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New solitary wave solution for the nonlinear autonomous equations using an efficient method

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Abstract

An efficient approach is used in this article to solve and analyze the nonlinear autonomous equations. The Tanh method provides more general, precise traveling wave solutions with little additional work. The Tanh technique is demonstrated to be an effective mathematical tool for solving nonlinear autonomous equations and any model equation in mathematical physics. Additionally, the obtained solutions are physically represented and interpreted. Furthermore, the authors present the visual effects of the behavior of wave structures using various figures and contour plots, in the results to enhance our comprehensive understanding.

Subject Classification: *37Dxx, 34Dxx, 76Fxx.*

Keywords: *PDEs, Soliton, Breaking soliton, Nonlinear autonomous equation, Tanh method.*

1. Introduction

In many areas of mathematics, fluid dynamics, plasma physics, and engineering, the nonlinear autonomous equations play an important role. Recently, a number of organized, effective methods have been used to obtained solutions of non-linear partial differential equations (NLPDEs), including Painleve analysis [1], Bäcklund transformation [2], Ansatz method [3], Hirota bilinear formalism [4], Lie symmetry [5], Homoclinic test method [6], Jacobi elliptic function method [7], HPM [8], ADM [9], and semi-analytic technique [10]. A powerful method for symbolically calculating wave-traveling solutions to nonlinear wave and autonomous equations is the hyperbolic tangent (tanh)

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method. According to Hussein and Tawfiq [11], the approach is particularly effective when dispersion, convection, and reaction-diffusion phenomena are significant. The analytical and numerical approaches to the nonlinear autonomous equations have been extensively studied using a variety of methods. This article describes the Tanh method and then applies it to solve the breaking soliton equation and autonomous equations. This equation has an important property in engineering, sciences, and mathematical physics. These equations include the breaking soliton equation and the nonlinear autonomous equation.

2. Tanh Method

We'll introduce the tanh approach as described by Authors [10] and [11]. Consider the following non-linear PDEs:

$$\mathcal{F}\left(\mathcal{v}, \mathcal{v}_t, \mathcal{v}_y, \mathcal{v}_z, \mathcal{v}_x, \mathcal{v}_{xx}, \mathcal{v}_{yy}, \mathcal{v}_{zz}, \mathcal{v}_{xy}, \dots\right) = 0 \tag{1}$$

Where $\mathcal{F}$ defined as a polynomial of the variable $\mathcal{v}$ and its derivatives. Suppose $\mathcal{v}(\mathfrak{P} - t) = \mathcal{V}(\mathcal{E})$, where $\mathfrak{P} = x + y + z, \mathcal{E}r(\mathfrak{P} - st) or\ \mathcal{E} = r(\mathfrak{P} + st)$. We will use the following form:

$$\pm \mathrm{sr}\frac{d}{d\mathcal{E}} = \frac{d}{d\mathrm{t}}, \mathrm{r}\frac{d}{d\mathcal{E}} = \frac{d}{d\mathrm{x}}, \mathrm{r}\frac{\mathrm{d}}{\mathrm{d}\mathcal{E}} = \frac{d}{d\mathrm{y}}, \mathrm{r}\frac{d}{\mathrm{d}\mathcal{E}} = \frac{d}{\mathrm{dz}}, \mathrm{r}^2\frac{d^2}{\mathrm{d}\mathcal{E}^2} = \frac{d^2}{d\mathrm{x}^2}, \mathrm{r}^3\frac{d^3}{d\mathcal{E}^3} = \frac{d^3}{d\mathrm{x}^3}$$

And so on, then equation (1) changed to an ordinary differential equation (ODE) say:

$$Q(V, V', V'', V''', \dots) = 0 \tag{2}$$

Reduced ODEs (2) are what they are called. The integration constants in equation (2) are supposed to be 0 (zero) in light of the localized solutions, that is, all terms have derivatives. Now, obtaining the solution to the reduced ODE (2) is identical to getting the traveling wave solutions to equation (1). Suppose the new independent variable for the tanh method.

