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Fibrewise h-topological spaces

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Abstract

In this paper, we introduce and study fibrewise h*-topological spaces over $\mathscr{B}$. We define this class of spaces and investigate related notions, including fibrewise h*-closed sets and fibrewise h*-open sets in h*-topological spaces over $\mathscr{B}$. Several propositions concerning these concepts are established and proved. In addition, a substantial part of this work is devoted to μ.h*-topology and Ƒ.Ꮚμ.h*-Ʈ-ς in the setting of micro-topology. The results presented here contribute to the development of the theory of fibrewise h*-topological structures and their associated properties.

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1. Introduction and Preliminaries

To initiate the classification of fibrewise structures, we first summarize the concept of sets distributed over a specific base space, denoted by the symbo (Ƒ.ƓƊ) and referred to as the fundamental set. Let $\mathscr{B}$ represent this fundamental set. A fibrewise space over $\mathscr{B}$ consists of a total set W and a mapping $\wp$: W→$\mathscr{B}$ known as the projection. For any point $b \in \mathscr{B}$, the fibre over b is the subset W_b=$\wp^{-1}$(b)⊆ W. Because the projection $\wp$ is not strictly required to be surjective, it is possible for these fibres to be empty. Furthermore, for any subset $\mathscr{B}$ *⊆ $\mathscr{B}$ we consider the preimage $\wp^{-1}(\mathscr{B}$ *) alternatively denoted as W| $\mathscr{B}$ * which forms a fibrewise set over $\mathscr{B}$ *. The projection for this space is determined via the restriction of the original map $\wp^{-1}$ to W| $\mathscr{B}$ * [1-7].

This work contains, we were able to establish new ideas, such as fibrewise h*-topological spaces, fibrewise h*-open, h*-closed from h*-topological space.and fibrewise micro h*-topological spaces. Finaly for a universal set $\mathfrak{U}$, the symbol $\mathfrak{U}$ / $\mathfrak{R}$ means the equivalence classes of $\mathfrak{U}$ due to the equivalence relation $\mathfrak{R}$, while the symbol $\mathcal{T}_{\mathfrak{R}}(\mathcal{H})$ means the nano topology of the subset $\mathcal{H}$ related to the relation $\mathfrak{R}$ [7]. Now, we will present the following definitions that we need in our work.

Definition 1.1: [8] A subset $\mathcal{X}$ of a toplogical space (W, τ), is called h-open set if for every non-empty set U in W, U ≠ W and U ∈ τ, $\mathcal{X} \subseteq$ Int($\mathcal{X}$∪U).

The complement of the h-open set is called h-closed. The family of all h-open sets of (W, τ) will denoted by τ^h.

Definition 1.2: A set $\mathcal{X}$ be h-closure is indicated by the symbol h-cl($\mathcal{X}$), which is defined as h-cl($\mathcal{X}$) = ∩{ $\mathcal{Y}$; $\mathcal{Y}$ be h-closed, and $\mathcal{X} \subseteq \mathcal{Y}$ }. A set $\mathcal{X}$ is h-interior be indicated by the notation h-int($\mathcal{X}$), which has the definition h-int($\mathcal{X}$) = ∪{ $\mathcal{Y}$: $\mathcal{Y}$ be h-open and $\mathcal{X} \supseteq \mathcal{Y}$}.

Definition 1.3: [9] Assume that $\mathcal{G}$ and ℵ are topological spaces, then the function ψ: $\mathcal{G} \to$ ℵ is known as h- irresolute, if Ψ^{-1}(U) is h-open set in $\mathcal{G}$ for every h-open set U in ℵ.

Definition 1.4: Assume that G and H are fibrewise sets over B, with a projection $\wp_G$: G→ $\mathscr{B}$, and $\wp_H$: H →$\mathscr{B}$, respectively. Function φ: G→ H is referred to as fibrewise if $\wp_H$o φ=$\wp_G$, Alternatively, if $\varphi(G_b) \subset H_b$ each from the points b from $\mathscr{B}$.

Keep in mind that a fibrewise function φ: G→ H over $\mathscr{B}$ determined via constraint, a fiberwise function $\varphi_{\mathcal{B}^*}$:$G_{\mathcal{B}^*}$→$H_{\mathcal{B}^*}$ over $\mathscr{B}$* from $\mathscr{B}$.

