfrak{t}, \mathfrak{m} \geq N$, the set $\left\{\mathfrak{t}, \mathfrak{m} \in \mathbb{N} : \dot{\mu}(\varepsilon, \| \mathfrak{n}_{\mathfrak{t}} - \mathfrak{n}_{\mathfrak{m}}, \mathfrak{f} \|) > \frac{1}{2}\right\}$ has natural density zero.

Theorem 3.4: *If* $(\mathfrak{n}_{\mathfrak{t}})_{\mathfrak{t} \in \mathbb{N}}$ *and* $(\mathfrak{m}_{\mathfrak{t}})_{\mathfrak{t} \in \mathbb{N}}$ *be two sequences in F2-NRS*$(\mathfrak{X}, \|\bullet,\bullet\|)$*, Then*

i. If $Fsc_n^2 - \lim_{\mathfrak{t}} \| \mathfrak{n}_{\mathfrak{t}} - \xi, \mathfrak{f} \| = 0$ and $Fsc_n^2 - \lim_{\mathfrak{t}} \| \mathfrak{m}_{\mathfrak{t}} - \psi, \mathfrak{f} \| = 0$, then $Fsc_n^2 - \lim_{\mathfrak{t}} \| \mathfrak{n}_{\mathfrak{t}} \vee \mathfrak{m}_{\mathfrak{t}} - \xi \vee \psi, \mathfrak{f} \| = 0$ and $Fsc_n^2 - \lim_{\mathfrak{t}} \| \mathfrak{n}_{\mathfrak{t}} \wedge \mathfrak{m}_{\mathfrak{t}} - \xi \wedge \psi, \mathfrak{f} \| = 0$.

ii. If $Fsc_n^2 - \lim_{\mathfrak{t}} \| \mathfrak{n}_{\mathfrak{t}} - \xi, \mathfrak{f} \| = 0$, then $Fsc_n^2 - \lim_{\mathfrak{t}} \| \mathfrak{n}_{\mathfrak{t}}^+ - \xi^+, \mathfrak{f}^+ \| = 0$, $Fsc_n^2 - \lim_{\mathfrak{t}} \| \mathfrak{n}_{\mathfrak{t}}^- - \xi^-, \mathfrak{f}^- \| = 0$ and $Fsc_n^2 - \lim_{\mathfrak{t}} \| |\mathfrak{n}_{\mathfrak{t}}| - |\xi|, |\mathfrak{f}| \| = 0$.

iii. If $Fsc_n^2 - \lim_{\mathfrak{t}} \| \mathfrak{n}_{\mathfrak{t}} - \xi, \mathfrak{f} \| = 0$, $Fsc_n^2 - \lim_{\mathfrak{t}} \| \mathfrak{s}_{\mathfrak{t}} - \psi, \mathfrak{f} \| = 0$ and $\mathfrak{n}_{\mathfrak{t}} \perp \mathfrak{s}_{\mathfrak{t}}$ then $\xi \perp \psi$.

iv. If $Fsc_n^2 - \lim_{\mathfrak{t}} \| \mathfrak{n}_{\mathfrak{t}} - \xi, \mathfrak{f} \| = 0$ and $Fsc_n^2 - \lim_{\mathfrak{t}} \| \mathfrak{n}_{\mathfrak{t}} - \psi, \mathfrak{f} \| = 0$, then $\xi = \psi$.

Proof:

i. By assumption, therefore, for each $\varepsilon > 0$ and nonzero $\mathfrak{f} \in \mathfrak{X}$ we have the sets $\left\{\mathfrak{t} \in \mathbb{N} : \dot{\mu}(\varepsilon, \| \mathfrak{n}_{\mathfrak{t}} - \xi, \mathfrak{f} \|) > \frac{1}{2}\right\}$ and $\left\{\mathfrak{t} \in \mathbb{N} : \dot{\mu}(\varepsilon, \| \mathfrak{m}_j - \psi, \mathfrak{f} \|) > \frac{1}{2}\right\}$ have both natural density zero. From this, we obtained that, $\left\{\mathfrak{t} \in \mathbb{N} : \dot{\mu}(\varepsilon, \| \mathfrak{n}_{\mathfrak{t}} \vee \mathfrak{m}_j - \xi \vee \psi, \mathfrak{f} \|) > \frac{1}{2}\right\} \subset \left\{\mathfrak{t} \in \mathbb{N} : \dot{\mu}(\varepsilon, \| \mathfrak{n}_{\mathfrak{t}} - \xi, \mathfrak{f} \|) > \frac{1}{2}\right\} \cup \left\{\mathfrak{t} \in \mathbb{N} : \dot{\mu}(\varepsilon, \| \mathfrak{m}_{\mathfrak{t}} - \psi, \mathfrak{f} \|) > \frac{1}{2}\right\}$ since the right-hand has natural density zero. Thereby the set $\left\{\mathfrak{t} \in \mathbb{N} : \dot{\mu}(\varepsilon, \| \mathfrak{n}_{\mathfrak{t}} \vee \mathfrak{m}_{\mathfrak{t}} - \xi \vee \psi, \mathfrak{f} \|) > \frac{1}{2}\right\}$ has natural density zero and the statement follows. Similarly, we have that $Fsc_n^2 - \lim_{\mathfrak{t}} \| \mathfrak{n}_{\mathfrak{t}} \wedge \mathfrak{m}_{\mathfrak{t}} - \xi \wedge \psi, \mathfrak{f} \| = 0$.

