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New analytical results for Laguerre polynomials based on the Emad–Sara transform

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Abstract

An Emad–Sara transform be an effective mathematical technique for simplifying and solving differential equations arising in applied mathematics and engineering sciences. By converting differential operators into algebraic forms, this transform facilitates the analytical

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treatment of complex problems and improves solution efficiency. In this paper, the Emad–Sara transform is employed to analyze Laguerre polynomials of arbitrary order, and several of their transform properties are derived. Furthermore, the proposed approach is apply to solutions for ordinary differential equation involving Laguerre polynomials, demonstrating the practical utility of the transform in handling polynomial-based differential models. The obtained results confirm that the Emad–Sara transform provides a systematic and reliable framework for simplifying the solution process while preserving analytical accuracy. This study highlights the potential of the Emad–Sara transform as a valuable tool for solving polynomial-based differential equations and supports its applicability in mathematical and engineering applications.

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1. Introduction

The Emad–Sara integral transforms was successfully apply in various area for the engineering, natural sciences, and modern technology. Owing to its mathematical structure, the transform provides an efficient framework for solving linear differential equations, particularly by simplifying differential operators into manageable algebraic forms [1]. Linear ordinary differential equations with both constant and variable coefficients are commonly addressed using different integral transform techniques, which allow solutions to be obtained without explicitly determining their general forms [2–4]. In this context, Laguerre polynomials of order n are traditionally analyzed using classical transforms like as the Laplace's, Elzaki, Aboodh integral transforms [5–8]. This article, differential equations involving Laguerre polynomials are investigated using the Emad–Sara integral transform. The proposed approach demonstrates the effectiveness of this transform as an alternative analytical tool for handling polynomial-based differential equation.

2. Basic Concept

2.1 *Emad-Sara Integral Transforms*

Where a map $f(t), t \geq 0$ has the exponentials orders and be the piecewise continuous on any intervals, then Emad-Sara transforms [1] for the $f(t)$ be

$$ES\{f(H)\} = T(\pi) = \frac{1}{\pi^2}\int_{t=0}^{\infty} e^{-\pi t} f(t)dt\,.$$

Emad-Sara integral transform [1] of some basic function are given by:

$$ES\,\{t^n\} = \frac{n!}{\pi^{n+3}},\ \text{where } n = 0,1,2,3,\ldots$$

$$ES\,\{e^{\gamma t}\} = \frac{1}{\pi^2(\pi-\gamma)}, \text{where } \gamma \text{ is a constant.}$$

$$ES\,\{\sin(\gamma t)\} = \frac{\gamma}{\pi^2(\pi^2+\gamma^2)}\,.$$

$$\mathrm{ES}\{\cos(\gamma t)\} = \frac{\gamma}{\pi(\pi^2+\gamma^2)}.$$

$$\mathrm{ES}\{\sinh(\gamma t)\} = \frac{\gamma}{\pi^2(\pi^2-\gamma^2)}.$$

$$\mathrm{ES}\{\cosh(\gamma t)\} = \frac{1}{\pi(\pi^2-\gamma^2)}.$$

$$\mathrm{ES}\{\delta(t)\} = \frac{1}{\pi^2}.$$

2.2 *Inverse ES-Transforms*

An inverse Emad-Sara transforms [1] for the some functional, function be the

$$\mathrm{ES}^{-1}\{\pi^{n+3}\} = \frac{t^n}{n!} \ , n = 0,1,2,3, \dots$$

$$\mathrm{ES}^{-1}\left\{\frac{1}{\pi^2(\pi-\gamma)}\right\} = e^{\gamma t} \ , \gamma \text{ is a constant.}$$

$$\mathrm{ES}^{-1}\left\{\frac{1}{\pi^2(\pi^2+\gamma^2)}\right\} = \frac{1}{\gamma}\sin(\gamma t).$$

$$\mathrm{ES}^{-1}\left\{\frac{1}{\pi(\pi^2+\gamma^2)}\right\} = \cos(\gamma t).$$

$$\mathrm{ES}^{-1}\left\{\frac{1}{\pi^2(\pi^2-\gamma^2)}\right\} = \frac{1}{\gamma}\sin\mathrm{h}(\gamma t).$$

$$\mathrm{ES}^{-1}\left\{\frac{1}{\pi(\pi^2-\gamma^2)}\right\} = \cosh(\gamma t).$$

2.3 *Emad-Sara Transform of Derivatives*

Supper that $\mathrm{T}(\pi)$ be Emad-Sara integrals transforms $\mathrm{ES}\{\mu(\varkappa)\} = \mathrm{T}(\pi)$, so:

$$\mathrm{ES}\{\mu'(\varkappa)\} = \frac{-\mu(0)}{\pi^2} + \pi\mathrm{T}(\pi).$$

$$\mathrm{ES}\{\mu''(\varkappa)\} = \frac{-\mu'(0)}{\pi^2} + \pi\,\mathrm{ES}\{\mu'(\varkappa)\}.$$