Definition 1.5. [10] Assume that $\mathscr{B}$ is a topological space such that, the fibrewise topology on fibrewise N over $\mathscr{B}$, purpose whatever topology on N where in projection $\wp$ is a continuous.

Definition 1.6: [11] A fibrewise function φ: G→ H, in which G and H are fibrewise topological spaces over $\mathscr{B}$, the referred to as:

(I) A mapping is continuous if, for every point g ∈ G_b (where b ∈ $\mathscr{B}$), the inverse image of any open set containing φ(g).

(II) A mapping is open if, for every point g ∈ G_b(where b ∈ $\mathscr{B}$), the direct image of any open set containing g results in an open set containing φ(g).

Definition 1.7: [12] If a projection $\wp$ an open (resp., closed) function, the fibrewise topological space over $\mathscr{B}$ the referred to as fibrewise open (resp., closed).

2. Fibrewise h*-Topological Spaces

This part provides explanations of the fibrewise h*-topology and various topological characteristics that can be gleaned from it.

Definition 2.1: Assume that $\mathcal{G}$ and ℵ are topological spaces, then the function ψ: G → ℵ is known as h*-continuous if Ψ^{-1}(U) is open set in $\mathcal{G}$ for every h-open set U in ℵ.

Definition 2.2: Let $\mathscr{B}$ be a topological space. We define a fibrewise h*-topological space denoted by the symbol (Ƒ.ＧƊ.h*-Ʈ-ς) on a fibrewise set W over $\mathscr{B}$ as any topology assigned to W such that the associated projection map $\wp$: W ⟶ $\mathscr{B}$ is h* continuous. Consequently, a fibrewise h* topological space over $\mathscr{B}$ simply consists of a fibrewise set W paired with this specific topology.

For example, a h*-topological product $\mathscr{B}$ × T for each h*-topological space T, may be considered as the Ƒ.ＧƊ.h*-Ʈ-ς over $\mathscr{B}$, making use from comparable to the first projection and for each subspace from $\mathscr{B}$ × T.

Example 2.3: Let W = {w_1, w_2, w_3}. τ_w = {ϕ, W, {w_1}, {w_2}, {w_1, w_2}, {w_2, w_3}}, and $\mathscr{B}$ = {b_1, b_2, b_3}, τ_B = {ϕ, $\mathscr{B}$, {b_1}, {b_1, b_2}}, τ_B^h = {ϕ, B, {b_1}, {b_2}, {b_1, b_2}, {b_2, b_3}}. If $\wp_W$: W →$\mathscr{B}$; $\wp_W$(w_1) = b_1, $\wp_W$(w_2) = b_2 $\wp_W$(w_3) = b_3. It is clear that $\wp_W$ is Ƒ.ＧƊ.h*-Ʈ-ς.

Example 2.4: Let W = {w_1, w_2, w_3}. τ_w = {ϕ, W, {w_1}, {w_1, w_3}}, and $\mathscr{B}$ = {b_1, b_2, b_3}, τ_B = {ϕ, $\mathscr{B}$, {b_1}, {b_1, b_2}}, τ_B^h = {ϕ, B, {b_1}, {b_2}, {b_1, b_2}, {b_2, b_3}}. If $\wp_W$: W →$\mathscr{B}$; $\wp_W$(w_1) = b_1, $\wp_W$(w_2) = b_3, $\wp_W$(w_3) = b_2. It is clear that $\wp_W$ is not Ƒ.ＧƊ.h*-Ʈ-ς, while it is Ƒ.ＧƊ.-Ʈ-ς.

Remark 2.5 Every fibrewise h*-topological spaces is fibrewise topological space and visa versa is not true by above example:

Definition 2.6: If $\mathcal{G}$ and ℵ are fibrewise h*-topological spaces over $\mathscr{B}$, then the fibrewise function ψ: $\mathcal{G}$ → ℵ is known as:

(I) h*-continuous, when each point g ∈ $\mathcal{G}_b$where b ∈ $\mathscr{B}$, inverse image from any h-open set from ψ(g) be open set from g.

(II) h*-open if for any point, g ∈ $\mathcal{G}_b$ where b ∈ $\mathscr{B}$, a direct image from any open set from g is a h- open set from ψ(g).

Let ψ: $\mathcal{G} \rightarrow \aleph$ be a fibrewise function where $\mathcal{G}$ is fibrewise set and $\aleph$ is a Ƒ.ƓƊ.h*-Ʈ-ς over $\mathscr{B}$. We could offer $\mathcal{G}$ the h*-induced topology, in the conventional sense, therefore this must be a fibrewise h*-topology. We can say to it, as the h*-induced Ƒ.ƓƊ. h*-Ʈ-ς.