$$\mathcal{G}(\mathfrak{P}) = tanh(\mathcal{E}) \tag{3}$$

That is, we obtained the change of variables:

$$\frac{d}{d\mathcal{E}} = (1 - \mathcal{G}^2)\frac{d}{d\mathcal{G}}$$

$$\frac{d^2}{d\mathcal{E}^2} = -2\mathcal{G}(1 - \mathcal{G}\mathcal{G}^2)\frac{d}{d\mathcal{G}} + (1 - \mathcal{G}^2)^2\frac{d^2}{d\mathcal{G}^2}$$

$$\frac{d^3}{d\mathcal{E}^3} = 2(1 - \mathcal{G}^2)(3\mathcal{G}^2 - 1)\frac{d}{d\mathcal{G}} - 6(1 - \mathcal{G}^2)^2\frac{d^2}{d\mathcal{G}^2} + (1 - \mathcal{G}^2)^3\frac{d^3}{d\mathcal{G}^3} \tag{4}$$

The next critical step is to express the desired outcome in the form that we are looking for.

$$\mathcal{v}(x, y, z, t) = \mathcal{V}(\mathcal{E}) = \sum_{i=0}^{m} a_i\, \mathcal{G}^i \tag{5}$$

Where $r, s, a_0, a_1, a_2, \dots$ has to calculate, m is calculate by balance highests orders for the nonlinear parts with the linear part in Eq. (2). Then, substitute Eqs. (5) in Eqs. (2) to gotten a results in the series for the algebraic equation of $r, s, a_0, a_1, a_2, \dots$ because all coefficients of $\mathcal{G}^i$ have to vanish. From these relations, $r, s, e_0, e_1, e_2, \dots$ can be obtained. We are able to derive analytical

solutions $v(x, y, z, t)$ in a closed form by determining these parameters, knowing that m is typically a positive integer, the tanh approach appears to be an effective tool for handling nonlinear physical models.

3. Application Problems

To illustrate the efficiency and precision of the Tanh method, the authors present three problems.

Problem 1: A nonlinear fourth-order autonomous equation has the form

$$u_{zxxx} + u_{yxxx} - u_{tx} + 2u_y u_{xx} + 2u_z u_{xx} + 4u_x u_{xy} + 4u_x u_{xz} = 0 \quad (6)$$

$$u(x, y, z, t) = \mathcal{V}(\mathcal{E}), \text{ where } \mathcal{E} = r(x + y + z + st) \quad (7)$$

Substituting (7) into (6), we have ODE

$$\mathrm{r}^4\mathcal{V}^{(4)} + \mathrm{r}^4\mathcal{V}^{(4)} - \mathrm{sr}^2\mathcal{V}'' + 2\mathrm{r}^3\mathcal{V}''\mathcal{V}' + 2\mathrm{r}^3\mathcal{V}''\mathcal{V}' + 4\mathrm{r}^3\mathcal{V}'\mathcal{V}'' + 4\mathrm{r}^3\mathcal{V}'\mathcal{V}'' = 0$$

$$2r^4\mathcal{V}^{(4)} - cr^2\mathcal{V}'' + 12r^3\mathcal{V}'\mathcal{V}'' = 0 \quad (8)$$

Integrating eq. (8) gives

$$2r^4\mathcal{V}''' - sr^2\mathcal{V}' + 6r^3(\mathcal{V}')^2 = 0 \quad (9)$$

For simplicity assume a constants for unbounded integrations are 0. By balancing higher orders derivatives components $\mathcal{V}'''$ with non - linear part $(\mathcal{V}')^2$ in equation (9), obtain avalued m = 1, as results,

$$u(x, y, z, t) = \mathcal{V}(\mathcal{E}) = \sum_{i=0}^{m=1} a_i \mathcal{G}^i = a_0 + a_1\mathcal{G} \quad (10)$$

