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New results in bornological algebra of smooth mappings

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Abstract

This paper presents new results in bornological algebra, focusing on smooth mappings under certain conditions. We apply our results to solve dynamic problems using the derived bornological algebra theorems.

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1. Introduction

A bornological algebra is a well-known and traditional realm of mathematics with several applications that present a framework for studying algebraic and analytic structure. This is accomplished mainly by focusing on boundedness properties, rather than the conventional reliance on open-set topologies. Bounded sets viewed through the lens of duality originated in explorations of duality in locally convex spaces during the 1950s, specifically in a 1959 publication [1]. Many generalizations of differentiable and topological properties on bornological algebra have been discussed in many papers and books ([2–13]). Nonetheless, the monograph in [4] provided the initial comprehensive exploration of bornological space. This study rigorously defined a bornology, essentially a collection of subsets identified as “bounded” and adhering to axiomatic principles that parallel the behavior of neighborhoods in topology. This work showed that this perspective provides a versatile alternative for the established theories concerning (TVS). The goal of this paper is to establish new bornological algebra theorems for contraction mappings in

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Frechet spaces. Furthermore, we use our findings to demonstrate the existence of solutions to dynamic problems.

2. Preliminaries

Definition 2.1: If K is one for field R for the real number or C for the complex number and X the sets. The family β for the subset for the set X named *bornology* on the set X when:

(i) β is covering of X, that is the set $X = \bigcup_{B\epsilon\beta} B$;

(ii) β be the hereditary under inclusions, that is, when $A\epsilon\beta$ & B be the subsets for the X included in A, so $B\epsilon\beta$;

(iii) β be the stables under finite unions, i.e. $\bigcup_{i=1}^{n} B_i\epsilon\beta, \forall\, B_i\epsilon\beta$.

the (X,β) consist for X & bornology β on X named bornological spaces, & element for β named bounded sets.

In particulars, if X is linear spaces over K & the bornology β on X satisfies a below condition:

(iv) $B_1, B_2\epsilon\beta$, then $B_1 + B_2\epsilon\beta$,

(v) $\alpha\epsilon$K and $B\epsilon\beta$ then $\alpha B\epsilon\beta$.

(vi) If $B\epsilon\beta$, then $\bigcup_{|\alpha|\leq 1} \alpha B\epsilon\beta$.

So, (X,β) named bornological linear spaces over K. if bornology β on X satisfies, in additions to condition $\big((iv) - (vi)\big)$ a conditions.

(vii) If $x\epsilon X$, and $B\epsilon\beta$, then $xB, Bx\epsilon\beta$.

so (X,β) named bornological algebras.

Definition 2.2: Webbed spaces be Hausdorff locally convexly vector spaces equipped with the web, a specific hierarchical system of absolutely convex sets that ensures desirable properties for functional analytic theorems, such as the Open Mapping Theorem and Closed Graph Theorem in a non-metrizable space.

Lemma 2.3: *If E is linear C^∞- closed sub - spaces for F. so traces for C^∞-topology for the F on E be C^∞-topology on E.*

Lemma 2.4: *The sub - sets be bounded in the $L(E;\ F) \subseteq C^\infty(E;\ F)$ iff it uniformly bounded on bounded sub - set for the E.*

Corollary 2.5: *If E, F, G, etc. is the locally convexly space, & U, V is the C^∞-open sub - set for the such. Then followed canonicals map. Is the smooth.*

1- $(\,)^\wedge : C^\infty\big(U, C^\infty(V,G)\big) \to C^\infty(U\times V, G)$,

2- $(\,)^\vee : C^\infty(U\times V, G) \to C^\infty\big(U, C^\infty(V,G)\big)$.

3. Main Results

Differentiable Curves 3.1: Assume $\mathcal{M}$ is bornological algebra (e.g., von a Neumann bounded sets), A curve $f: \mathbb{R} \to \mathcal{M}$ be differentiable if a derivative

$f'(t) := \lim_{\gamma\to 0}\frac{f(t+\gamma)-f(t)}{\gamma}$, exist in a bornology induce topology of $\mathcal{M}, \forall t$, where f' is continuous. A curve is called smooth if the every iterated derivatives exist, and derivatives of C^n exist and are continuous.