Proposition 2.7: Let ψ: $\mathcal{G} \rightarrow \aleph$ a fibrewise function when $\aleph$ a Ƒ.ƓƊ.h*-Ʈ-ς over $\mathscr{B}$, and $\mathcal{G}$ is a fibrewise set that contains induced fibrewise set h*-topology. Then for every Ƒ.ƓƊ.h*-Ʈ-ς $\mathcal{W}$, fibrewise function λ: $\mathcal{W} \rightarrow \mathcal{G}$ be h*-continuous if and only if the composition ψo λ: $\mathcal{W} \rightarrow \aleph$ is h*-continuous.

Proof: Consider a diagram shown in Figure 1:

$$\begin{tikzcd}
\mathcal{W} \arrow[r, "\lambda"] \arrow[dr, "\wp_{\mathcal{W}}"'] & \mathcal{G} \arrow[r, "\psi"] \arrow[d, "\wp_{\mathcal{G}}"] & \aleph \arrow[dl, "\wp_{\aleph}"] \\
 & \mathscr{B} &
\end{tikzcd}$$

Figure 1
Composition diagram

(Part if) Assume that λ is h*-continuous. Let w ∈ $\mathcal{W}_b$, where b ∈ $\mathscr{B}$ and ϑ h-open set from (ψo λ)(w) = n ∈ $\aleph_b$ in $\aleph$. Since ψ be h*-continuous, $\psi^{-1}(\vartheta)$ is an open and hence h-open [9] set containing λ(w) = g ∈ $\mathcal{G}_b$ in $\mathcal{G}$. Given that λ is h*- continuous, then $\lambda^{-1}(\psi^{-1}(\vartheta))$ is a h-open set include w ∈ $\mathcal{W}_b$in $\mathcal{W}$ and $\lambda^{-1}(\psi^{-1}(\vartheta))$ = $(\lambda \text{o} \psi)^{-1}(\vartheta)$ is h-open set containing w ∈ $\mathcal{W}_b$in $\mathcal{W}$.

(Part only if) Assume that ψo λ is h*-continuous. Let w ∈ $\mathcal{W}_b$, where b ∈ $\mathscr{B}$ and Ʊ h-open set from λ(w)= g ∈ $\mathcal{G}_b$ in $\mathcal{G}$. Since ψ is h*-open, ψ(Ʊ) is h-open set containing ψ(g)= ψ(λ(w)) = n ∈ $\aleph_b$ in $\aleph$. Since ψo λ is h*-continuous, then $(\psi \text{o} \lambda)^{-1}(\psi$(Ʊ)) =$\lambda^{-1}$(Ʊ) is open set containing w ∈ $\mathcal{W}_b$in $\mathcal{W}$.

Proposition 2.8: Let ψ: $\mathcal{G}\rightarrow \aleph$ a fibrewise function when $\aleph$ be Ƒ.ƓƊ.h*-Ʈ-ς over $\mathscr{B}$, and $\mathcal{G}$ is fibrewise set that possesses h*-induced Ƒ.ƓƊ.h*-Ʈ-ς. If at all any Ƒ.ƓƊ.h*-Ʈ-ς $\mathcal{W}$, the fibrewise function λ: $\mathcal{W}\rightarrow \mathcal{G}$ is h*-open, surjection if and only if the composition ψo λ: $\mathcal{W}\rightarrow \aleph$ is h*-open.

Proof: Evident.

3. Fibrewise micro h*- closed and micro h*- open from fibrewise micro h*- topological spaces

Within this section introduces notions from fibrewise micro h*-closed, and micro h*-open from micro h*-topological spaces, many micro h*-topological properties from the resulting ideas are being investigated. During this research the symbol μ-closed (resp. μ-open) means micro closed (resp. micro-open) see [13]. The symbol μ. h-closed (resp. μ. h-open) refers to h-closed (resp.h-open) set but in micro topological spaces.

Definition 3.1: A function ψ: $\mathcal{G} \rightarrow \aleph$ is called micro h*-closed function (μ. h*-closed) in which $\mathcal{G}$ and $\aleph$ are h*-topological spaces,when it is map micro h-closed (μ.h- closed) sets [10] onto (μ.closed)sets. The collection of μ-open, μ. h- open a subset $\mathcal{H}$ will denoted by $\mu_{\Re}(\mathcal{H})$, $\mu_{\Re}{}^{h}(\mathcal{H})$ respectively.

