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Negative theorems for co-positive L_p^1, $0< p < 1$ interpolating approximation

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Abstract

This work provides a rigorous mathematical analysis of the relationship between Hermite interpolation and function approximation, such that at a certain point $A_q \in \mathbb{A}_q$, where $\mathbb{A}_q$, is the set of all collections $A_q := \{\alpha_i\}_{i=1}^{q}$ of points α_i such that $-1 \le \alpha_q < \cdots < \alpha_1 \le 1$, $q \in \mathbb{N}$. Approximation polynomials (if possible) must interpolate the function and its derivatives, where at such points the function value matches not only the polynomial but also its derivatives. This kind of interpolation is known as Hermite interpolation also investigates whether the interpolation conditions can be extended to derivatives of the functionf. The accuracy of approximation does not just rely on the smoothness of *f*, but also the location points of interpolation, such that the points in A_q lie in the interior of the interval $I = [-1, 1]$ or on the boundary of interval *I*, and so depend on the odd or even for highest derivatives for the f. With the points from $Y_s \in \mathbb{Y}_s$, where $\mathbb{Y}_s$ a sets for every collection $Y_s := \{y_i\}_{i=1}^{s}$ for the point y_i, such that $-1 < y_s < \cdots < y_1 < 1$. The interpolation condition of derivatives is more complex than the usual approximation. In previous papers, we examined the interpolation condition with positive and co-positive approximation. We do this condition under some constrains

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here we shall introduce a negative theorem for piecewise positive approximation with an interpolation condition for functions in L_p^1 Space, 0 < p< 1.

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1. Introduction

In this paper, we need the following notations and definitions.

On $I := [-1, 1]$, for $0 < p < 1, L_p(I)$, be the function space on I such that

$$L_p := L_p(I) := \left\{ f: I \to \mathbb{R}, \|f\|_P = (\int_I \|f\|^p)^{1/p} < \infty \right\}. \text{ [1], [2]}$$

And let $L_p^j := L_p^j(I)$ be the function space, those that are $j -$ fold integrals of $L_p(I)$ functions [8].

Let $s \in \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers. And $\mathbb{Y}_s$ a sets for each class $Y_s := \{y_i\}_{i=1}^s$ for point y_i, such that $-1 < y_i < \cdots < y_1 < 1$.

So defined $y_0 := 1$ & $y_{s+1} := -1$ & denoted as $\Delta^{(0)}(Y_s)$ is function set f which are non-negative near 1 and have $0 \le s \le \infty$ sign changes at the points in Y_s for $\in [y_r, y_{r+1}], r = 0, \cdots, s$, such that $(-1)^{s-r} f(x) \ge 0$.

Particularly, $\Delta^{(0)} := \Delta^{(0)}(Y_s)$ is the set of functions that are non-negative on I.

The function f is said to be co-positive if $f \in \Delta^{(0)}(Y_s)$. For $f \in L_p(I)$,

A degrees for the unconstrained approximations for f be

$$E_n(f)_p :=\inf\{\|f - P_n\|_p \mid P_n \in \mathbb{P}_n\}.$$

Co-positive approximation is the functions approximation $f \in \Delta^{(0)}(Y_s)$ by the polynomials that are co-positive with f, denote the co-positive polynomial approximation degree of f is

$$E_n^{(0)}(f, Y_s)_p :=\inf\{\|f - P_n\|_p \mid P_n \in \mathbb{P}_n \cap \Delta^{(0)}(Y_s)\}. \text{ [3]}$$

In particular, the positive polynomial approximation degree of f is

$$E_n^{(0)}(f, Y_0)_p :=\inf\{\|f - P_n\|_p \mid P_n \in \Delta^{(0)}(Y_0) \cap \mathbb{P}_n\}. \text{ [3]}$$

Where $\mathbb{P}_n$ be space for the all polynomial that have degree $\le n$.

if $\mathbb{A}_q$, a sets for the all collection $A_q := \{\alpha_i\}_{i=1}^q$ for points α_i such that $-1 \le \alpha_q < \cdots < \alpha_1 \le 1, q \in \mathbb{N}$. [4]

We denote by $\mathcal{J}_p^{(r)}(f, A_q)$ is the functions set satisfying the interpolation constraint at points A_q, such that

$\mathcal{J}_p^{(r)}(f, A_q) := \{ g \in L_p^{(r)}(I) \mid f^{(i)}(\alpha) = g^{(i)}(\alpha)\}$, for all $\alpha \in A_q$ and $0 \le i \le r\}$.