ii. Since $Fsc_n^2 - \lim_{\mathfrak{t}} \| \mathfrak{n}_{\mathfrak{t}} - \xi, \mathfrak{f} \| = 0$, then for every $\varepsilon > 0$ the set $\left\{\mathfrak{t} \in \mathbb{N} : \dot{\mu}(\varepsilon, \| \mathfrak{n}_{\mathfrak{t}} - \xi, \mathfrak{f} \|) > \frac{1}{2}\right\}$ has natural density zero for each $\mathfrak{f} \in \mathfrak{X}$. Firstly, to show that $Fsc_n^2 - \lim_{\mathfrak{t}} \| \mathfrak{n}_{\mathfrak{t}}^+ - \xi^+, \mathfrak{f}^+ \| = 0$. For $\varepsilon > 0$, & nonzero $\mathfrak{f} \in \mathfrak{X}$, we see that the set $\left\{\mathfrak{t} \in \mathbb{N} : \dot{\mu}(\varepsilon, \| \mathfrak{n}_{\mathfrak{t}}^+ - \xi^+, \mathfrak{f}^+ \|) > \frac{1}{2}\right\} \subset \left\{\mathfrak{t} \in \mathbb{N} : \dot{\mu}(\varepsilon, \| \mathfrak{n}_{\mathfrak{t}} - \xi, \mathfrak{f} \|) > \frac{1}{2}\right\}$. Therefore $\left\{\mathfrak{t} \in \mathbb{N} : \dot{\mu}(\varepsilon, \| \mathfrak{n}_{\mathfrak{t}}^+ - \xi^+, \mathfrak{f}^+ \|) > \frac{1}{2}\right\}$ has natural density zero. Thus $Fsc_n^2 - \lim_{ţ}\|\text{ȵ}_{ţ}^{+} - \xi^{+}, f^{+}\| = 0$. To prove assertion that $Fsc_n^2 - \lim_{ţ}\|\text{ȵ}_{ţ}^{-} - \xi^{-}, f^{-}\| = 0$, we can use the same discussion at the above part. Finally, to prove that $Fsc_n^2 - \lim_{ţ}\||\text{ȵ}_{ţ}| - |\xi|, |f|\| = 0$. Since we can write, for every $\varepsilon > 0$, and nonzero$f \in \mathfrak{T}$, $\left\{ţ \in \mathbb{N}: \dot{\mu}(\in, \||\text{ȵ}_{ţ}| - |\xi|, |f|\|) > \frac{1}{2}\right\} \subset \left\{ţ \in \mathbb{N}: \dot{\mu}(\in, \|\text{ȵ}_{ţ} - \xi, f\|) > \frac{1}{2}\right\}$. For $\varepsilon > 0$& nonzero$f \in \mathfrak{T}$, then $\left\{ţ \in \mathbb{N}: \dot{\mu}(\in, \||\text{ȵ}_{ţ}| - |\xi|, |f|\|) > \frac{1}{2}\right\}$ have natural density zero.

iii. Assume that $Fsc_n^2 - \lim_{ţ}\|\text{ȵ}_{ţ} - \xi, f\| = 0$

$Fsc_n^2 - \lim_{ţ}\|\text{ɱ}_{ţ} - \psi, f\| = 0$ and $\text{ȵ}_{ţ} \perp \text{ɱ}_{ţ}$. By using (ii), implies that $Fsc_n^2 - \lim_{ţ}\||\text{ȵ}_{ţ}| - |\xi|, |f|\| = 0$ and $Fsc_n^2 - \lim_{ţ}\||\text{ɱ}_{ţ}| - |\psi|, |f|\| = 0$ respectively. Consequently, for each $f \in \mathfrak{T}$ we write by using (i), $Fsc_n^2 - \lim_{ţ}\||\text{ȵ}_{ţ}| \wedge |\text{ɱ}_{ţ}| - |\xi| \wedge |\psi|, |f|\| = 0$. From hypothesis $|\text{ȵ}_{ţ}| \wedge |\text{ɱ}_{ţ}| = 0$, we have that $|\xi| \wedge |\psi| = 0$.