Definition 3.2: A function ψ: $\mathcal{G} \rightarrow \aleph$ is called micro h*-open (μ.h*-open) function in which $\mathcal{G}$ and $\aleph$ are h*-topological spaces, when it is map μ.h-open sets onto μ-open sets.

Definition 3.3: A Ƒ.GƆ μ.h*-Ʈ-ς N over $\mathscr{B}$ is known as fibrewise μ.h*closed if the projection $\wp$ is μ.h*-closed.

Example 3.4: Let $\mathfrak{U}$ = {x, y, z, w, q}, $\mathfrak{U}$ / $\Re$ = {w, z, y}, {x}, {q}}, $\mathcal{H}$ = {x, y} $\subseteq \mathfrak{U}$.The $\mathcal{T}_{\Re}(\mathcal{H})$ ={ $\mathfrak{U}$, ϕ, {x}, {x, y, z, w}, {y, z, w}}, then M={y}$\notin$ $\mathcal{T}_{\Re}(\mathcal{H})$. Then $\mu_{\Re}(\mathcal{H})$ = {$\mathfrak{U}$, ϕ, {y}, {x}, {x, y}, {x, y, z, w}, {y, z, w}} = $\mu_{\Re}{}^{h}(\mathcal{H})$, so the collection of μ.h-closed set is { $\mathfrak{U}$, ϕ, {x, y, z, q}, { y, z, w, q}, {z, w, q}, { q}, {x, q}}.

Let $\mathscr{B}$ ={1,2,3}, $\mathscr{B}/\ \Re = \{\{2\},\{1,3\}\}, \vartheta = \{1,2\} \subseteq \mathscr{B}$, then $\acute{\mathcal{T}}_{\Re}(\vartheta)$={ ϕ, $\mathscr{B}$},{2},{1,3}}, then $\acute{\mu}$={3}$\notin \acute{\mathcal{T}}_{\Re}(\vartheta)$. Thus $\acute{\mu}_{\Re}(\vartheta)$= {$\mathscr{B}$, ϕ, {2,3}, {1,3}, {2}, {3}}, hence the micro h-closed is the set {$\mathscr{B}$, ϕ, {1,3}, {1,2}, {2}, {1}}. Define a projection $\wp$: $(\mathfrak{U}, \mu_{\Re}(\mathcal{H})) \rightarrow (\mathscr{B}, \acute{\mu}_{\Re}(\vartheta))$ such that $\wp$(x)= $\wp$(q)=2, $\wp$(y)= 3, $\wp$(z) = $\wp$(w)= 1, then the projection $\wp$ is μ. h*- closed, hence $\mathfrak{U}$ is fibrewise μ. h*- closed.

Proposition 3.5: Let ψ: $\mathcal{G} \rightarrow \aleph$ be a fibrewise μ. h*- closed a fibrewise function, where both $\mathcal{G}$ and $\aleph$ represent (Ƒ.GƆ. μ. h*- Ʈ-ς) spaces over the fundamental set $\mathscr{B}$. If the space $\aleph$ is fibrewise μ. h*- closed, then $\mathcal{G}$ must also be fibrewise μ. h*- closed.

Proof: Consider a diagram shown in Figure 2:

$$\begin{tikzcd} \mathcal{G} \arrow[rr, "\psi"] \arrow[dr, "\wp_{\mathcal{G}}"'] & & \aleph \arrow[dl, "\wp_{\aleph}"] \\ & \mathscr{B} & \end{tikzcd}$$