So the notation $\mathcal{J}_p(f, A_q) := \mathcal{J}_p^{(0)}(f, A_q)$, if $r = 0$. And $\mathbb{R}$ is the set of real numbers. And $\|\cdot\|_p$ is L_p – norm on I such that

$$\|f\|_P = (\int_I \|f\|^p)^{1/p} < \infty. \text{ [1]}$$

In our work we need the following property:

Let g and f function in $L_p(I)$, such that $g \neq 0$, then

$$\|g+f\|_p \leq \left(\|g\|_p^p + \|f\|_p^p\right)^{1/p} \leq 2^{\frac{1}{p}-1}\left(\|g\|_p + \|f\|_p\right). \text{[1] (1)}$$

And

$\tau_t(f,.;I)_p := \tau_t(f,.)_p$ is $\tau-$modulus of smoothness of f in $L_p(I)$, such that

$$\tau_t(f,\delta)_p := \tau_t(f,\delta;I)_{p(I)} = \left\|\omega_t(f,x;\delta)_p\right\|_{L_p(I)}, \delta \geq 0, \text{[1]}$$

where

$$\omega_t(f,x;\delta)_p = \sup\{|\Delta_h^t(f,y)|: y + \tfrac{th}{2} \in [x - \tfrac{t\delta}{2}, x + \tfrac{t\delta}{2}] \cap I\}, \text{[1]}$$

is the local modulus of smoothness of f of order t.

Co-positive approximation, where the approximation polynomials are required to maintain the same sign pattern as the original function, represents an important branch of shape-preserving approximation theory.

Previously, some papers introduced the study of the constrained positive and co-positive approximation. They proved direct theorems for functions in $L_{p[a,b]}$, and $C[a,b]$, using polynomials of degree $\leq n, n \in \mathbb{N}$.

Then, for strength, we need to prove that the direct theorems are in terms of ω_1 not ω_2 or ω_i, $i \geq 3$, The authors [5] in introduced a negative theorem to strengthen the above results.

Then in [6], [7] he authors proved direct theorems for interpolating polynomials. The above works lead us to answer the question: Can we prove direct theorems for positive and co-positive interpolating approximation without any conditions, or what conditions are needed for this? This leads us to introduce [8], and find conditions to have interpolating positive and co-positive approximation for functions in $L_{p[0,1]}$, $0 < p < 1$.

We study positive and co-positive approximations under an interpolation constraint. Also, we prove a direct theorem for positive intertwining. Here we strengthen our result in [8], and prove a negative theorem for co-positive approximation for functions in the $L_{p,}^1 0 < p < 1$, space.

2. The Main Result.

Theorem 2.1: *For any $n_0 \in \mathbb{N}, D > 1,$& for positive real sequences $(X_n)_{n\in\mathbb{N}}$ which is converge to zero, and $\beta \in [0,1)$, there exists $f = f_n \in L_p^1(I), n \in \mathbb{N}$, and $n \geq n_0$, s.t $f \equiv 0$ on $[-1, -1/2]$, $f \geq 0$ on I, and for any polynomial $P_n \in \mathbb{P}_n$, which is non-negative on $\left(\beta - \frac{1}{n}, \beta + \frac{1}{n}\right)$ and satisfies $f(\beta) = P_n(\beta)$, so*

$$\|f - P_n\|_p > DX_n n^{-2/3}\, \tau_4(f', n^{-1})_p.$$

Proof: if $n \geq C(1-\beta)^{-1}$, and $\varepsilon = n^{-4/3}$, where C is positive constant greater than 1.

And define

$$h(x) = \begin{cases} 0, \text{if } \beta \leq x \leq \beta + \varepsilon, \\ S(x), \text{if } x < \beta, \\ S(x) + \varepsilon^4, \text{if } x \geq \beta + \varepsilon, \end{cases}$$

where $S(x) := 3(x-\beta)^4 - 4\varepsilon(x-\beta)^3$.

See that $h \in L^1_{p,}$so

$$h'(x) = \begin{cases} S'(x\,), \text{if } x < \beta \text{ or } x > \beta + \varepsilon \\ 0, \text{if } \beta \leq x \leq \beta + \varepsilon, \end{cases}$$

with $S'(x\,) = 12(x-\beta)^3 - 12\varepsilon(x-\beta)^2 = 12(x-\beta)^2(x-\beta-\varepsilon)$.