- A map $g: \mathbb{R}^n \to \mathcal{M}$ be smooth if any iterated a partial derivatives $\partial_{j_1,\dots,j_p} g$ exist.
- A curve $f: \mathbb{R} \to \mathcal{M}$ be a smooth if any iterated derivatives $\frac{d^k f}{dt^k}$ exist.

And f be a bornologically Lipschitzian, if any bounded interval $[a, b]$ $\left(\frac{f(t)-f(s)}{t-s}: s \neq t; s, t\epsilon[a, b]\right)$ is a bounded in the bornology of $\mathcal{M}$.

A curve f is $\left(C^{(k+1)}\right)$ if the derivative order of k exist and are bornologically Lipschitzian.

Lipschitz impels the uniform continuity on the bounded interval.

Compatibility Axiom (Every bornologically bounded set in $\mathcal{M}$ is topologically bounded) e.g., this hold for Frechet space, Banach Algebra, …

Lemma 3.2: *If* $\mathcal{J}: \mathcal{M} \to \mathcal{N}$ *bounded linear mappings between a convex bornological algebra with seminorm* $\{p_\alpha\}_{\alpha\epsilon M}, \{q_\beta\}_{\beta\epsilon N}$, *and* $c: \mathbb{R} \to \mathcal{M}$ $(0 \leq k \leq 1)$ *be* Lip^k*-curve then* $\mathcal{J}oc: \mathbb{R} \to \mathcal{N}$ *is* Lip^k*-curve in* $\mathcal{N}$, *and for* $k > 0$, *the derivative satisfies:*

$$(\mathcal{J}oc)'(t) = \mathcal{J}\big(c'(t)\big), q_\beta\big((\mathcal{J}oc)'(t)\big) \leq C_\beta p_\alpha\big(c'(t)\big).$$

Proof: If $c: \mathbb{R} \to \mathcal{M}$ the Lip^k- curves. from definition, any seminorm p_α exist $\Upsilon_\alpha > 0$ so that:

$$p_\alpha(c(t) - c(s)) \leq \Upsilon_\alpha\, |t - s|^k \text{ for all } t, s\epsilon\mathbb{R}.$$

Since $\mathcal{J}$ is bounded linear map, for any seminorm q_β on $\mathcal{N}$ we can find $\alpha\epsilon M$ and $C_\beta > 0$, so that $q_\beta(\mathcal{J}(x)) \leq C_\beta p_\alpha(x)\ \forall x\epsilon\mathcal{M}$. We can use $\mathcal{J}$ to $c(t) - c(s)$:

$$q_\beta\big((\mathcal{J}oc)(t) - (\mathcal{J}oc)(c)\big) \leq C_\beta \Upsilon_\alpha |t - s|^k.$$

So, $\mathcal{J}oc$ the Lip^k-curve in $\mathcal{N}$. For $k > 0$, the derivative $c'(t)$ in $\mathcal{M}$ is defined bornlogically as:

$$c'(t) = \lim_{h\to 0}\frac{c(t+h)-c(t)}{h},$$

Since $\mathcal{J}$ is a linear bounded, it commute with bornological limits:

$$(\mathcal{J}oc)'(t) = \lim_{h\to 0}\frac{\mathcal{J}\big(c(t+h)\big) - \mathcal{J}(c(t))}{h} = \mathcal{J}\left(\lim_{h\to 0}\frac{c(t+h) - c(t)}{h}\right) = \mathcal{J}\big(c'(t)\big).$$

By use boundedness of $\mathcal{J}$:

$$q_\beta\big((\mathcal{J}oc)'(t)\big) = q_\beta\left(\mathcal{J}\big(c'(t)\big)\right) \leq C_\beta p_\alpha\big(c'(t)\big).$$

Hint: (bornologically bounded $\Longrightarrow$ topologically bounded) the derivative $(\mathcal{J}oc)'$ is continuous. This follows from $\mathcal{J}$ conserving bornological boundedness, hence a topologically boundedness.