iv. Assume that $Fsc_n^2 - \lim_{ţ}\|\text{ȵ}_{ţ} - \xi, f\| = 0$

$Fsc_n^2 - \lim_{ţ}\|\text{ȵ}_{ţ} - \psi, f\| = 0$, means that for every $\varepsilon > 0$, and nonzero $f \in \mathfrak{T}$, the sets $\left\{ţ \in \mathbb{N}: \dot{\mu}(\varepsilon, \|\text{ȵ}_{ţ} - \xi, f\|) > \frac{1}{2}\right\}$ and $\left\{ţ \in \mathbb{N}: \dot{\mu}(\varepsilon, \|\text{ȵ}_{ţ} - \psi, f\|) > \frac{1}{2}\right\}$. Consequently, by Theorem 2.7, we have $\left\{ţ \in \mathbb{N}: \dot{\mu}(0, \|\text{ȵ}_{ţ} - \xi, f\| - \|\text{ȵ}_{ţ} - \psi, f\|) > \frac{1}{2}\right\}$. It follows that, $\left\{ţ \in \mathbb{N}: \dot{\mu}(|\|\text{ȵ}_{ţ} - \xi, f\| - \|\text{ȵ}_{ţ} - \psi, f\||, \|\xi - \text{ȵ}_{ţ} + \text{ȵ}_{ţ} - \psi, f\|) > \frac{1}{2}\right\}$ Thereby we write that, $\left\{ţ \in \mathbb{N}: \dot{\mu}(0, \|\xi - \psi, f\|) > \frac{1}{2}\right\}$. Hence $\xi = \psi$.

Theorem 3.5: *If* $(\text{ȵ}_{ţ})_{ţ \in \mathbb{N}}$, $(\text{ɱ}_{ţ})_{ţ \in \mathbb{N}}$ *and* $(s_{ţ})_{ţ \in \mathbb{N}}$ *be a three sequences in F2-NRS* $(\mathfrak{T}, \|\bullet,\bullet\|)$ *such that* $\dot{\mu}(\text{ȵ}_{ţ}, \text{ɱ}_{ţ}) > \frac{1}{2}, \dot{\mu}(\text{ɱ}_{ţ}, s_{ţ}) > \frac{1}{2}$ *for every* $ţ \in \mathbb{N}$, $Fsc_n^2 - \lim_{ţ}\|\text{ȵ}_{ţ}, f\| = \|\xi, f\|$ *and* $Fsc_n^2 - \lim_{ţ}\|s_{ţ}, f\| = \|\xi, f\|$, *then* $Fsc_n^2 - \lim_{ţ}\|\text{ɱ}_{ţ}, f\| = \|\xi, f\|$.

Proof: By Definition 3.2, we have that for every $\varepsilon > 0$ and nonzero $f \in \mathfrak{T}$, the sets $K_1(\frac{\varepsilon}{2}) = \left\{ţ \in \mathbb{N}: \dot{\mu}(\|\text{ȵ}_{ţ} - \xi, f\|, \frac{\in}{2}) > \frac{1}{2}\right\}$ and $K_2(\frac{\varepsilon}{2}) = \left\{ţ \in \mathbb{N}: \dot{\mu}(\|s_{ţ} - \xi, f\|, \frac{\in}{2}) > \frac{1}{2}\right\}$ has natural density 1. It follows that $K(\varepsilon) = K_1\left(\frac{\varepsilon}{2}\right) + K_2\left(\frac{\varepsilon}{2}\right) = \left\{ţ \in \mathbb{N}: \dot{\mu}(\|\text{ȵ}_{ţ} - \xi, f\| + \|s_{ţ} - \xi, f\|, \varepsilon) > \frac{1}{2}\right\}$ Since for every $ţ \in \mathbb{N}$, that $\dot{\mu}(\text{ȵ}_{ţ}, \text{ɱ}_{ţ}) > \frac{1}{2}, \dot{\mu}(\text{ɱ}_{ţ}, s_{ţ}) > \frac{1}{2}$, then by condition (i) of the Definition 2.6 we write $\dot{\mu}(\text{ȵ}_{ţ} - \xi, \text{ɱ}_{ţ} - \xi) > \frac{1}{2}$ and $\dot{\mu}(\text{ɱ}_{ţ} - \xi, s_{ţ} - \xi) > \frac{1}{2}$. It follows that for every $\varepsilon >$

0, the set $\left\{ţ \in \mathrm{N}: \dot{\mu}(\|m_ţ - \xi, f\|, \| n_ţ - \xi, f\| + \|s_ţ - \xi, f\|) > \frac{1}{2}\right\}$ for each $f \in \mathfrak{T}$ Also the set $K(\varepsilon)$, for $\varepsilon > 0$ & nonzero$f \in \mathfrak{T}$, then $\left\{ţ \in \mathrm{N}: \dot{\mu}(\varepsilon, \|m_ţ - \xi, f\|) > \frac{1}{2}\right\}$ has natural density 1. Hence $Fsc_n^2 - \lim_ţ \|m_ţ, f\| = \|\xi, f\|$.

