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Homotopy perturbation analysis of descriptor nonlinear MFSS with ALE distribution extension

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Abstract

The use of fractional integro-differential models has gained prominence in describing complex dynamical systems with memory and hereditary properties. A set of nonlinear descriptor systems with multi-fractional integro-differential operators is studied. The mathematical model uses the Caputo fractional derivative and Riemann-Liouville fractional integral to model the underlying dynamics. A Homotopy Perturbation Method (HPM)-based analytical framework is formulated to obtain approximate solutions to the proposed system. A number of illustrative examples are given to demonstrate the method's efficiency and convergence behavior and to attest to its ability to serve as an efficient analytical tool for studying nonlinear multi-fractional dynamical models arising in applied mathematics and other fields. Secondly, the paper proposes a new statistical distribution that is based on the sine transformation framework. The proposed model extends the Alpha Leg-Exponential (ALE) distribution to construct a new composite distribution, called the Sine-Alpha Leg-

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Exponential Distribution (SALED). Significant distributional functions are calculated and discussed, such as probability density function, cumulative distribution function, survival function, and hazard rate function. Parameter estimation is performed via simulation experiments to assess the model's flexibility and performance. The results show that the SALED distribution offers a competitive alternative to available sine-based distributions with a similar parameter structure.

Subject Classification: *34A08, 34A34.*

Keywords: *Homotropy perturbation method, Nonlinear systems, Integro-differential, Riemann–Liouville, Alpha leg-exponential, Parameter, Survival function, Hazard function, Sine family.*

1. Introduction

Descriptor systems, also known as differential algebraic systems, are a generalization of dynamical models that can express processes involving a mixture of differential and algebraic relationships [1]. Fractional differential equations have received growing interest in recent years due to their effectiveness in modeling complex phenomena involving memory and nonlocal effects. These equations occur in a wide variety of mathematical and physical models, such as fractional convection-dispersion processes, time-fractional diffusion models, generalized Burgers-type fluids, KdV-type nonlinear systems, the Whitham-Broer-Kaup equations, and fractional heat, wave, and backward Kolmogorov formulations [2-5]. To address such nonlinear fractional models, various semi-analytical and analytical methods have been developed. One such method is the Homotopy Perturbation Method (HPM), which has become an effective means of finding approximate solutions to nonlinear differential and integro-differential equations. Recent work has generalized homotopy-based frameworks and associated variational methods to classes of more general fractional systems, showing their applicability to both theoretical study and modeling problems [6-9]. With these advances in mind, the given work is devoted to the study of nonlinear descriptor equations describing multi-fractional integro-differential operators [10-13]. The proposed framework is based on the Homotopy Perturbation Method to construct analytical approximations of the system under consideration. To be more precise, the paper is structured as follows: Section one provides the context of the research, Section two describes the mathematical background of fractional descriptor systems, and Section three presents the homotopy perturbation formulation applied to the solution of these systems.

2. Frac Derivatives and Frac Intro.

Definition 2.1: The Frac Intro in the sense of Riemann–Liouville with order□α> 0 is defined as: $I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} f(x) dx$, $\alpha > 0, x > 0.$

Definition 2.2: The survival function is the complementary cumulative distribution function of the lifetime. Sometimes complementary cumulative distribution functions are called survival functions (t). The term survival function is usually used in medical and life studies and its mathematical formula is: [14] $S(t) = \int_t^{t\,max} f(u)du$, $S(t) = 1 - F(t)$ In other words, the survival function is inversely proportional to time. $\lim_{n\to 0} S(t) = 1$, $\lim_{n\to\infty} S(t) = 0$.

Definition 2.3: Based on its total age or lifespan, the hazard rate measures an object's propensity to malfunction or die. It falls under the larger field of survival analysis, which is a group of statistical methods used to forecast how long it will take for a specific event to occur, like the failure or demise of an engineering system or component. The hazard rate, often known as the failure rate, is represented by h(t) and is computed from [14, 15]

$$h(t) = \frac{f(t)}{1-F(t)} = \frac{f(t)}{S(t)}.$$

3. Parameters Estimation of the SALED (Method of Solution)

In this section, the SALED model parameters are estimated using the MLE method. Let T_1, T_2, …, T_n represent a random sample of size n of the SALED. Next, the probability distribution of SALED results from replacing the PDF. (10) into Equation (11)

$$L(T_1, T_2, \dots, T_n, \alpha, \lambda) = f(t_1, \lambda, \alpha).f(t_2, \lambda, \alpha).\cdots.f(t_n, \lambda, \alpha)\,, L = \prod_{i=1}^{n} f(t_i, \lambda, \alpha)\ (11)$$