The equation in (9) reduces to

$$2\mathrm{r}^4[2(1-\mathcal{G}^2)(3\mathcal{G}^2-1)\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}\mathcal{G}} - 6(1-\mathcal{G}^2)^2\frac{\mathrm{d}^2\mathcal{V}}{\mathrm{d}\mathcal{G}^2} + (1-\mathcal{G}^2)^3\frac{\mathrm{d}^3\mathcal{V}}{\mathrm{d}\mathcal{G}^3}] -$$
$$\mathrm{sr}^2\left[(1-\mathcal{G}^2)\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}\mathcal{G}}\right] + 6\mathrm{r}^3[(1-\mathcal{G}^2)\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}\mathcal{G}}]^2 = 0 \quad (11)$$

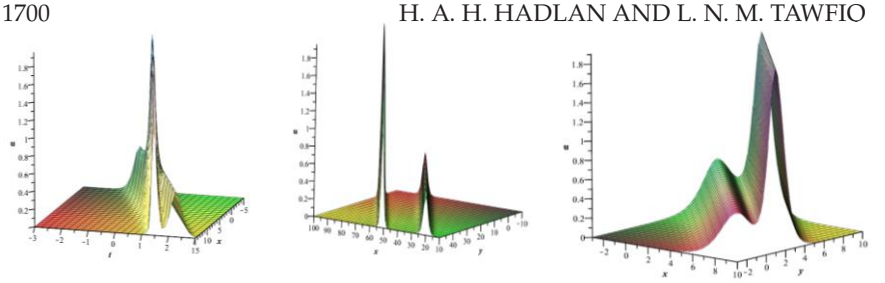
Substituting $\mathcal{V}', \mathcal{V}'''$ from equation (10) into equation (11) produce sets for the algebraic equation of variable a_0, a_1, s, r

$$\mathcal{G}^0: -4r^4a_1 - sr^2a_1 + 6r^3a_1^2 = 0\ ; \mathcal{G}^2: 16r^4a_1 + sr^2a_1 - 12r^3a_1^2 = 0$$
$$\mathcal{G}^4: -12r^4a_1 + 6r^3a_1^2 = 0$$

We have $a_1 = 2r, s = 8r^2$

If $r = \frac{1}{2} \rightarrow a_1 = 1$ and $s = 2$, let $a_0 = 0 \therefore u(x, y, z, t) = \mathcal{V}(\mathcal{E}) = \sum_{\mathrm{i}=0}^{\mathrm{m}=1} \mathrm{a_i}\,\mathcal{G}^{\mathrm{i}} = \mathrm{a_0} + \mathrm{a_1}\mathcal{G} = \tanh\left(\frac{1}{2}(x + y + z + 2t)\right)$ Since $\mathcal{G} = tanh(\mathcal{E})$. This is exact solution.

This is the same result obtained by using the New Extended Decomposition Method and sine Gordon expansion method but the suggested method is easy implementation. The two-soliton solution illustrated in Figure 1, as three-dim plots when $r_1 = \mathrm{p}_1 = \mathrm{q}_1 = 1, \mathrm{r}_2 = \mathrm{p}_2 = \mathrm{q}_2 = 2, \mathrm{z} = 0, \mathrm{s}_1 = 7, \mathrm{s}_2 = 20$, the figure illustrated in three cases: (a) $y = 1, -5 \leq x \leq 15, -3 \leq t \leq 3$, (b) $\mathrm{t} = 10, 10 \leq \mathrm{x} \leq 100, -10 \leq \mathrm{y} \leq 50$, and (c) when $\mathrm{t} = 1, -3 \leq \mathrm{x} \leq 10, -3 \leq \mathrm{y} \leq 10$.