Example 3.3: Suppose that $\mathcal{M} = \mathcal{N} = C^{\infty}(\mathbb{R})$, involving seminorm $p_l(f) = \sup_{x,|\beta|\le l}\left|\partial^{\beta} f(x)\right|$, so that the curve $c(t)(x) = te^{-x^2}$, that is Lip^1.

Proposition 3.4: Let $\mathcal{M}$ be a locally convex bornological algebra with a continuous seminorms $\{p_\alpha\}_{\alpha\epsilon M}$ defining the structure of its bornology, $c: I = [a,b] \to \mathcal{M}$ a continuous curve and is bornologically differentiable except at points in a countable subset $D \subseteq I$ and $k: I \to \mathbb{R}$ is a continuous monotonicity, differentiable on $I\backslash D$, for some a convex, bornologically closed $M\epsilon\mathcal{M}$, exist $C_\alpha > 0$ such that $p_\alpha\big(c'(t)\big) \le k'(t).\sup_{a\epsilon M} p_\alpha(a)\,, \forall \quad t\epsilon I\,\backslash D, \forall\alpha$. Then $c(b) - c(a)\epsilon(k(b) - k(a)).\bar{M}$.

Proof: Suppose that $c(b) - c(a) \notin (k(b) - k(a)).\bar{M}$.

Since $(k(b) - k(a)).M$ be a scalar convex bornologically closed subset, from separation Hahn Banach theorems: $\exists$continuous bounded linear function $\emptyset: \mathcal{M} \to \mathbb{R}$ where $M_1, M_2\epsilon\mathcal{M}$. Let $M_1 = \{c(b) - c(a)\}$ compact convex $, M_2 = (k(b) - k(a)).M$ bornologically closed convex, then there exists bounded linear function $\emptyset\epsilon\mathcal{M}'$ s.t:

$$\emptyset\big(c(b) - c(a)\big) > \sup_{y\epsilon M_2} \emptyset(y) = (k(b) - k(a)).\sup_{a\epsilon M_1} \emptyset(a).$$

Now by growth condition on $c'(t)$ for a seminorm $p_\emptyset$ corresponding of the $\emptyset$, and by the chain rule $\frac{d}{dt}\emptyset\big(c(t)\big) = \emptyset\big(c'(t)\big), \forall t\epsilon I\backslash D$

$$\left|\emptyset\big(c'(t)\big)\right| \le p_\emptyset\big(c'(t)\big) \le k'(t).\sup_{a\epsilon M} p_\emptyset(a) \le k'(t).\sup_{a\epsilon\bar{M}}|\emptyset(a)|.$$

By the integral, we get:

$$\int_a^b \emptyset\big(c'(t)\big)\,dt \le \sup_{a\epsilon\bar{M}} \emptyset(a)\,.\int_a^b k'(t)\,dt = \sup_{a\epsilon\bar{M}} \emptyset(a)\,.\,(k(b) - k(a)).$$

By the fundamental theorem of calculus (FTC), and $\emptyset oc$ be absolutely continuous, we get:

$$\int_a^b \emptyset\big(c'(t)\big)\,dt = \emptyset\big(c(b) - c(a)\big).$$

From above get

$$\emptyset\big(c(b) - c(a)\big) \le \sup_{a\epsilon\bar{M}} \emptyset(a)\,.\,(k(b) - k(a)) < \emptyset\big(c(b) - c(a)\big),$$

which is a contradiction of the inequality,

There for get $c(b) - c(a)\epsilon(k(b) - k(a))\,.\bar{M}$.