$L = \prod_{i=1}^{n} f(t_i, \lambda, \alpha) = \prod_{i=1}^{n}[\frac{\pi}{2} g(t) \cos\left(\frac{\pi}{2} G(t)\right)]$ Where $g(t) = \dfrac{\alpha e^{\frac{-t}{\lambda}}}{\lambda ln(\alpha+1)(\alpha-\alpha e^{\frac{-t}{\lambda}}+1)}$,

$G(t) = \dfrac{ln(\alpha-\alpha e^{\frac{-x}{\lambda}}+1)}{ln(\alpha+1)}$ The function of log-likelihood can be expressed as:

$\ell(\alpha, \lambda) = nln\frac{\pi}{2} + nln\alpha + \frac{1}{\lambda}\sum_{i=1}^{n} t_i - nln\lambda - \sum_{i=1}^{n} \ln(\alpha - \alpha e^{\frac{-t}{\lambda}} + 1) - nInIn(\alpha+1) + \sum_{1}^{n} In\ cos(\frac{\pi}{2} G(t_i)\ )$ By differentiating the log-likelihood function partially with respect to the unknown parameters (λ, α) and equating to zero, respectively, we obtain the following nonlinear equation [16, 17].

$$\frac{\partial \ell}{\partial \lambda} = \frac{1}{\lambda^2} - \sum_{i=1}^{n} t_i - \frac{n}{\lambda}\text{-}\sum_{i=1}^{n} \frac{\alpha t_i e^{\frac{-t_i}{\lambda}}}{\lambda^2(\alpha-\alpha e^{\frac{-t_i}{\lambda}}+1)} + \sum_{i=1}^{n}\left[-tan(\frac{\pi}{2} G(t_i))\frac{\pi}{2}\frac{\partial G(t_i)}{\partial \lambda})\right]$$

Where $\dfrac{\partial G(t_i)}{\partial \lambda} = \dfrac{-\alpha t_i e^{\frac{-t_i}{\lambda}}}{\lambda^2 ln(\alpha+1)(\alpha-\alpha e^{\frac{-t_i}{\lambda}}+1)}$, $\dfrac{\partial \ell}{\partial \alpha} = \dfrac{n}{\alpha} - \sum_{i=1}^{n} \dfrac{1-e^{\frac{-t_i}{\lambda}}}{(\alpha-\alpha e^{\frac{-t_i}{\lambda}}+1)} - \dfrac{n}{(\alpha+1)In(\alpha+1)} + \sum_{i=1}^{n}\left[-tan(\frac{\pi}{2} G(t_i))\frac{\pi}{2}\frac{\partial G(t_i)}{\partial \alpha})\right]$

Where $\dfrac{\partial G(t_i)}{\partial \alpha} = \dfrac{[\frac{1-e^{\frac{-t}{\lambda}}}{(\alpha-\alpha e^{\frac{-t}{\lambda}}+1)}]In(\alpha+1) - In(\alpha+1) - In(\alpha-\alpha e^{\frac{-t}{\lambda}}+1)(\frac{1}{\alpha+1})}{[In(\alpha+1)]^2}$

It is impossible to solve equations (12) and (13) analytically since they are nonlinear. Therefore, to determine the estimated values of the coefficients (λ, α), numerical techniques such the Newton-Raphson method are employed.

3.1. *Simulation data*

The convergence of the MLEs is confirmed in this section through a Monte Carlo simulation. For every combination of the following parameters, this study is performed 1000 times I: $\alpha = 0.1$, and $\lambda = 2$, II: $\alpha = 0.5$, $\lambda = 0.5$, and III: $\alpha = 1.5$, $\lambda = 3$. The sample sizes are n = 50, 100, 150, 200, and 250. R software is used. We see that as the sample size increases, subjective biases and mean squared errors (MSE) diminish. Additionally, the notion is reinforced by the fact that the confidence intervals' CPs are rather close to 95% nominal values. As shown in Tables 1, 2, and 3, the MLE and its asymptotic results can be used to estimate and create confidence intervals for the model parameters.

Table 1

Results MSE and Bias for the compound probability distribution's parameters for model I

n	MSE (α)	Bias (α)	MSE (λ)	Bias (λ)
50	2.20945	0.391012	0.208143	-0.0117971
100	1.78024	0.454721	0.117142	0.0699320
150	0.49672	0.043912	0.071900	-0.010922
250	0.33965	0.109432	0.055151	0.010917
500	0.18431	0.080301	0.023521	0.0025901

Table 2

Results MSE and Bias for the compound probability distribution's parameters for model II

n	MSE (α)	Bias (α)	MSE (λ)	Bias (λ)
50	3.78530	0.55011	0.0132010	0.0172320
100	1.43650	0.31234	0.0085913	0.0050722
150	0.79000	0.19353	0.0063154	0.0060321
250	0.36421	0.12431	0.0032027	0.0088681
500	0.16714	-0.017689	0.0017542	-0.0027621