And, define

$$f(x) = \begin{cases} 0, x \in [-1, {}^{-1}/_{2}\,], \\ h(x)\widetilde{M}\,(x; [{}^{-1}/_{2}, {}^{-1}/_{4}]), x \in ({}^{-1}/_{2}, {}^{-1}/_{4}), \\ h(x), x \in [{}^{-1}/_{4}, 1], \end{cases}$$

where $\widetilde{M}\,(x; [a,b])$, is a function s.t

$$\widetilde{M}(x) := \widetilde{M}\,(x; [a,b]) := \frac{M\,(x; [a,b])}{M\,(1; [a,b])},$$

Where $M\,(x; [a,b]) := \int_{-1}^{x} h(u; (a+b)/2, (a-b)/2)\,du$,

$$h(x; x_0, d) = \begin{cases} \exp(d^2/((x-x_0)^2 - d^2), x \in (x_0 - d, x_0 + d), \\ 0, \text{otherwise}, \end{cases}$$

and $[a,b] \subset [-1,1]$. If $x \leq a$, then $\widetilde{M}(x) = 0$, and $h(x; x_0, d) = 0$, and $\widetilde{M}(x) := \frac{M\,(1;[a,b])}{M\,(1;[a,b])} = 1$, If $x \geq b$, and If $a \leq x \leq b$, then $0 \leq \widetilde{M}(x) \leq 1$.

Now let us estimate $\|f' - S'\|_p$ on $[\beta, \beta+\varepsilon]$:

$$\|f' - S'\|_{p_{[-1/_{4},1]}} = \|S'\|_{p_{[\beta,\beta+\varepsilon]}} := \left(\int_{\beta}^{\beta+\varepsilon} |(S'(x))|^p\,dx\right)^{1/p}$$

$$= \left(\int_{\beta}^{\beta+\varepsilon} |12(x-\beta)^2(x-\beta-\varepsilon)|^p\,dx\right)^{\frac{1}{p}},$$

$= (\int_{\beta}^{\beta+\varepsilon} |12(x^2 - 2x\beta + \beta^2)(x-\beta-\varepsilon)|^p\,dx)^{\frac{1}{p}}$

$= (\int_{\beta}^{\beta+\varepsilon} |12(x^3 - \beta x^2 - \varepsilon x^2 - 2\beta x^2 + 2\beta^2 x + 2\beta\varepsilon x + \beta^2 x - \beta^3 - \beta^2\varepsilon)|^p\,dx)^{\frac{1}{p}}$

since $\beta, \varepsilon\; > 0, \beta \in [0,1], \varepsilon \in \left(0, \frac{\beta}{2}\right)$, Assume $\beta\varepsilon = C$,

we get

$\|f' - S'\|_{p_{[-1/_{4},1]}} \leq 12\left(\int_{\beta}^{\beta+\varepsilon}(x^{3p} + \beta^p x^{2p} + \varepsilon^p x^{2p} + 2^p\beta^p x^{2p} + 2^p\beta^{2p}x^p + \right.$

$\left. 2^p\varepsilon^p\beta^p x^p + \beta^{2p}x^p + \beta^{3p} + \varepsilon^p\beta^{2p})dx\right)^{\frac{1}{p}}$

$= 12\left(\frac{x^{3p+1}}{3p+1} + \beta^p \frac{x^{2p+1}}{2p+1} + \varepsilon^p \frac{x^{2p+1}}{2p+1} + 2^p\beta^p \frac{x^{2p+1}}{2p+1} + 2^p\beta^{2p} \frac{x^{p+1}}{p+1} + 2^p\beta^p\varepsilon^p \frac{x^{p+1}}{p+1} + \beta^{2p} \frac{x^{p+1}}{p+1} + \beta^{3p}x + \beta^{2p}\varepsilon^p x|_{\beta}^{\beta+\varepsilon}\right)^{\frac{1}{p}}$

$= 12\left(\frac{(\beta+\varepsilon)^{3p+1}}{3p+1} + \beta^p \frac{(\beta+\varepsilon)^{2p+1}}{2p+1} + \varepsilon^p \frac{(\beta+\varepsilon)^{2p+1}}{2p+1} + 2^p\beta^p \frac{(\beta+\varepsilon)^{2p+1}}{2p+1}\right.$

$+2^p\beta^{2p} \frac{(\beta+\varepsilon)^{p+1}}{p+1} + 2^p\beta^p\varepsilon^p \frac{(\beta+\varepsilon)^{p+1}}{p+1} + \beta^{2p} \frac{(\beta+\varepsilon)^{p+1}}{p+1} + \beta^{3p}(\beta+\varepsilon) +$

$\beta^{2p}\varepsilon^p(\beta+\varepsilon) - \frac{(\beta)^{3p+1}}{3p+1} - \frac{(\beta)^{3p+1}}{2p+1} - \varepsilon^p \frac{(\beta)^{2p+1}}{2p+1} - 2^p \frac{(\beta)^{3p+1}}{2p+1} - 2^p \frac{(\beta)^{3p+1}}{p+1} -$