Definition 3.5: A subset $M \subseteq \mathcal{M}$ is called C^{∞}-open where $\mathcal{M}$ is a convex bornological algebra with a family of seminorm $\{p_\alpha\}_{\alpha\epsilon M}$, the C^{∞}- topology on $\mathcal{M}$ is a final topology with respect to every bornologically smooth curves $c: \mathbb{R} \to \mathcal{M}$ and their higher-order derivatives.

Example 3.6: Suppose that $\mathcal{M} = C^{\infty}(\mathbb{R})$ with seminorm $p_m(f) = \sup_{x\epsilon\mathbb{R},|\gamma|\leq m}|\partial^{\gamma} f(x)|$. A C^{∞}-open set M is $M = \{f\epsilon\mathcal{M} \mid \sup_{m\leq n} p_m(f) < 1 \text{ for some } n\epsilon\mathbb{N}\}$.

Lemma 3.7: *Let $\mathcal{M}$ is a locally convex bornological algebra with the class for seminorm $\{p_{\gamma}\}_{\gamma\epsilon M}$. the subset $L \subseteq C^{\infty}(\mathbb{R};\mathcal{M})$ the bounded iff $\forall k\epsilon\mathbb{N}^0$, compacts $K \subset \mathbb{R}$ & seminorm p_{γ}, exist $C_{\gamma,k,K} > 0$ s.t $sup_{t\epsilon K}\, p_{\gamma}\left(c^{(k)}(t)\right) \leq C_{\gamma,k,K}$ $\forall c\epsilon L$. Moreover, the bornology of $C^{\infty}(\mathbb{R};\mathcal{M})$ be generated by a family map $\{\mathcal{J}_*: C^{\infty}(\mathbb{R};\mathcal{M}) \to C^{\infty}(\mathbb{R};\mathbb{R}) |\, \mathcal{J}\epsilon\mathcal{M}'\}$, where $\mathcal{J}_*(c)(t) = \mathcal{J}\left(c(t)\right)$.*

Proof: $\Longrightarrow$ Suppose that L be a bound in $C^{\infty}(\mathbb{R};\mathcal{M})$, then by the definition of a bornology created by $\{\mathcal{J}_*\}_{\mathcal{J}\epsilon\mathcal{M}'}$, the set $\mathcal{J}_*(L) = \{\mathcal{J}oc: c\epsilon L\}$ is bounded in $C^{\infty}(\mathbb{R};\mathbb{R}), \forall\mathcal{J}\epsilon\mathcal{M}'$.

A set in the $C^{\infty}(\mathbb{R};\mathbb{R})$ is bounded iff for any $k\epsilon\mathbb{N}^0$, compact $K \subset \mathbb{R}$, exsite $C_{\mathcal{J},k,K} > 0$, such that $\sup\limits_{t\epsilon K}\left|(\mathcal{J}oc)^{(k)}(t)\right| \leq C_{\mathcal{J},k,K}\ \forall c\epsilon L$. By using chains rules $(\mathcal{J}oc)^{(k)}(t) = \mathcal{J}\left(c^{(k)}(t)\right)$, thus $\sup\limits_{t\epsilon K}\left|\mathcal{J}\left(c^{(k)}(t)\right)\right| \leq C_{\mathcal{J},k,K}\ \forall c\epsilon L$. Since $\mathcal{J}\epsilon\mathcal{M}'$, the set $\left\{c^{(k)}(t) |\, c\epsilon L, t\epsilon K\right\}$ be the weakly bounded in $\mathcal{M}$. from Hana-Banach theorem, weak bounded implies bornological bounded in $\mathcal{M}$, then for every p_{γ}, $\sup_{t\epsilon K} p_{\gamma}\left(c^{(k)}(t)\right) \leq C_{\mathcal{J},k,K}\ \forall c\epsilon L$.