In general:

$$\mathrm{ES}\{\mu^n(\varkappa)\} = \frac{-\mu^{n+1}(0)}{\pi^2} + \pi\,\mathrm{ES}\{\mu^{n-1}(\varkappa)\}.$$

2.4 *Laguerre Polynomial*

The Laguerre polynomial is defined by:

$$\mathrm{L}_n(u) = \frac{e^u d^n}{n!\ du^n}(e^{-u}u^n).$$

3. Result

This section presents the analytical results derived using the Emad–Sara transform. By employing its formal definition, several properties and expressions are obtained, which are essential for the subsequent applications.

$$\mathrm{ES}\{f(H)\} = \mathrm{T}(\pi) = \frac{1}{\pi^2}\int_{\varkappa=0}^{\infty} e^{-\pi\varkappa} f(\varkappa)dt\ .$$

Therefore,

$$\mathrm{ES}\{L_n(\varkappa)\} = \frac{1}{\pi^2}\int_{\varkappa=0}^{\infty} e^{-\pi\varkappa}\left[\frac{e^{\varkappa}d^n}{n!\ d\varkappa^n}(e^{-\varkappa}\varkappa^n)\right]d\varkappa,$$

$$=\frac{1}{n!\,\pi^2}\int_{\varkappa=0}^{\infty} e^{-(\pi-1)\varkappa}\left[\frac{d^n}{d\varkappa^n}(e^{-\varkappa}\varkappa^n)\right]d\varkappa,$$

$$=\frac{1}{n!\,\pi^2}\left[(\pi-1)\int_{\varkappa=0}^{\infty} e^{-(\pi-1)\varkappa}\left[\frac{d^{n-1}}{d\varkappa^{n-1}}(e^{-\varkappa}\varkappa^n)\right]d\varkappa\right],$$

Integrating again,

$$=\frac{(\pi-1)^2}{n!\,\pi^2}\int_{\varkappa=0}^{\infty} e^{-(\pi-1)\varkappa}\left[\frac{d^n}{d\varkappa^{n-2}}(d^{n-2})\right]d\varkappa,$$

Also, integrating,

$$\frac{(\pi-1)^n}{n!\,\pi^2}\int_{\varkappa=0}^{\infty} e^{-(\pi-1)\varkappa}\ (e^{-\varkappa}\,\varkappa^n)d\varkappa,$$

$$=\frac{(\pi-1)^n}{n!\,\pi^2}\int_{\varkappa=0}^{\infty} e^{-\pi\varkappa}\varkappa^n\ d\varkappa = \frac{(\pi-1)^n}{n!}\ \mathrm{ES}\{\varkappa^n\}.$$

But, by the definition for the Emad-Sara integrals transforms,

$$\mathrm{ES}\{f(\varkappa)\} = \frac{1}{\pi^2}\int_{\varkappa=0}^{\infty} e^{-\pi\varkappa} f(\varkappa)d\varkappa = T(\pi).$$

Hence,

$$\frac{(\pi-1)^n}{n!}\ \mathrm{ES}\{\varkappa^n\} = \frac{(\pi-1)^n}{n!}\cdot\frac{n!}{\pi^{n+3}}.$$

Hence,

$$\mathrm{ES}\{L_n(\varkappa)\} = \frac{(\pi-1)^n}{\pi^{n+3}}.$$

4. Applications

Solving an Ordinary Differential Equation Using the Emad–Sara Transform.

Application 4.1: Solve the differential equation $\frac{d^2y}{dt^2} + 4y = L_2(t)$, & $y(0) = 0$, $y'(0) = 1$, taking Emad-Sara transform on both sides, we have:

$$\mathrm{ES}\{y''\} + 4\mathrm{ES}\{y\} = \mathrm{ES}\{L_2(t)\}.$$

Because Leguerre polynomial of 2^{nd}order is:

$$L_2\{t\} = \frac{1}{2}(2 - 4t + t^2).$$

Now,

$$\frac{-y'(0)}{\pi^2} - \frac{y(0)}{\pi} + \pi^2 T(\pi) + 4T(\pi) = \frac{(\pi-1)^2}{\pi^5},$$

$$\frac{-1}{\pi^2} + \pi^2 T(\pi) + 4T(\pi) = \frac{(\pi-1)^2}{\pi^5},$$

$$(\pi^2 + 4)T(\pi) = \frac{(\pi-1)^2}{\pi^5} + \frac{1}{\pi^2},$$

$$T(\pi) = \frac{(\pi-1)^2}{\pi^5(\pi^2+4)} + \frac{1}{\pi^2(\pi^2+4)},$$

$$\mathrm{T}(\pi) = \frac{\pi^2-2\pi+1+\pi^2}{\pi^5(\pi^2+4)} = \frac{\pi^3+\pi^2-2\pi+1}{\pi^5(\pi^2+4)},$$

$$\mathrm{T}(\pi) = \frac{16\,\alpha^3+16\pi^2-32\pi+16}{16\pi^5(\pi^2+4)},$$

$$\mathrm{T}(\pi) = \frac{3}{16\pi^3} + \frac{1}{4\pi^5} - \frac{1}{2\pi^4} - \frac{3}{16\pi(\pi^2+4)} + \frac{24}{16\pi^2(\pi^2+4)}.$$