Figure 2
The diagram shows the relationship between $\aleph$ and $\mathcal{G}$

Assume that ψ: $\mathcal{G}\rightarrow\aleph$ be μ. h*- closed fibrewise function and $\aleph$ is fibrewise μ. h*- closed such that the projection $\wp_{\aleph}$: $\aleph \rightarrow \mathscr{B}$ be μ. h*- closed. To demonstrate that $\mathcal{G}$ a fibrewise μ. h*- closed such that the projection $\wp_{\mathcal{G}}$: $\mathcal{G} \rightarrow \mathscr{B}$ be μ. h*- closed. Suppose $\mathcal{F}$ be a μ.h-closed sub set from $\mathcal{G}_b$, when b $\in \mathscr{B}$. Given that ψ be μ. h*- closed, then $\psi(\mathcal{F})$ be μ. h- closed subset from $\aleph_b$, since $\wp_{\aleph}$ be μ. h*- closed, and each μ-closed is μ.h-closed [9] thereafter $\wp_{\aleph}(\psi(\mathcal{F}))$ be μ - closed in $\mathscr{B}$, but $\wp_{\aleph}(\psi(\mathcal{F}))$ = $(\wp_{\aleph}o\psi)(\mathcal{F})$ = $\wp_{\mathcal{G}}(\mathcal{F})$ a μ - closed in $\mathscr{B}$. Hence, $\wp_{\mathcal{G}}$ μ. h*- closed and $\mathcal{G}$ be a fibrewise μ. h*- closed.

Proposition 3.6: Assume that N a Ƒ.Ꮚ. μ. h*- Ʈ-ς over $\mathscr{B}$. Let N_i fibrewise μ. h*- closed from a finite covering for N, thereafter N a fibrewise μ. h*-closed.

Proof: Suppose that N a Ƒ.Ꮚ.h*-Ʈ-ς. over $\mathscr{B}$, such that a projection $\wp$: N$\rightarrow\mathscr{B}$ exist. For demonstrate that, $\wp$ be μ. h*- closed. Since N_i be fibrewise μ. h*- closed, then the restricted projection $\wp_i$: $N_i \rightarrow \mathscr{B}$ is μ. h*- closed for each member N_i of a finite cover of N. Let us assume that $\mathcal{F}$ is μ. h*- closed subset of N, thereafter, $\wp(\mathcal{F}) = \cup\, \wp_i(N_i \cap \mathcal{F})$, this be finite union from μ-closed sets [10], and $\wp$ be μ. h*- closed. So, N is fibrewise μ. h*-closed.

Proposition 3.7: Let N a Ƒ.Ꮚ. μ. h*- Ʈ-ς over $\mathscr{B}$, thereafter N be fibrewise μ. h*- closed if and only if for evey fibre N_b from N, and any h-open set Ʊ from N_b in N, In existence an h-open set $\mathcal{O}$ from b, such that $N_{\mathcal{O}}$ ⊂ Ʊ.

Proof: (Part if) Assume that N is fibrewise μ. h*- closed such that, the projection $\wp$: N$\rightarrow$ $\mathscr{B}$ be μ. h*- closed. Suppose b ∈ $\mathscr{B}$ and Ʊ is μ.h-open from N_b in N, then N-Ʊ is μ.h-closed in N, this means $\wp$(N-Ʊ) is μ-closed in $\mathscr{B}$. Let $\mathcal{O}$= $\mathscr{B}$-$\wp$(N-Ʊ), thereafter $\mathcal{O}$ is a μ-open set from b in $\mathscr{B}$ and $N_{\mathcal{O}} = \wp^{-1}(\mathcal{O})$ = N-$\wp^{-1}$($\wp$(N-Ʊ)) ⊂ Ʊ.

(Part only if) Assume that the assumption is correct and $\wp$: N$\rightarrow$ $\mathscr{B}$. Suppose $\mathcal{F}$ a μ.h-closed subset from N, and b ∈ $\mathscr{B}$-$\wp(\mathcal{F})$ and any μ.h-open set Ʊ from fibre N_b in N. Assuming they exist a μ-open $\mathcal{O}$ from b, such that $N_{\mathcal{O}}$ ⊂ Ʊ. It is simple to demonstrate this $\mathcal{O} \subset \mathscr{B}$- $\wp(\mathcal{F})$, thus $\mathscr{B}-\wp(\mathcal{F})$ a μ- open in $\mathscr{B}$ means $\wp(\mathcal{F})$ μ- closed in $\mathscr{B}$, and $\wp$ μ. h*- closed. Hence, N is fibrewise μ. h*- closed.

Definition 3.8: A Ƒ.Ꮚ. μ. h*- Ʈ-ς N over $\mathscr{B}$ a known as fiberwise μ. h*- open, if a projection $\wp$ be μ. h*- open.