Theorem 3.6: *Let* $(n_ţ)_{ţ \in \mathrm{N}}$ *be a sequence in F2-NRS*$(\mathfrak{T}, \|\bullet,\bullet\|)$. *The sequence* $(n_ţ)_{ţ \in \mathrm{N}}$ *is* Fsc_n^2- *Convergent if and only if for any* $\varepsilon > 0$ *& nonzero*$f \in \mathfrak{T}$, $\left\{ţ, n \in \mathrm{N}: \dot{\mu}(\varepsilon, \| n_ţ - n_{ţ_n}, f\|) > \frac{1}{2}\right\}$ *has natural density zero, where* $(n_{ţ_n})_{n \in \mathrm{N}}$ *is a subsequence of* $(n_ţ)_{ţ \in \mathrm{N}}$ *such that* $Fsc_n^2 - \lim_n \|n_{ţ_n}, f\| = \|\xi, f\|$.

Proof: Suppose that $(n_ţ)_{ţ \in \mathrm{N}}$ is Fsc_n^2- Convergent. Now to prove that when $\varepsilon > 0$, $\left\{ţ, n \in \mathrm{N}: \dot{\mu}(\varepsilon, \| n_ţ - n_{ţ_n}, f\|) > \frac{1}{2}\right\}$ have natural density zero Consequently, when $\varepsilon > 0$ & nonzero $f \in \mathfrak{T}$, $\left\{ţ \in \mathrm{N}: \dot{\mu}(\varepsilon, \| n_ţ - \xi, f\|) > \frac{1}{2}\right\}$ has natural density zero, $\xi \in \mathfrak{T}$.

From this it follows that, $\left\{ţ, n \in \mathrm{N}: \dot{\mu}(2\varepsilon, \| n_ţ - n_{ţ_n}, f\|) > \frac{1}{2}\right\} = \left\{ţ, n \in \mathrm{N}: \dot{\mu}(2\varepsilon, \|(n_ţ - \xi) + (\xi - n_{ţ_n}), f\|) > \frac{1}{2}\right\} \leq \left\{ţ \in \mathrm{N}: \dot{\mu}(\varepsilon, \| n_ţ - \xi, f\|) > \frac{1}{2}\right\} + \left\{n \in \mathrm{N}: \dot{\mu}(\varepsilon, \|\xi - n_{ţ_n}, f\|) > \frac{1}{2}\right\}$ by Theorem 4.1 [2],

$\leq \max\left[\left\{ţ \in \mathrm{N}: \dot{\mu}(\varepsilon, \| n_ţ - \xi, f\|) > \frac{1}{2}\right\}, \left\{n \in \mathrm{N}: \dot{\mu}(\varepsilon, \|n_{ţ_n} - \xi, f\|) > \frac{1}{2}\right\}\right]$

implies, $\leq \max\left[0, \left\{n \in \mathrm{N}: \dot{\mu}(\varepsilon, \|n_{ţ_n} - \xi, f\|) > \frac{1}{2}\right\}\right]$

Thereby, we obtain that $Fsc_n^2 - \lim_n \|n_{ţ_n}, f\| = \|\xi, f\|$. Then for $\varepsilon > 0$ & nonzero $f \in \mathfrak{T}$, $\left\{n \in \mathrm{N}: \dot{\mu}(\varepsilon, \|n_{ţ_n} - \xi, f\|) > \frac{1}{2}\right\}$.