$\Longleftarrow$ $\forall k\epsilon\mathbb{N}^0$, compact $K \subset \mathbb{R}$ & seminorm p_{γ}, exist $C_{\gamma,k,K} > 0$ s.t $\sup_{t\epsilon K} p_{\gamma}\left(c^{(k)}(t)\right) \leq C_{\gamma,k,K}\ \forall c\epsilon L$. For every $\mathcal{J}\epsilon\mathcal{M}'$, the continuities of c results in for $\gamma\epsilon M$ and $C_{\mathcal{J}} > 0$ s.t $|\mathcal{J}(x)| \leq C_{\mathcal{J}} p_{\gamma}(x)\ \forall x\epsilon\mathcal{M}$. By apply $x = c^{(k)}(t)$, we get $\sup\limits_{t\epsilon K}\left|\mathcal{J}\left(c^{(k)}(t)\right)\right| \leq C_{\mathcal{J}}\ \sup_{t\epsilon K} p_{\gamma}\left(c^{(k)}(t)\right) \leq C_{\mathcal{J}} C_{\gamma,k,K}$. from this inequality the $\mathcal{J}_*$ is bounded in $C^{\infty}(\mathbb{R};\mathbb{R})$. So the set L is bounded in $C^{\infty}(\mathbb{R};\mathcal{M})$ by the $\{\mathcal{J}_*\}_{\mathcal{J}\epsilon\mathcal{M}'}$.

Definition 3.8: A bornological algebra be a vector space M with multiply operator, equipped a topology defined by a seminorm, so that

1- For all bounded linear from M to another locally convex is continuous.
2- Multiplying two elements is "boundedness" any two sets of elements are bounded, the products also bounded, by the seminorm.

Definition 3.9: Suppose that $\mathcal{M}$ be a bornological algebra (i.e., a locally convex topological algebra equipped with a jointly continuous multiplication and whose vector space structure is bornological). Then for a convex subset $M \subseteq \mathcal{M}$:

1- M is an open in a locally convexly topologies $\mathcal{M}$ iff be open in C^{∞}-topologies.
2- If M be absolutely convex, then M 0-neighborhood in a locally convex topology iff it a 0-neighborhood in a C^{∞}-topology.

Example 3.10: The Schwartz space $S(\mathbb{R}^n)$, of a smooth function $c: \mathbb{R}^n \to \mathbb{C}$ where c and all the derivatives are fading faster than any polynomial at infinity.

Lemma 3.11: *Let $\mathcal{M}$ and ρ are bornological algebras, adorned with seminorms $\{p_\alpha\}_{\alpha\epsilon M}$, $\{q_\beta\}_{\beta\epsilon N}$ respectively, suppose that:*

1- $U \subseteq \mathcal{M}$ be a C^∞-open subset.

2- $\{\omega_n\}_{n\epsilon\mathbb{N}} \subseteq R$, be a sequence endowed with ω_n converge to ∞.

3- $c: U \to \rho$ be a map where for any ω-converging sequence $\{x_k\} \subseteq U$, the sequence $\{f(x_k)\} \subseteq \rho$ is bounded. Then c be a bounded on any bornologically compact subset $K \subseteq U$.

Proof: Assume that $K \subseteq U$, be a bornologically compact (i.e., compact in the Banach algebra $\mathcal{M}_N \hookrightarrow \mathcal{M}$. Let c is unbounded on K, so there exists $\{x_n\} \subseteq K$ and $\beta\epsilon N$where $q_\beta\big(f(x_n)\big) \to \infty$. As a result of K is compact in the $\mathcal{M}_N$, it has the property of being sequentially compact, where the subsequence $\{x_{n_k}\} \subseteq K$ with $x_{n_k} \to x\epsilon K$ in $\mathcal{M}_N$. Form the $\mathcal{M}_N \hookrightarrow \mathcal{M}$, we can get for all $\alpha\epsilon M$ exist $C_\alpha > 0$, where $p_\alpha\big(x_{n_k} - x\big) \le C_\alpha \big\|x_{n_k} - x\big\|_{\mathcal{M}_N}$,where $\|.\|_{\mathcal{M}_N}$ continuous to $\{p_\alpha\}$. Hence $\omega_{n_k} p_\alpha\big(x_{n_k} - x\big) \le C_\alpha \omega_{n_k}.\frac{1}{\omega_{n_k}} = C_\alpha$, where $\big\|x_{n_k} - x\big\|_{\mathcal{M}_N} \le \frac{1}{\omega_{n_k}}$ for large K, so $\{x_{n_k}\}$ ω -converging in $\mathcal{M}$. By an assumption $\{f(x_k)\} \subseteq \rho$ has to be bounded in ρ (i.e., $\mathrm{suq}_k\, q_\beta\left(f\big(x_{n_k}\big)\right) < \infty, \forall\beta\epsilon N$), but we have $q_\beta\left(f\big(x_{n_k}\big)\right) \to \infty$, a contradiction of $\{x_{n_k}\}$. Thus c is bounded.