Example 3.9: Assume 𝔘 = {1, 2, 3, 4}}, 𝔘 / ℜ = {{3, 4},{1}, {2}}, $\mathcal{H}$ = {3, 1} ⊆ 𝔘. $\mathcal{T}_{\Re}(\mathcal{H})$={𝔘,$\phi$,{1},{3,4},{1,3,4}},let ʍ={2}. Then $ʍ_{\Re}(\mathcal{H})$= {𝔘,ϕ,{1,2}, {3,4},{1,3,4}, {2,3,4},{1}, {2}}=$ʍ_{\Re}{}^{h}(\mathcal{H})$. Let $\mathscr{B}$ = {x, y, z, w}, $\mathscr{B}$/ℜ={{w}, {x},{y, z}}, $\mathcal{G}$ ={z, w}⊆$\mathscr{B}$, then $\acute{\mathcal{T}}_{\Re}(\mathcal{G})$ ={ $\mathscr{B}$, ϕ,{w},{y, z},{y, z, w}, then ʍ́ ={x }. Then $ʍ́_{\Re}(\mathcal{G})$= {$\mathscr{B}$, ϕ, {y, z}, {y, z, w},{x}, {w},{w,x},{x,y,z}}. Define a projection $\wp$: (𝔘, $ʍ_{\Re}(\mathcal{H})$)$\rightarrow$ ($\mathscr{B}$, $ʍ́_{\Re}(\mathcal{G})$) such that $\wp$(1) = x, $\wp$(2) = w, $\wp$(3) =

y, $\wp(4)$ = z, then the projection $\wp$ is μ. h*- open. Hence, $\mathfrak{U}$ is fibrewise μ.h*-open.

Example 3.10: Assume $\mathfrak{U}$ = {w_1, w_2, w_3}, $\mathfrak{U}$ / $\mathfrak{R}$={{w_1}, {w_2}, {w_3}}, $\mathcal{H} = \phi \subseteq \mathfrak{U}$, thereafter $\mathcal{T}_{\mathfrak{R}}(\mathcal{H})$={ $\mathfrak{U}, \phi$}, let ᴍ={ w_1}}$\notin \mathcal{T}_{\mathfrak{R}}(\mathcal{H})$. Then $\mu_{\mathfrak{R}}(\mathcal{H})$ = {$\mathfrak{U}$, ϕ, {w_1}}, hence $\mu_{\mathfrak{R}}{}^{\mathbf{h}}(\mathcal{H})$={ $\mathfrak{U}$, ϕ, {w_1}, {w_2,w_3}}. If $\mathscr{B}$={ȡ1, ȡ2, ȡ3}, $\mathscr{B}$/ $\mathfrak{R}$ ={{ ȡ1}, {ȡ2, ȡ3}}, ϑ = { ȡ1} ⊆ $\mathscr{B}$, thereafter $\acute{\mathcal{T}}_{\mathfrak{R}}(\vartheta)$= { $\mathscr{B}$, ϕ, { ȡ1}}, then ḿ={ ȡ2}$\notin \acute{\mathcal{T}}_{\mathfrak{R}}(\vartheta)$. Thus $\acute{\mu}_{\mathfrak{R}}(\vartheta)$= {$\mathscr{B}$, ϕ, {ȡ1}, { ȡ2}, { ȡ1, ȡ2}}. Define a projection $\wp$: ($\mathfrak{U}$, $\mu_{\mathfrak{R}}(\mathcal{H})$)→ ($\mathscr{B}$, $\acute{\mu}_{\mathfrak{R}}(\vartheta)$) such that $\wp$(w_1)= ȡ1, $\wp$(w_2)= ȡ2, $\wp$(w_3)= ȡ3, clear that the projection $\wp$ is not μ. h*-open while it is μ-open.

Proposition 3.11: Let ψ: $\mathcal{G}$→ ℵ be μ. h*-open fiberwise function, where both $\mathcal{G}$ and ℵ are (Ƒ.Ꮻ. μ. h*- Ʈ-ς) spaces over $\mathscr{B}$. Whether ℵ be fibrewise μ. h*-open, thereafter $\mathcal{G}$ be fibrewise μ. h*- open.