Have natural density zero. So, we have where $\varepsilon > 0$, $\left\{ţ, n \in \mathrm{N}: \dot{\mu}(\varepsilon, \| n_ţ - n_{ţ_n}, f\|) > \frac{1}{2}\right\}$ has natural density zero and for each nonzero$f \in \mathfrak{T}$. Conversely, let for every$\varepsilon > 0$ and nonzero$f \in \mathfrak{T}$, so $\left\{ţ, n \in \mathrm{N}: \dot{\mu}(\varepsilon, \| n_ţ - n_{ţ_n}, f\|) > \frac{1}{2}\right\}$ has natural density zero. To show that the sequence $Fsc_n^2 - \lim_ţ \| n_ţ, f\| = \|\xi, f\|$. We will write for every $\varepsilon > 0$ and nonzero $f \in \mathfrak{T}$, the set $\left\{ţ \in \mathrm{N}: \dot{\mu}(2\varepsilon, \| n_ţ - \xi, f\|) > \frac{1}{2}\right\} = \left\{ţ, n \in \mathrm{N}: \dot{\mu}(2\varepsilon, \|(n_ţ - n_{ţ_n}) + (n_{ţ_n} - \xi), f\|) > \frac{1}{2}\right\} \leq \left\{ţ, n \in \mathrm{N}: \dot{\mu}(\varepsilon, \| n_ţ - n_{ţ_n}, f\|) > \frac{1}{2}\right\} + \left\{n \in \mathrm{N}: \dot{\mu}(\varepsilon, \|n_{ţ_n} - \xi, f\|) > \frac{1}{2}\right\} \leq \max\left[\left\{ţ, n \in \mathrm{N}: \dot{\mu}(\varepsilon, \| n_ţ - n_{ţ_n}, f\|) > \frac{1}{2}\right\}, \left\{n \in \mathrm{N}: \dot{\mu}(\varepsilon, \|n_{ţ_n} - \xi, f\|) > \frac{1}{2}\right\}\right]$

$\leq \max\left[0, \left\{n \in \mathrm{N}: \dot{\mu}(\varepsilon, \|n_{ţ_n} - \xi, f\|) > \frac{1}{2}\right\}\right]$.

Also, by (4) we have for every $\in\ > 0$ and nonzero$\mathfrak{f} \in \mathfrak{T}$, the set $\left\{ \mathfrak{t} \in \mathbb{N}: \dot{\mu}(2\varepsilon, \| \mathfrak{n}_{\mathfrak{t}} - \xi, \mathfrak{f}\|) > \frac{1}{2}\right\}$ has natural density zero. Hence$\mathrm{Fsc}_{\mathrm{n}}^2 - \lim_{\mathfrak{t}} \| \mathfrak{n}_{\mathfrak{t}}, \mathfrak{f}\| = \|\xi, \mathfrak{f}\|$.

Theorem 3.7: *Let* $(\mathfrak{n}_{\mathfrak{t}})_{\mathfrak{t}\in\mathbb{N}}$ *be a sequence in an F2-NRS*$(\mathfrak{T}, \|\bullet,\bullet\|)$. *A sequence* $(\mathfrak{n}_{\mathfrak{t}})_{\mathfrak{t}\in\mathbb{N}}$ *is* Fsc_n^2*- Convergent iff there are subsets such as* $\wp = \{\mathfrak{t}_1 < \mathfrak{t}_2 < \ldots\} \subset \mathbb{N}$ *s.t* $\delta(\wp) = 1$ *&* $\lim_{\substack{\mathfrak{t}\to\infty \\ \mathfrak{t}\in\wp}} \mathfrak{n}_{\mathfrak{t}} = \xi$.

Proof: Assume that $(\mathfrak{n}_{\mathfrak{t}})_{\mathfrak{t}\in\mathbb{N}}$ is $\mathrm{Fsc}_{\mathrm{n}}^2$- Convergent to an element ξ. Put $\wp_{\mathrm{f}} = \left\{\mathfrak{t} \in \mathbb{N}: \dot{\mu}(\frac{1}{\mathrm{f}}, \| \mathfrak{n}_{\mathfrak{t}} - \xi, \mathfrak{f}\|) > \frac{1}{2}\right\}$ and $\mathfrak{J}_{\mathrm{f}} = \left\{\mathfrak{t} \in \mathbb{N}: \dot{\mu}(\| \mathfrak{n}_{\mathfrak{t}} - \xi, \mathfrak{f}\|, \frac{1}{\mathrm{f}}) > \frac{1}{2}\right\}$, (f=1,2, ...). From this we have that $\delta(\wp_{\mathrm{f}}) = 0$. And$\mathfrak{J}_1 \supset \mathfrak{J}_2 \supset \ldots \supset \mathfrak{J}_{\mathfrak{t}} \supset \mathfrak{J}_{\mathfrak{t}+1} \supset \ldots \delta(\mathfrak{J}_{\mathrm{f}}) = 1$, f=1,2, 3,...