$\left. 2^p\varepsilon^p \frac{(\beta)^{2p+1}}{p+1} - \frac{(\beta)^{3p+1}}{p+1} - \beta^{3p+1} - \beta^{2p+1}\varepsilon^p|_{\beta}^{\beta+\varepsilon}\right)^{\frac{1}{p}}$

$$\begin{aligned}\le 12\big((\beta+\varepsilon)^{3p+1} &+ (\beta+\varepsilon)^{2p+1} + \varepsilon^p(\beta+\varepsilon)^{2p+1} + 2^p(\beta+\varepsilon)^{2p+1} \\ &+ 2^p(\beta+\varepsilon)^{p+1} + 2^p\varepsilon^p(\beta+\varepsilon)^{p+1} + (\beta+\varepsilon)^{p+1} + (\beta+\varepsilon) \\ &+ \varepsilon^p(\beta+\varepsilon)\big)^{\frac{1}{p}}\end{aligned}$$

$$\le 12\big((1+1+2^p+2^p+2^p+1+1)(\beta+\varepsilon) + \varepsilon^p(2^p+1+1)(\beta+\varepsilon)\big)^{\frac{1}{p}},$$

$$\le 12(10\,\varepsilon + 4\,\varepsilon)^{\frac{1}{p}} = 12(10^{\frac{1}{p}}\varepsilon^{\frac{1}{p}}) + 12(4^{\frac{1}{p}}\varepsilon^{\frac{1}{p}}) \le (48)^{\frac{1}{p}}\varepsilon^{\frac{1}{p}} \le (48\,)^{\frac{1}{p}}\varepsilon.$$

Now estimate $\tau_4(f', n^{-1})_{p,}$
s.t

$$\tau_4(f', n^{-1})_{p[-1,\frac{-1}{8}]} \le \frac{C(p)}{n^4}\left\|f^{(5)}\right\|_{p\left[-1,\frac{-1}{8}\right]} \le \frac{C(p)}{n^4},$$

when $C(p)$ be constant depend on p & it positive, so follows froms the behavior of $S'(x)$ on $\left[{}^{-1}/_{4}, 1\right]$, and the smoothness of f on $\left[-1, \frac{-1}{8}\right]$, then $\left\|f^{(5)}\right\|_{p\left[-1,\frac{-1}{8}\right]} \le C(p)$.

Also we have

$$\tau_4(f', n^{-1})_{p[{}^{-1}/_{4},1]} = \tau_4(f' - S', n^{-1})_{p[{}^{-1}/_{4},1]} \le C(p)\|f' - S'\|_p \le C(p)(48\,)^{\frac{1}{p}}\varepsilon.$$

Hence

$$\tau_4(f', n^{-1})_p = \max\left\{C(p)\frac{1}{n^4}, C(p)\varepsilon\right\} = C(p)\frac{1}{n^4}. \quad (2)$$

Then estimate $\|h(x) - S(x)\|_{p[{}^{-1}/_{4},1],}$ such that

$$\|h - S\|_{p[{}^{-1}/_{4},1]} = \|S\|_{p[\beta,\beta+\varepsilon]} = \left(\int_{\beta}^{\beta+\varepsilon}\left|(S(x))\right|^p dx\right)^{\frac{1}{p}}$$

$$= \left(\int_{\beta}^{\beta+\varepsilon}|3(x-\beta)^4 - 4\varepsilon(x-\beta)^3|^p\,dx\right)^{\frac{1}{p}}$$

since $\beta, \varepsilon > 0, \beta \in [0,1], \varepsilon \in \left(0, \frac{\beta}{2}\right)$, Assume $\beta\varepsilon = C$,
we get