Proof: Assume that ψ: $\mathcal{G}$→ ℵ a μ. h*- open fibrewise function, and ℵ a fibrewise μ. h*- open such that the projection $\wp_{\aleph}$: ℵ → $\mathscr{B}$ is μ. h*- open. To demonstrate that $\mathcal{G}$ is fibrewise μ. h*-open such that the projection $\wp_{\mathcal{G}}$: $\mathcal{G}$ → $\mathscr{B}$ is μ. h*-open. Let $\mathcal{O}$ be a μ.h-open sub set from $\mathcal{G}_b$, where b ∈ $\mathscr{B}$. Since ψ is μ. h*- open, thereafter $\psi(\mathcal{O})$ be μ - open subset from $\aleph_b$, since $\wp_{\aleph}$ a μ. h*- open and each μ - open μ.h-open [9], thereafter $\wp_{\aleph}(\psi(\mathcal{O}))$ is μ- open in $\mathscr{B}$, but $\wp_{\aleph}(\psi(\mathcal{O}))$ = $(\wp_{\aleph} o \psi)(\mathcal{O})$ = $\wp_{\mathcal{G}}(\mathcal{O})$ a μ- open in $\mathscr{B}$. Thus $\wp_{\mathcal{G}}$ must be μ. h*- open and $\mathcal{G}$ a fibrewise μ. h*-open.

Proposition 3.12: Let $\{N_j\}$ constitute finite family from fibrewise μ. h*- open spaces over $\mathscr{B}$, thereafter the Ƒ.Ꮻ. μ. h*- Ʈ-ς product N=$\prod_{\mathcal{B}} N_j$ also has μ. h*-open.

Proof: Assume that N=$\prod_{\mathcal{B}} N_j$ a Ƒ.Ꮻ. μ. h*- Ʈ-ς over $\mathscr{B}$, thereafter $\wp$: N=$\prod_{\mathcal{B}} N_j$→ $\mathscr{B}$ is exists. To demonstrate that $\wp$ is μ. h*- open. Since $\{N_j\}$ constitute a finite family from fibrewise μ. h*-open spaces over $\mathscr{B}$, thereafter the projection $\wp_j$: N_j→ $\mathscr{B}$ is μ. h*- open for each j. Suppose $\mathcal{O}$ be a μ. h- open subset from N, then $\wp(\mathcal{O})$ = $\wp(\prod_{\mathcal{B}}(N_j \cap \mathcal{O}))$ = $\prod_{\mathcal{B}} \wp_j(N_j \cap \mathcal{O})$, this is finite product from μ- open sets [11], and therefore $\wp$ μ. h*- open. Hence, the fibrewise μ. h*-topological product N=$\prod_{\mathcal{B}} N_j$ is a fibrewise μ. h*- open.

In other words, the category from fibrewise μ. h*- open spaces has a finitely multiplicative. In reality, Proposition (3.12) holds true for infinite families if each member from the family is fibrewise nonempty in the sense that the projection is surjective.

Definition 3.13: Let $\mathcal{G}$ and ℵ be micro spaces. A function ψ: $\mathcal{G}$ → ℵ is termed μ-h*- irresolute, if for every μ-h- open set Ʊ in ℵ. its preimage ψ^{-1}(Ʊ) is μ-h-open set in $\mathcal{G}$.

Our next three findings apply equally to fibrewise μ. h*- closed and fibrewise μ. h*- open spaces.

Proposition 3.14: Let ψ: $\mathcal{G}\rightarrow \aleph$ be a μ. h*-irrosolute fibrewise surjection, when $\mathcal{G}$, and $\aleph$ are Ƒ.ƇƊ. μ. h*- Ţ-ς over $\mathscr{B}$. If $\mathcal{G}$ a fibrewise μ. h*- closed (resp., μ. h*- open), then $\aleph$ is fibrewise μ. h*-closed (resp., μ. h*- open).

Proof: Assume that ψ: $\mathcal{G}\rightarrow \aleph$ is μ. h*-irrosolute fibrewise surjection and $\mathcal{G}$ is fibrewise μ. h*-closed (resp., μ. h* -open) such that the projection $\wp_{\mathcal{G}}: \mathcal{G} \rightarrow \mathscr{B}$ is μ. h*-closed (resp., μ. h*-open). To demonstrate that $\aleph$ is fibrewise μ. h*-closed (resp., μ. h*-open) such that the projection $\wp_{\aleph}: \aleph\rightarrow \mathscr{B}$ is μ. h*-closed (resp., μ. h*- open). Let $\mathcal{C}$ be a μ. h-closed (resp., μ. h-open) subset from $\aleph_b$, when b ∈ $\mathscr{B}$. Since ψ a μ. h*-irrosolute map, thereafter $\psi^{-1}(\mathcal{C})$ is μ.h-closed (resp., μ.h- open) subset from $\mathcal{G}_b$, since $\wp_{\mathcal{G}}$ is μ. h*-closed (resp., μ. h*-open), then $\wp_{\mathcal{G}}(\psi^{-1}(\mathcal{C}))$ be μ - closed (resp., μ- open) in $\mathscr{B}$, but $\wp_{\mathcal{G}}(\psi^{-1}(\mathcal{C})) = (\wp_{\mathcal{G}} o \psi^{-1})(\mathcal{C}) = \wp_{\aleph}(\mathcal{C})$ is μ -closed (resp., μ -open) in $\mathscr{B}$. Hence, $\wp_{\aleph}$ a μ. h*-closed, (resp., μ. h*- open), and $\aleph$ is fibrewise μ. h*-closed (resp., μ. h*-open).