Table 3

Results MSE and Bias for the compound probability distribution's parameters for model III

n	MSE (α)	Bias (α)	MSE (λ)	Bias (λ)
50	8.051	0.80631	0.0066310	0.0051702
100	2.8443	0.10411	0.0034210	-0.0084813
150	2.0231	-0.11122	0.0024510	-0.0080933
250	0.79125	-0.053750	0.0013601	-0.0054754
500	0.37791	-0.073399	0.00057212	-0.0034696

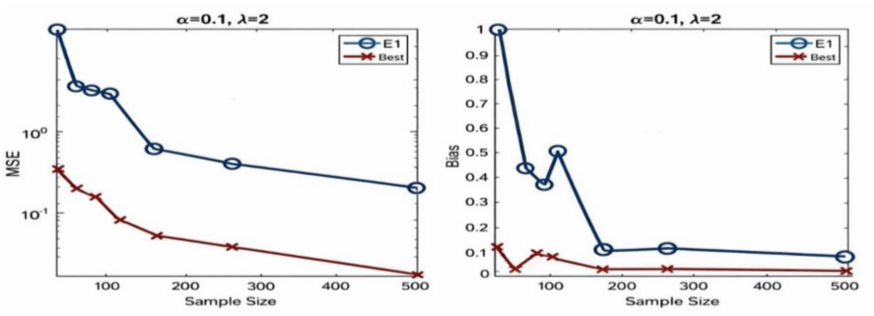

Figure 1
MSE and Bias of SALED for model I

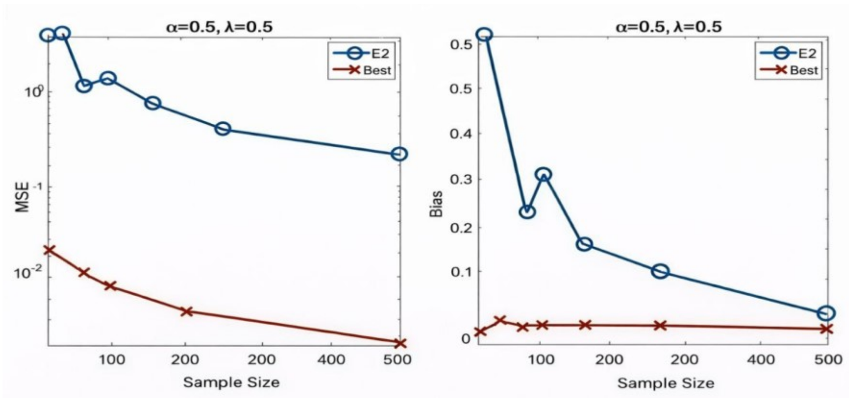

Figure 2
MSE and Bias of SALED for model II

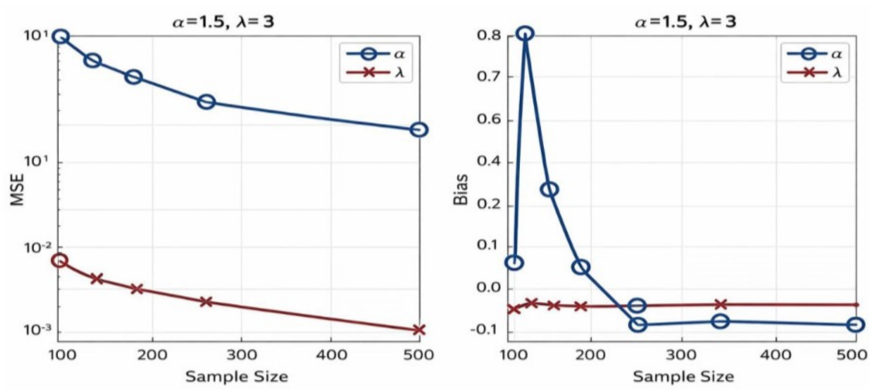

Figure 3
MSE and Bias of SALED for model III

4. Results

The Result shows the performance of the estimators, Figures 1–3 present the mean squared error (MSE) and bias of the parameter estimates α and λ under the three models for different sample sizes.

Model I: A clear, gradual decrease in the MSE values for both α and λ is observed as the sample size increases from 50 to 500. The bias for λ is very small and approaches zero as n increases, demonstrating asymptotic unbiasedness. The bias for α starts with a relatively large positive value for small sample sizes and then gradually decreases. Performance clearly improves at $n \geq 250$.

Model II: The decrease in MSE is even more pronounced compared to Model I. The MSE values for λ They are very small, even in small samples. The bias for both parameters decreases steadily and approaches zero at $n = 500$. The numerical performance is more stable compared to Model 1.