Lemma 3.12: *Suppose that $\mathcal{M}$ is a not normable bornological algebra with a countable number of seminorms $\{p_\alpha\}_{\alpha\epsilon\mathbb{N}}$, definition of it, bornology, and locally convex topology, assume $\mathcal{M}$ has a countable basis, then exist a double sequence $\big(b_{m,n}\big)_{m,n\epsilon\mathbb{N}} \subset \mathcal{M}$ satisfying:*

1- For any fixed m, the sequence $\big(b_{m,n}\big)_{n\epsilon\mathbb{N}}$ be a bound in $\mathcal{M}$.

2- For any bounded set $B \subset \mathcal{M}$, exist $m\epsilon\mathbb{N}$ such that $b_{m,n} \notin B$ for an infinite number of n.

Proof: For $\mathcal{M}$ be a locally convex space where that, for any p_α, exist $C > 0$ and $s, r\epsilon\mathbb{N}$, s.t $p_\alpha(a.b) \le C.p_s(a).p_r(b)$, $\forall a, b\epsilon\mathcal{M}$, and bornology has an increasing sequence $B_1 \subseteq B_2 \subseteq \cdots$ of the absolutely convex sets $B_\alpha = \{x\epsilon\mathcal{M}: p_\alpha(x) \le 1\}$.

Since $\mathcal{M}$ is not normable and not a single seminorm p_α defines a bornology.

For any B_α, α does not absorb $B_{\alpha+1}$(i.e., there is on scalar $\beta > 0$ such that $B_{\alpha+1} \subseteq \beta B_\alpha$).

For $m, n\epsilon\mathbb{N}$ choose $b_{m,n}\epsilon\frac{1}{m}B_{n+1}$ where $b_{m,n} \notin B_n$. Now for the fixed m the $b_{m,n}\epsilon\frac{1}{m}B_{n+1} \subseteq \frac{1}{m}B_{m+1}$(since $n \leq m$ eventully), and $p_{m+1}(b_{m,n}) \leq \frac{1}{m}$, so $(b_{m,n})_{n\epsilon\mathbb{N}}$ be bounded by $\frac{1}{m}B_{m+1}$. Assume that $B \subset \mathcal{M}$ be a bounded set, since B_α be a basis, $B \subseteq \beta B_N$ for some $\beta > 0$ and $N\epsilon\mathbb{N}$. Now for $m > \beta$, consider that $b_{m,n}\epsilon\frac{1}{m}B_{n+1}$, if $n \geq N$, then we get $\frac{1}{m}B_{n+1} \subseteq \frac{1}{\beta}B_{n+1}$ but $b_{m,n} \notin B_n \supseteq B_N \supseteq \frac{1}{\beta}B$.

Thus $b_{m,n} \notin B$ for any $n \geq N$.

Proposition 3.13: Suppose that $\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_n$and ρ are bornological algebras, with a families on the seminorms $\left\{p_{\alpha_j}^{(j)}\right\}_{\alpha_j\epsilon A_j}$ and $\{q_\gamma\}_{\gamma\epsilon B}$, respectively, let the multiplication in any $\mathcal{M}_i$ be a joint continuous with a seminorm, then the injective $\Lambda(\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_n; \rho) \hookrightarrow C^\infty(\mathcal{M}_1 \times \mathcal{M}_2 \times \dots \times \mathcal{M}_n; \rho)$ bounded map to the smooth maps is a bornological embedding.