Proposition 3.15: Let N be a Ƒ.ƇƊ. μ. h*- Ţ-ς over $\mathscr{B}$. Assume that N a fibrewise μ. h*-closed (resp., μ. h*- open) over $\mathscr{B}$, then $N_{\mathcal{B}^*}$ is fibrewise μ. h*-closed (resp., μ. h*- open) over $\mathcal{B}^*$for every μ-closed (resp. μ-open) subset from $\mathscr{B}$.

Proof: suppose that N be a fibrewise μ. h*-closed (resp., μ. h*- open), such that the projection $\wp$:N→$\mathscr{B}$ a μ. h*-closed (resp., μ. h*-open). To demonstrate that $N_{\mathcal{B}^*}$ is fibrewise μ. h*-closed (resp., μ. h*-open) over $\mathcal{B}^*$; the projection $\wp_{\mathcal{B}^*}: N_{\mathcal{B}^*} \rightarrow \mathcal{B}^*$ be μ. h*- closed (resp., μ. h*- open). Let $\mathcal{C}$ be a μ. h-closed (resp., μ. h-open) subset from N, then $\mathcal{C} \cap N_{\mathcal{B}^*}$ μ. h- closed (resp., μ. h-open) in $N_{\mathcal{B}^*}$[12, 13], and $\wp_{\mathcal{B}^*}(\mathcal{C} \cap N_{\mathcal{B}^*}) = \wp(\mathcal{C} \cap N_{\mathcal{B}^*}) = \wp(\mathcal{C}) \cap \mathcal{B}^*$ which be μ-closed (resp., μ- open) set in $\mathcal{B}^*$. Hence, $\wp_{\mathcal{B}^*}$ μ. h*- closed (resp., μ. h*-open), and $N_{\mathcal{B}^*}$ is fibrewise μ. h*-closed (resp., μ. h*-open) over $\mathcal{B}^*$.

Proposition 3.16: Let N be an (Ƒ.ƇƊ. μ. h*- Ţ-ς) space over $\mathscr{B}$. If $N_{\mathcal{B}j}$ is a fibrewise μ-h*-closed (resp., μ. h*-open) space over $\mathcal{B}_j$ for every set $\mathcal{B}_j$ in a μ-open cover of $\mathscr{B}$ then N itself is fibrewise μ-h*-closed (resp., μ. h*- open) over $\mathscr{B}$.

Proof: Let N be Ƒ.ƇƊ. μ. h*- Ţ-ς over $\mathscr{B}$, then projection $\wp$:N→ $\mathscr{B}$ exists. For demonstrate that $\wp$ a μ. h*-open. Since $N_{\mathcal{B}_j}$ is fibrewise μ. h*-open over $\mathcal{B}_j$, thereafter projection $\wp_{\mathcal{B}_j}: N_{\mathcal{B}_j} \rightarrow \mathcal{B}_j$ be μ. h*-open for every member $\mathcal{B}_j$ from an μ-open covering from $\mathscr{B}$. Assume $\mathcal{C}$ is a μ.h-open subset from N, thereafter $\wp(\mathcal{C}) = \cup \wp_{\mathcal{B}}(N_{\mathcal{B}_j} \cap \mathcal{C})$, this is a union from μ-open sets, and so $\wp$ is μ.h*-open. Hence, N be fibrewise μ. h*-open over $\mathscr{B}$.

4. Conclusion

Topological spaces have been presented in a wide variety of ways over the years. Considering transparency leads to a number of intriguing issues. We present μ.h-open sets, μ.h-closed sets and μ.h*-continuity in fibrewise μ.h*-topological spaces in this article and look into some of the fundamental properties. Its significance is important in many fields of mathematics and related sciences. Future research will expand on this and add some uses.

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