Conversely, to prove that $\lim_{\substack{\mathfrak{t}\to\infty \\ \mathfrak{t}\in\mathfrak{J}_{\mathrm{f}}}} \mathfrak{n}_{\mathfrak{t}} = \xi$. We will assume that the sequence$(\mathfrak{n}_{\mathfrak{t}})_{\mathfrak{t}\in\mathbb{N}}$ is not convergent toξ. Thereby there is $\varepsilon > 0$ and nonzero $\mathfrak{f} \in \mathfrak{T}$, the set$\left\{\mathfrak{t} \in \mathbb{N}: \dot{\mu}(\varepsilon, \| \mathfrak{n}_{\mathfrak{t}} - \xi, \mathfrak{f}\|) > \frac{1}{2}\right\}$. In particular, we can write for every $\varepsilon > 0$ and nonzero $\mathfrak{f} \in \mathfrak{T}$, the set $\mathfrak{J}(\varepsilon) = \left\{\mathfrak{t} \in \mathbb{N}: \dot{\mu}(\| \mathfrak{n}_{\mathfrak{t}} - \xi, \mathfrak{f}\|, \varepsilon) > \frac{1}{2}\right\}$, also $\dot{\mu}(\frac{1}{\mathrm{f}}, \varepsilon) > \frac{1}{2}$, f=1, 2, 3, ... From this, it follows that $\delta(\mathfrak{J}(\varepsilon)) = 0$.

So we obtain that $\mathfrak{J}_{\mathrm{f}} \subset \mathfrak{J}(\varepsilon)$. Implies that $\delta(\mathfrak{J}_{\mathrm{f}}) = 0$. Therefore $\lim_{\substack{\mathfrak{t}\to\infty \\ \mathfrak{t}\in\wp}} \mathfrak{n}_{\mathfrak{t}} = \xi$.

Conversely, assume there is $\wp = \{\mathfrak{t}_1 < \mathfrak{t}_2 < \ldots\} \subset \mathbb{N}$ s.t $\delta(\wp) = 1$ & $\lim_{\substack{\mathfrak{t}\to\infty \\ \mathfrak{t}\in\wp}} \mathfrak{n}_{\mathfrak{t}} = \xi$, that is, there is $\mathrm{k} \in \mathbb{N}$ s.t for $\varepsilon > 0$ & nonzero $\mathfrak{f} \in \mathfrak{T}$,$\dot{\mu}(\| \mathfrak{n}_{\mathfrak{t}} - \xi, \mathfrak{f}\|, \varepsilon) > \frac{1}{2}$, for every$\mathfrak{t} \geq \mathrm{k}$. Consequently, so $\wp(\varepsilon) = \left\{\mathfrak{t} \in \mathbb{N}: \dot{\mu}(\varepsilon, \| \mathfrak{n}_{\mathfrak{t}} - \xi, \mathfrak{f}\|) > \frac{1}{2}\right\} \subset \mathbb{N} - \{\mathfrak{t}_{k+1} < \mathfrak{t}_{k+2} < \ldots\}$. Thus $\dot{\mu}(\delta(\wp(\varepsilon)),0) > \frac{1}{2}$. Hence $\mathrm{Fsc}_{\mathrm{n}}^2 - \lim_{\mathfrak{t}} \| \mathfrak{n}_{\mathfrak{t}}, \mathfrak{f}\| = \|\xi, \mathfrak{f}\|$.

Theorem 3.8: *Let* $(\mathfrak{n}_{\mathfrak{t}})_{\mathfrak{t}\in\mathbb{N}}$ *and* $(\mathfrak{m}_{\mathfrak{t}})_{\mathfrak{t}\in\mathbb{N}}$ *be two sequences in F2-NRS and* $\xi \in \mathfrak{T}$. $\lim_{\mathfrak{t}\to\infty} \mathfrak{n}_{\mathfrak{t}} = \xi$ *and* $Fsc_n^2 - \lim_{\mathfrak{t}} \|\mathfrak{m}_{\mathfrak{t}} - 0, \mathfrak{f}\| = 0$*, then* $Fsc_n^2 - \lim_{\mathfrak{t}} \| \mathfrak{n}_{\mathfrak{t}} + \mathfrak{m}_{\mathfrak{t}}, \mathfrak{f}\| = \|\xi, \mathfrak{f}\|$.

Proof: Let $\lim_{\mathfrak{t}\to\infty} \mathfrak{n}_{\mathfrak{t}} = \xi$, i.e., $\lim_{\mathfrak{t}\to\infty} \| \mathfrak{n}_{\mathfrak{t}} - \xi, \mathfrak{f}\| = 0$

Also, from $\mathrm{Fsc}_{\mathrm{n}}^2 - \lim_{\mathfrak{t}} \|\mathfrak{m}_{\mathfrak{t}} - 0, \mathfrak{f}\| = 0$, where $\varepsilon\ > 0$,