Applying inverse Emad-Sara integral transform:

$$y(t) = \frac{3}{16} + \frac{t^2}{8} - \frac{t}{2} - \frac{3}{16}\cos(2t) + \frac{3}{4}\sin(2t).$$

Application 4.2: Solve the differential equation $\frac{d^2y}{dt^2} + \frac{dy}{dt} = L_1(t)$, the initials condition $y(0) = 0$, $y'(0) = 1$, take Emad –Sara integrals transforms on both sides, we get:

$$\mathrm{ES}\{y''\} + 4\mathrm{ES}\{y'\} = \mathrm{ES}\{L_1(t)\}.$$

Because Leguerre polynomial of 1^{st} order is:

$$L_1\{t\} = 1 - t$$

$$\frac{-y'(0)}{\pi^2} - \frac{y(0)}{\pi} + \pi^2\mathrm{T}(\pi) - \frac{y(0)}{\pi^2} + \pi\mathrm{T}(\pi) = \frac{(\pi-1)}{\pi^4}.$$

Applying initial conditions, we get:

$$\mathrm{T}(\pi) = \frac{2}{\pi^4} - \frac{1}{\pi^3} + \frac{1}{2\pi^4} - \frac{3}{\pi^2(\pi+1)} - \frac{1}{\pi^5}.$$

Applying inverse Emad-Sara transform, so

$$y(t) = 2t - 1 + e^{-t} - \frac{t^2}{2}.$$

Application 4.3: Solve the differential equation $\frac{d^2y}{dt^2} + b^2\frac{dy}{dt} = L_1(t)$, with initials condition $y(0) = y'(0) = 0$, taking Emad-Sara integral transform on both sides, we get:

$$\mathrm{ES}\{y''\} + b^2\mathrm{ES}\{y'\} = \mathrm{ES}\{L_1(t)\}.$$

Because Leguerre polynomial of 1^{st} order is:

$$L_1\{t\} = \{1 - t\}.$$

$$\frac{-y'(0)}{\pi^2} - \frac{y(0)}{\pi} + \pi^2\mathrm{T}(\pi) + [\,-b^2\frac{y(0)}{\pi^2} + b^2\pi\mathrm{T}(\pi) = \frac{(\pi-1)}{\pi^4}.$$

Apply initials condition, we have:

$$\mathrm{T}(\alpha) = -\left(\frac{1}{\pi^3\,b^2}\right) - \left(\frac{1}{\pi^3\,b^4}\right) + \left(\frac{1}{\pi^4\,b^2}\right) + \left(\frac{1}{\pi^4\,b^4}\right) - \left(\frac{1}{\pi^5\,b^2}\right) + \frac{1}{\pi^2\,b^4\,(\pi+b^2)} + \frac{1}{\pi^2\,b^6\,(\pi+b^2)}.$$

Apply inverse Emad –Sara transforms:

$$y(t) = -\frac{1}{b^6} - \frac{1}{b^5} + \frac{t}{b^2} + \frac{t}{b^4} - \frac{t^2}{2b^2} + \frac{1}{b^4}e^{-b^2t} + \frac{e^{-b^2t}}{b^6},$$

Or

$$y(t) = -\left(\frac{1}{b^6} + \frac{1}{b^4}\right) + \left(\frac{1}{b^2} + \frac{1}{b^4}\right)t - \frac{t^2}{2b^2} + \left(\frac{1}{b^4} + \frac{1}{b^6}\right)e^{-b^2 t}.$$

Or

$$y(t) = \left(\frac{1}{b^6} + \frac{1}{b^4}\right)\left(e^{-b^2 t} - 1\right) + \left(\frac{1}{b^6} + \frac{1}{b^4}\right)t - \frac{t^2}{2b^2}.$$

5. Conclusion

In this paper, the Emad–Sara integral transform of Laguerre polynomials of order n has been derived and analyzed. The study demonstrates the effectiveness of the Emad–Sara transforms as a systematic tool for handling polynomial-based mathematical structures. An applicability for Emad–Sara transforms in solve ordinary linear differential equation involving Laguerre polynomials has been established. The obtained results confirm that the proposed transform-based approach simplifies the solution procedure and provides accurate analytical solutions with reduced computational complexity. These findings highlight the potential of the Emad–Sara transform as a valuable tool for addressing a broad class of differential equations encountered in applied mathematics and engineering. Future work may extend this approach to partial differential equations and other classes of special functions.

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