(a) **(b)** **(c)**

Figure 1
Three-dim plots of the two-soliton solution of Problem 1

Problem 2: Consider the following nonlinear autonomous equation (NEE) is given by

$$3w_{xy} - (2w_t + w_{xxx} - 2ww_x)_y + 2(w_x\partial_x^{-1}w_y)_x = 0 \tag{12}$$

Where ∂_x^{-1} is the inverse of ∂_x with $\partial_x^{-1}\partial_x = 1$

Firstly suppose that

$$w_x(x, y, z, t) = v(x, y, z, t) \tag{13}$$

$$3v_{xxy} - (2v_{xt} + v_{xxxx} - 2v_xv_{xx})_y + 2(v_{xx}v_y)_x = 0 \tag{14}$$

$$v(x, y, z, t) = \mathcal{V}(\mathcal{E}), \text{ where } \mathcal{E} = \mathfrak{P} - st = x + y + z - st \tag{15}$$

Substituting equation (15) into equation (14), we have ODE

$$3\mathcal{V}''' - (-2s\mathcal{V}'' + \mathcal{V}^{(4)} - 2\mathcal{V}'\mathcal{V}'')' + 2(\mathcal{V}''\mathcal{V}')' = 0 \tag{16}$$

Integrating eq. (16) gives

$$3\mathcal{V}'' - (-2s\mathcal{V}'' + \mathcal{V}^{(4)} - 2\mathcal{V}'\mathcal{V}'') + 2(\mathcal{V}''\mathcal{V}') = 0 \tag{17}$$

Again integrating eq. (17) gives

$$3\mathcal{V}' + 2s\mathcal{V}' - \mathcal{V}''' + (\mathcal{V}')^2 + (\mathcal{V}')^2 = 0$$

$$(3 + 2s)\mathcal{V}' - \mathcal{V}''' + 2(\mathcal{V}')^2 = 0 \tag{18}$$

Now, for simplicity, the authors suppose a constants for unbounded integrations is 0.

By balancing higher-orders derivatives components $\mathcal{V}'''$ with non-linear part $(\mathcal{V}')^2$ in equation (18), we obtain valued M = 1, as the results,

$$v(x, y, z, t) = \mathcal{V}(\mathcal{E}) = \sum_{i=0}^{m=1} a_i\,\mathcal{G}^i = a_0 + a_1\mathcal{G} \tag{19}$$

The equation in (18) reduces to

$$(3 + 2s)\left[(1 - \mathcal{G}^2)\frac{d\mathcal{V}}{d\mathcal{G}}\right] - [2(1 - \mathcal{G}^2)(3\mathcal{G}^2 - 1)\frac{d\mathcal{V}}{d\mathcal{G}} - 6(1 - \mathcal{G}^2)^2\frac{d^2\mathcal{V}}{d\mathcal{G}^2} + (1 - \mathcal{G}^2)^3\frac{d^3\mathcal{V}}{d\mathcal{G}^3}] + 2[(1 - \mathcal{G}^2)\frac{d\mathcal{V}}{d\mathcal{G}}]^2 = 0 \tag{20}$$

Substituting $\mathcal{V}', \mathcal{V}'''$ from equation (19) into equation (20) produces sets for algebraic equation of variable a_0, a_1, s

$\mathcal{G}^0$: $(3+2s)a_1 + 2a_1 + 2a_1^2 = 0$; $\mathcal{G}^2$: $-(3+2s)a_1 - 8a_1 - 4a_1^2 = 0$
$\mathcal{G}^4$: $6a_1 + 2a_1^2 = 0$

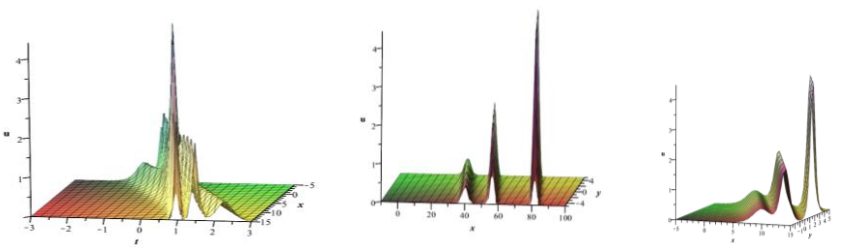
We have $a_1 = -3, s = \frac{1}{2}$, let $a_0 = 0$