$\left\{\mathfrak{t} \in \mathbb{N}: \dot{\mu}(\varepsilon, \|\mathfrak{m}_{\mathfrak{t}}, \mathfrak{f}\|) > \frac{1}{2}\right\}$ have the natural density which is zero for each $\mathfrak{f} \in \mathfrak{T}$. Now, put $\mathrm{Fsc}_{\mathrm{n}}^2 - \lim_{\mathfrak{t}} \| \mathfrak{n}_{\mathfrak{t}} + \mathfrak{m}_{\mathfrak{t}}, \mathfrak{f}\| = \|\psi, \mathfrak{f}\|$ so for $\varepsilon > 0$ & nonzero $\mathfrak{f} \in \mathfrak{T}$,

$\left\{ \mathfrak{t} \in \mathbb{N}: \dot{\mu}(\varepsilon, \|(\mathfrak{n}_{\mathfrak{t}} + \mathfrak{m}_{\mathfrak{t}}) - \psi, \mathfrak{f}\|) > \frac{1}{2} \right\}$ have natural density zero. Consequently, when$\varepsilon > 0$ & nonzero $\mathfrak{f} \in \mathfrak{T} \left| \lim_{\mathfrak{t}} \| \mathfrak{n}_{\mathfrak{t}} - \psi, \mathfrak{f}\| + \left\{ \mathfrak{t} \in \mathbb{N}: \dot{\mu}(\varepsilon, \|\mathfrak{m}_{\mathfrak{t}}, \mathfrak{f}\|) > \frac{1}{2} \right\} \right| = 0$. So, for $\varepsilon > 0$&nonzero $\mathfrak{f} \in \mathfrak{T} \max \left[\lim_{\mathfrak{t}} \| \mathfrak{n}_{\mathfrak{t}} - \psi, \mathfrak{f}\|, \left\{ \mathfrak{t} \in \mathbb{N}: \dot{\mu}(\varepsilon, \|\mathfrak{m}_{\mathfrak{t}}, \mathfrak{f}\|) > \frac{1}{2} \right\} \right] = 0$ that is $\max \left[\lim_{\mathfrak{t}} \| \mathfrak{n}_{\mathfrak{t}} - \psi, \mathfrak{f}\|, 0 \right] = 0$, using (9) which implies that $\lim_{\mathfrak{t} \to \infty} \| \mathfrak{n}_{\mathfrak{t}} - \psi, \mathfrak{f}\| = 0$. Hence $\lim_{\mathfrak{t} \to \infty} \mathfrak{n}_{\mathfrak{t}} = \eta$. But $\lim_{j \to \infty} \| \mathfrak{n}_{\mathfrak{t}} - \xi, \mathfrak{f}\| = 0$. Thus, $\xi = \psi$.

So, we get the required which $Fsc_n^2 - \lim_{\mathfrak{t}} \| \mathfrak{n}_{\mathfrak{t}} + \mathfrak{m}_{\mathfrak{t}}, \mathfrak{f}\| = \|\xi, \mathfrak{f}\|$.

Theorem 3.10: *Let* $(\mathfrak{T}, \|\bullet,\bullet\|)$ *be a F2-NRS. If a sequence* $(\mathfrak{n}_{\mathfrak{t}})_{\mathfrak{t} \in \mathbb{N}}$ *in* $\mathfrak{T}$*is* Fsc_n^2*-Convergent, then* $(\mathfrak{n}_{\mathfrak{t}})_{\mathfrak{t} \in \mathbb{N}}$ *is* Fsc_n^2*- Cauchy.*

Proof: Suppose that $(\mathfrak{n}_{\mathfrak{t}})_{\mathfrak{t} \in \mathbb{N}}$ is Fsc_n^2- Convergent to an element $\xi \in \mathfrak{T}$. Then for every $\varepsilon > 0$, we have $F_1(\frac{\varepsilon}{2}) = \left\{ \mathfrak{t} \in \mathbb{N}: \dot{\mu}(\frac{\varepsilon}{2}, \| \mathfrak{n}_{\mathfrak{t}} - \xi, \mathfrak{f}\|) > \frac{1}{2} \right\}$ has natural density zero for each $\mathfrak{f} \in \mathfrak{T}$, and if $i \in \mathbb{N}$ is chosen so that the set $F_2(\frac{\varepsilon}{2}) = \left\{ \mathfrak{t} \in \mathbb{N}: \dot{\mu}(\frac{\varepsilon}{2}, \| \mathfrak{n}_{\mathfrak{t}} - \xi, \mathfrak{f}\|) > \frac{1}{2} \right\}$ has natural density zero for each $\mathfrak{f} \in \mathfrak{T}$. Put $F_3(\varepsilon) = \left\{ \mathfrak{t}, i \in \mathbb{N}: \dot{\mu}(\varepsilon, \| \mathfrak{n}_{\mathfrak{t}} - \mathfrak{n}_i, \mathfrak{f}\|) > \frac{1}{2} \right\}$ for each $\mathfrak{f} \in \mathfrak{T}$. Then we have $F_3(\varepsilon) \subseteq F_1(\frac{\varepsilon}{2}) \cup F_2(\frac{\varepsilon}{2})$. Hence $\delta(F_3(\varepsilon)) \leq \delta(F_1(\frac{\varepsilon}{2})) + \delta(F_2(\frac{\varepsilon}{2})) = 0$. Therefore $(\mathfrak{n}_{\mathfrak{t}})_{\mathfrak{t} \in \mathbb{N}}$ is Fsc_n^2-Cauchy.