Model III: At $n = 50$, the MSE values for α are relatively high, but they decrease significantly with increasing sample size. The bias for α changes from positive to negative as n increases, indicating a gradual correction in the estimate. The MSE values for λ are very small and decrease steadily. Stability is almost achieved at $n \geq 250$.

Example 1: Consider the singular descriptor or nonlinear multi-FIDS

$$E\left[D^{\delta+1}u(t) + D^{\gamma}u(t)\right] + FD^{\beta+1}u(t) = KG(t) + JI^{\alpha}P(t)\ ,\ t \in [0,6], \quad (17)$$

$\begin{bmatrix} G_{1,0}(0) \\ G_{4,0}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} D^{0.75}u_{1,0}(0) \\ D^{0.75}u_{4,0}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} D^{0.3}u_{1,0}(0) \\ D^{0.3}u_{4,0}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, The following is solution of the system (17), $u(t) = \begin{bmatrix} \frac{2}{\Gamma(2.3)} t^{1.3} \\ \frac{6}{\Gamma(3.5)} t^{2.5} \end{bmatrix}$, Such that $E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\delta = 0.90$, $\gamma = 0.6$, $\beta = 0.3$, $\alpha = 0.4$,

$\begin{bmatrix} r_1(t) \\ r_2(t) \end{bmatrix} = \begin{bmatrix} \frac{2}{\Gamma(2.25)} t^{1.25} \\ 0 \end{bmatrix}$, $\begin{bmatrix} M_1(t) \\ M_2(t) \end{bmatrix} = \begin{bmatrix} \frac{2}{\Gamma(2.3)} t^{1.3} \\ \frac{6}{\Gamma(3.5)} t^{2.5} \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\left[\begin{bmatrix} \varphi^{0.95}u_1(t) \\ \varphi^{0.95}u_2(t) \end{bmatrix} + \begin{bmatrix} \varphi^{0.89}G_1(t) \\ \varphi^{0.89}G_2(t) \end{bmatrix}\right] + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \varphi^{0.5}u_1(t) \\ \varphi^{0.5}u_2(t) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{2}{\Gamma(2.3)} {t_1}^{1.3} \\ \frac{6}{\Gamma(3.5)} {t_1}^{2.5} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} I^{0.2}M_1(t) \\ I^{0.2}M_2(t) \end{bmatrix}$$

$u(0) = u_0 = \begin{bmatrix} u_{1,0} \\ u_{2,0} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then,

$$\begin{bmatrix} D^{0.75}u_1(t) + D^{0.5}u_1(t) \\ D^{0.3}u_2(t) \end{bmatrix} = \begin{bmatrix} \frac{2}{\Gamma(2.25)} t^{1.25} + \frac{2}{\Gamma(2.5)} t^{1.5} \\ \frac{6}{\Gamma(3.7)} t^{2.7} \end{bmatrix}$$

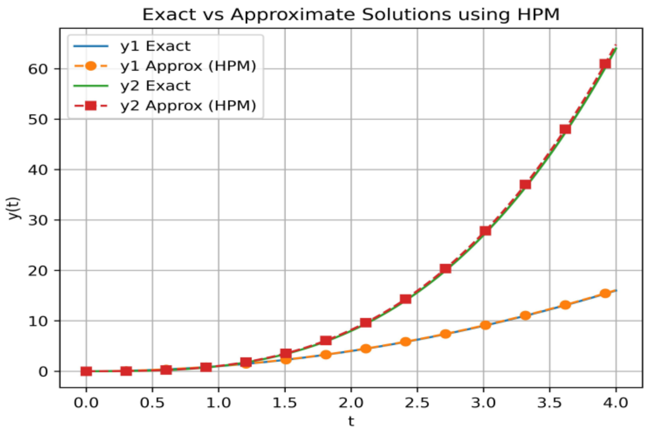

Figure 4
Exact solution.

The results obtained in Example 3 as shown in Figure 4, demonstrate that the proposed homotopy perturbation formulation efficiently handles nonlinear descriptor systems with fractional differential and integro-differential operators. For the selected benchmark problem, the HPM approximation reproduces the exact solution without deviation, which confirms the consistency of the constructed homotopy and the effectiveness of the method in treating coupled singular systems with nonlinear interactions.

Conclusion

The estimator is consistent, as both the MSE and the bias decrease as the sample size increases. Model II shows better performance in terms of accuracy, especially in estimating λ. In summary, the proposed HPM framework effectively solves nonlinear multi-fractional integro-differential systems, while the newly developed SALED distribution provides a flexible and competitive alternative for statistical modeling. The results confirm the accuracy, efficiency, and practical applicability of both approaches in addressing complex mathematical and real-world problems.

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