$\therefore v(x,y,z,t) = \mathcal{V}(\mathcal{E}) = \sum_{i=0}^{m=1} a_i \mathcal{G}^i = a_0 + a_1\mathcal{V} = -3tanh(x+y+z-\frac{1}{2}t)$, since $\mathcal{G} = tanh(\mathcal{E})$

Since $v(x,y,z,t) = w_x(x,y,z,t)$

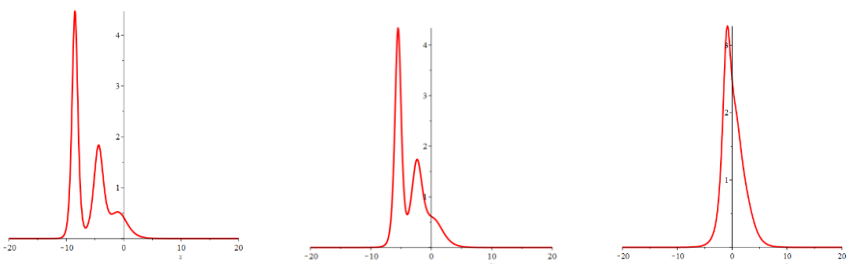
$w_x(x,y,z,t) = -3tanh(\mathfrak{P} - \frac{1}{2}t)$, by integral

$w(x,y,z,t) = -3\log\left(cosh\left(\mathfrak{P} - \frac{1}{2}t\right)\right) = -3\log\left(cosh\left(x+y+z-\frac{1}{2}t\right)\right)$. This is exact solution. This is the same result obtained by using the (G'/G) −expansion method [2, 3] and Exponential Function Method [11, 9] but the suggested method is easy implementation. In Figure 2, the three-soliton solutions illustrated as three-dim plots when $r_1 = p_1 = q_1 = 1$, $r_2 = p_2 = q_2 = 2$, $r_3 = p_3 = q_3 = 3$, $s_1 = 7$, $s_2 = 20, s_3 = 45, z = 0$, illustrated in three cases (a) $y = 1, -5 \le x \le 15, -3 \le t \le 3$; (b) $t = 10, -10 \le x \le 100, -5 \le y \le 5$; (c) $t = 1, -5 \le x \le 15, -3 \le y \le 5$.



(a) (b) (c)

Figure 2
Three-soliton solutions for Problem 2



(a) t = –0.5 **(b) t = –0.3** **(c) t = 0**

Figure 3
Two-dimensional plot of the three-soliton for Problem 2

But Figure 3, illustrate the three-soliton solutions as two-dim plots when $r_1 = p_1 = q_1 = 1, r_2 = p_2 = q_2 = 2, r_3 = p_3 = q_3 = 3, s_1 = 7, s_2 = 20, s_3 = 45, y = 1, z = 0,$ illustrated in 3 cases: (a) $t = -0.5, -20 \leq x \leq 20$; (b) $t = -0.3, -20 \leq x \leq 20$; (c) $t = 0, -20 \leq x \leq 20$.

4. Conclusions

This article presents the construction of solitary wave solutions of breaking type for the 3 dimensional 4th order PDEs. We constructed new solutions such as singular soliton solutions, bright solitons, and trigonometric function solutions, and the solutions obtained satisfy the original equation. Besides, 2D and 3D figures of bright and singular soliton and travelling wave solutions are presented for better clarification. Furthermore, the model's stability is assessed using modulation instability analysis. This article illustrates the method's effectiveness in recognizing analytical solutions. Consequently, it can be inferred that the algorithm has the potential to be applied to several partial differential equations within the realms of physical phenomena and applied sciences.

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