Proof:

1- For $n = 1$, a set $B \subseteq \Lambda(\mathcal{M}_1; \rho)$ be a bounded if any bounded $B_1 \subseteq \mathcal{M}_1$,
$$\sup_{g\epsilon B}\sup_{x\epsilon B_1} q_\gamma(g(x)) < \infty, \forall\, \gamma\epsilon B.$$
By (2.4 lemma) boundedness of the $C^1(\mathcal{M}_1; \rho)$ (be abounded derivative of a smooth map) agrees with boundedness in the $\Lambda(\mathcal{M}_1; \rho)$.
An inclusion $\Lambda(\mathcal{M}_1; \rho) \hookrightarrow C^\infty(\mathcal{M}_1; \rho)$ be a boundedness, hence, is a bornological embedding.

2- For $n \geq 2$
Assume is true for $(n-1)$-linear. For a n-linear map.
Let $B \subseteq \Lambda(\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_n; \rho)$ bounded, for all $B_i \subseteq \mathcal{M}_i$, $\sup_{g\epsilon B} \sup_{(x_1, x_2, \dots, x_n)\epsilon B_1 \times B_2 \times \dots \times B_n} q_\gamma(g(x)) < \infty$. By the joint continuous in ρ, implies a bounded in the $C^\infty(\mathcal{M}_1 \times \mathcal{M}_2 \times \dots \times \mathcal{M}_n; \rho)$.
Identification $\Lambda(\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_n; \rho) \cong \Lambda(\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_{n-1}; \Lambda(\mathcal{M}_n; \rho))$,
By inductive $\Lambda(\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_{n-1}; \Lambda(\mathcal{M}_n; \rho)) \hookrightarrow C^\infty(\mathcal{M}_1 \times \mathcal{M}_2 \times \dots \times \mathcal{M}_{n-1}; C^\infty(\mathcal{M}_n; \rho))$ be a bornological embedding. Now a bornology on the $C^\infty(\mathcal{M}_1 \times \mathcal{M}_2 \times \dots \times \mathcal{M}_n; \rho)$ be a equivalent to on $C^\infty(\mathcal{M}_1 \times \mathcal{M}_2 \times \dots \times \mathcal{M}_{n-1}; C^\infty(\mathcal{M}_n; \rho))$. (by 2.5 Corollary)
So, boundedness in $C^\infty(\mathcal{M}_1 \times \mathcal{M}_2 \times \dots \times \mathcal{M}_n; \rho)$ leads to boundedness in $\Lambda(\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_n; \rho)$.

Lemma 3.14: *Let B a system of bornological algebra with family of operation bounded, assume that $\mathcal{M}$ satisfy a uniform δ-bounbdedness principle for any point separating set $\delta \subseteq Mor(\mathcal{M}, \rho)$, then a following are equivalent:*

1- *Exist Ultrabornological topology on $\mathcal{M}$ weaker strictly than it is natural bornological topology.*

2- *Any point separating set $\delta \subseteq Mor(\mathcal{M}, \rho)$ satisfy the uniform δ-bounbdedness principle.*

Proof: $(1 \Longrightarrow 2)$ Assume the first point and let $\delta \subseteq \mathrm{Mor}(\mathcal{M}, \rho)$ point separating set of the structure preserving on the derivative and homomorphism. For any bounded subset $B \subseteq \mathcal{M}$, we introduce seminorm $q_B(x) = \sup_{s\epsilon\delta}\|s(x)\|_B$. a complete $\mathcal{M}$ towards q_B to a from complete bornologcal subsystem $\mathcal{M}_B$, and a topology τ_δ defined as a inductive limit $\lim_{\rightarrow} \mathcal{M}_B$, from a construction τ_δ be a ultrabornological.