$$\|h-S\|_{p[^{-1}/_4,1]} \le \left(\int_\beta^{\beta+\varepsilon}(3^P(x-\beta)^{4P}+4^P\varepsilon^P(x-\beta)^{3P})\,dx\right)^{\frac{1}{p}}$$
$$=\left(3^P\int_\beta^{\beta+\varepsilon}(x-\beta)^{4P}\,dx+4^P\varepsilon^P\int_\beta^{\beta+\varepsilon}(x-\beta)^{3P}\,dx\right)^{\frac{1}{p}}$$
$$=\left(3^P\frac{(x-\beta)^{4P+1}}{4p+1}\Big|_\beta^{\beta+\varepsilon}+4^P\varepsilon^P\frac{(x-\beta)^{3P+1}}{3p+1}\Big|_\beta^{\beta+\varepsilon}\right)^{\frac{1}{p}}$$
$$=\left(3^P\frac{(\varepsilon)^{4P+1}}{4p+1}-0+4^P\varepsilon^P\frac{(\varepsilon)^{3P+1}}{3p+1}-0\right)^{\frac{1}{p}}$$
$$=\left(3^P\frac{(\varepsilon)^{4P+1}}{4p+1}-0+4^P\frac{(\varepsilon)^{4P+1}}{3p+1}-0\Big|_\beta^{\beta+\varepsilon}\right)^{\frac{1}{p}}$$
$$\le(3^P\varepsilon+4^P\varepsilon)^{\frac{1}{p}}\le(3\varepsilon+4\varepsilon)^{\frac{1}{p}}=(7\,\varepsilon)^{\frac{1}{p}}=\;7^{\frac{1}{P}}\varepsilon\cdot \tag{3}$$

Suppose there exist a polynomial $P_n\in\mathbb{P}_n$, is non-negative on $\left(\beta-\frac{1}{n},\beta+\frac{1}{n}\right)$, satisfy $f(\beta)=P_n(\beta)=0$, and,

$$\|P_n(x)-f(x)\|_p\le DX_n n^{-2/3}\,\tau_4(f',n^{-1})_p\,. \tag{4}$$

By using (1), (2) and by the estimates (3) and (4)
we get:

$$\|P_n(x)-S(x)\|_{p[^{-1}/_4,1]}$$
$$\le 2^{\frac{1}{P}-1}(\|P_n(x)-f(x)\|_{p[^{-1}/_4,1]}+\|h(x)-S(x)\|_{p[^{-1}/_4,1]})$$
$$\le 2^{\frac{1}{P}-1}\left(\,DX_n n^{-2/3}\,\tau_4(f',n^{-1})_p\;+7^{\frac{1}{P}}\varepsilon\right)$$
$$\le 2^{\frac{1}{P}-1}\left(\,DX_n n^{-2/3}C(p)\,n^{-4}+7^{\frac{1}{P}}\varepsilon\right)$$
$$\le 2^{\frac{1}{P}-1}\left(\,DX_n n^{-14/3}C(p)\;+7^{\frac{1}{P}}\varepsilon\right). \tag{5}$$

By Markovs - Bernstein inequalities [9] on the $\left[^{-1}/_4,1\right]$, and by (5)
we get

$$\|P_n''-S''\|_{p[^{-1}/_4,1]}\le C(1-\beta)^{-1}\|P_n-S\|_{p[^{-1}/_4,1],}$$
$$\le C(1-\beta)^{-1}2^{\frac{1}{P}-1}\left(\,DX_n n^{-14/3}C(p)\;+7^{\frac{1}{P}}\varepsilon\right),$$

where $P_n^{(i)}(\beta)=S^{(i)}(\beta)=0, i=0{,}1$ and $P_n(\beta+\varepsilon)\ge 0$,
we have

$$1\le\varepsilon^{-4}[P_n(\beta-\varepsilon)-S(\beta+\varepsilon)]=\varepsilon^{-4}\int_\beta^{\beta+\varepsilon}(\beta+\varepsilon-t)[P_n''(t)-S''(t)]\,dt,$$
$$\le\varepsilon^{-4}\int_\beta^{\beta+\varepsilon}\|P_n''-S''\|_p,$$
$$=\varepsilon^{-4}(\beta+\varepsilon-\beta)\|P_n''-S''\|_p,$$

$$\leq C\varepsilon^{-3}(1-\beta)^{-1}2^{\frac{1}{P}-1}\left(DX_n n^{-14/3}\,C(p)\ +7^{\frac{1}{P}}\varepsilon\right),$$

$$\leq C\,n^4\,(1-\beta)^{-1}2^{\frac{1}{P}-1}\left(DX_n n^{-14/3}\,C(p)\ +7^{\frac{1}{P}}\varepsilon\right),$$

by taking n to be sufficiently large we arrive at a contradiction.

3. Conclusions

Interpolation constraints add additional complexity, especially when combined with shape-preserving requirements. We extend the study of polynomial approximation with interpolation constraints-particularly positive and co-positive approximations to include the framework of L_p, spaces, $0 < p < 1$.

The quasi-normed and locally non-convex structure of these spaces poses significant analytical challenges, as many of the traditional techniques available in the normed setting $p \geq 1$, are no longer applicable. Direct theorem for interpolating positive and co-positive approximation on L_P^1, quasi normed spaces is impossible in general under certain condition we make this possible then we strength our result by a negative theorem.

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