Conclusion

This paper introduces and investigates fuzzy statistically convergent sequences in 2-norm within fuzzy 2-normed Riesz spaces. The concept of Fsc_n²-Cauchy sequences is defined, and fundamental properties are established. These results contribute to the development of convergence theory in fuzzy functional analysis.

References

[1] H. Fast, "On statistical convergence," *Colloquium Mathematicae*, vol. 2, no. 3–4, pp. 241–244 (1951).

[2] I. J. Schoenberg, "The integrability of certain functions and related summability methods," *The American mathematical monthly*, vol. 66, no. 5, pp. 361–775 (1959).

[3] K. H. Khalaf, A. M. Salih, O. A. Abdullah, S. H. Asaad, and I. S. Ahmed, "Study of some properties for pre measure," *AIP Conf. Proc.*, vol. 2834, no. 1 (2023), doi: 10.1063/5.0162036.

[4] S. Gähler, "2-metric spaces and their topological structure," *Mathematische Nachrichten*, vol. 28, pp. 115–148 (1963).

[5] S. A. H. Kadhum and S. A. H. Kadhum, "Order-Convergence for Filters in a Dedekind Complete Riesz Spaces (K-Spaces)," *Journal of physics: conference series*, vol. 1294, no. 3 (Sep. 2019).

[6] I. S. Ahmed, H. H. Ebrahim, and A. Al-Fayadh, "γ–algebra of Sets and Some of its Properties," *Iraqi Journal of Science*, vol. 63, no. 11, pp. 4918–4927 (2022), doi: 10.24996/ijs.2022.63.11.28.

[7] S. Gähler, "Linear 2-normed spaces," *Mathematische Nachrichten,* vol. 28, no. 1–2, pp. 1–43 (1964), doi: 10.1002/mana.19640280102.

[8] Z. H. Abdulqader, R. B. Esmaeel, and N. E. Arif, "L-open set and some related concepts," *Journal of Discrete Mathematical Sciences and Cryptography*, vol. 27, no. 5, pp. 1715–1721 (2024).

[9] S. N. Al-Khafaji, A. Al-Fayadh, A. H. Hussain, and S. A. Abbas, "Toeplitz Determinant whose Its Entries are the Coefficients for Class of Non-Bazilević Functions," *Journal of Physics: Conference Series,* vol. 1660, no. 1 (Nov. 2020).

[10] M. Y. Abed, "On semi and quasi γρρ-orthogonality in real normed space," *Journal of Interdisciplinary Mathematics*, vol. 25, no. 8, pp. 2585–2590 (2022).

[11] N. E. Z. T. Abdulqader, R. B. Arif, and J. X. Esmaeel, "L – open and some of their generalizations," *Journal of Interdisciplinary Mathematics,* vol. 28, no. 3, pp. 1125–1130 (2025).

[12] A. R. S. Juma, A. Al-Fayadh, S. P. Vijayalakshmi, and T. V. Sudharsan, "Upper bound on the third hankel determinant for the class of univalent functions using an integral operator," *Afrika Matematika*, vol. 33, no. 2, p. 56 (2022).

[13] H. Gunawan and Mashadi, "On finite dimensional 2-normed spaces," *Soochow J. Math.*, vol. 27, no. 3, pp. 631–639 (2001).

[14] I. Beg and M. U. Islam, "Fuzzy Riesz spaces," *Journal of Fuzzy Mathematics*, vol. 2, no. 1, pp. 211–241 (1994).

[15] A. K. Shatha, "Fuzzy order convergence of double sequences in fuzzy Riesz spaces," *Journal of University of Babylon for Pure and Applied Sciences*, vol. 28, no. 2, pp. 99–128 (2020).

[16] S. A. H. Kadhim and S. A. H. Kadhim, "On Order-Convergence of Filters in a Riesz Spaces," *Kirkuk University Journal for Scientific Studies*, vol. 15, no. 2 (2020).

[17] W. Raymond, Y. Freese, and J. Cho, *Geometry of linear 2-normed spaces*. Huntington, USA: Nova Science Publishers (2001).

Received November, 2025