By the hypothesis τ_δ cannot hold weaker than $\mathcal{M}$ is a natural bornological topology. Thus τ_δ Synchronized with $\mathcal{M}$ is bornological. Consequently, if a $\sup_{s\epsilon\delta}\|s(x)\|_B < \infty$ for any $x\epsilon\mathcal{M}$, then δ be a uniformly bounded on the any bounded subsets of $\mathcal{M}$, so hold the uniform δ-bounbdedness principle.

$(2 \Longrightarrow 1)$ by contradiction, assume exist an ultrabornological topology τ on $\mathcal{M}$ strict weakers a natural bornological topologies. but τ be an ultrabornologicals, it inductives limits $\lim_{\rightarrow} V_i$, such that V_i be a complete bornological subsystem (Frechet space or the Banach algebra).

By the strict weakness of a τ implies that exist a complete bornological subsystem V and the τ-continuous morphism $g: V \rightarrow (\mathcal{M}, \tau)$ that is not continuous in $\mathcal{M}$ is a bornological topology. Assume $\delta = \{\mathrm{Id}: g: \mathcal{M} \rightarrow (\mathcal{M}, \tau)\}$, so δ be point separating because τ Hausdorff. But the discontinuous of g in $\mathcal{M}$ is bornology topological directly infringes a uniform δ-bounbdedness principle. Meanwhile g bounded in τ but not in $\mathcal{M}$ is stronger bornology, this contradiction.

Theorem 3.15: *Assume $\mathcal{M}$ is a webbed bornological algebra equipped with a seminorm $\{q_\gamma\}_{\gamma\epsilon A}$, where the multiplication be a joint bounded, then $\mathcal{M}$ holds uniform δ-bounded principle for each points separation sets δ for the bounded algebra homomorphism $\delta \subseteq Hom(\mathcal{M}, \mathbb{C})$.*

Proof: From the [14] for the bornological algebra. Consider of the bornologicalness $\mathcal{M}^{\mathrm{born}}$ of the $\mathcal{M}$, is ensures of the properties of a webbed structure and bornological algebra. Since $\mathcal{M}^{\mathrm{born}}$ is a bornological this meaning any bounded set in $\mathcal{M}$ is absorbing by the neighborhood of 0 in the $\mathcal{M}^{\mathrm{born}}$, such that in $\mathcal{M}^{\mathrm{born}}$ any homomorphism bounded in δ be continuous in the $\mathcal{M}^{\mathrm{born}}$.

Next, defined an embedding $J: \mathcal{M}^{\mathrm{born}} \rightarrow \prod_{f\epsilon\delta} \mathbb{C}, J(a) = \big(f(a)\big)_{f\epsilon\delta}$ where $\prod_{f\epsilon\delta} \mathbb{C}$ is product topology generated by seminorm $\|.\|_f = |f(.)|$.

Now, let $\big(a_n, J(a_n)\big)\epsilon\mathcal{M}^{\mathrm{born}} \times \prod_{f\epsilon\delta} \mathbb{C}$, since that δ be separates points, then the $y = \big(f(a)\big)_{f\epsilon\delta}, J(a) = yJ$ has close graph this impels that J is continuous. [15] Such that the continuity impels that equicontinuity of the δ there exist q_γ and neighborhood $V \subseteq \mathcal{M}^{\mathrm{born}}$, where $\sup_{f\epsilon\delta}|f(a)| \leq$

$q_\gamma(a)\ \forall a \epsilon V$, then by the bounded set in $\mathcal{M}^{\mathrm{born}}$ are absorbing by V, δ be a uniformly bounded on the all bounded subsets on the $\mathcal{M}$. Hence, $\mathcal{M}$ verify the uniform δ boundedness principle.

Conclusion

New theorems concerning contraction mappings in bornological algebras over Fréchet spaces were proved in this paper. The results obtained are an extension of the study of boundedness structures and are useful to understand algebraic and analytic properties in bornological settings. In addition, the developed results were applied successfully to provide a proof of existence of solutions to some dynamic problems.

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