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Convergent sequence spaces defined by positive function

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Abstract

This work presents the new result obtained by studying many sequences space define as positive functions. This functions are constructed to show the convergence of these spaces. The space of these functions is studied, and it is shown to be linear, total, and complete in a paranorm. Some new definitions and properties related to these sequence spaces are introduced and discussed in this field.

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Keywords: *Sequence space, Positive function, Convergence, P-convergence.*

1. Introduction

The concept of a sequence plays an important role in real analysis, where many studies focused on summability methods with arithmetic mean; after that, it was generalized by Cesaro as $(C,k)-$methods for the summability. To achieve convergence for the sequence, in 1951, Fast [13] introduced convergence for the sequence, and also in 1959, Schoenberg [10] independently studied convergence. It found in Zygmund [11]. Further, it was studied and proved in the sequences space and linked to summability by [14], Salat [12], and introduced by Maddox [15], [20], Tripathy [16, 18].

In 1971, [2] defined sequence space in much more details; they established each sequence space consists of subspaces isomorphic to $l_p(1 \leq p < \infty)$. Thus, each infinite-dimensional Banach space consists of closed subspaces isomorphic to c_0 or some l_p, positively of family for the space (see Lindenstrauss [1], Milman [8]) in 1970. Further, the sequence space is a generalization of the l_p space.

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In 2004, a positive sequence space was further investigated from a sequence space perspective, related to summability theory, in [5], [9], and many others. In [6], some vector-valued sequence spaces were introduced by using positive functions defined over a seminormed space in 2009. Later on, the summability method was studied by more mathematicians, such as Tripathy and Borgohain [4] in 2013.

A general concept for a single sequence is a double sequence, and all double sequences are infinite matrices. The initial works on over double sequences were done by Browmich [7] in 2005. Later, it was introduced at the beginning of the nineteenth century by an eminent mathematician [17] in 1904, where an investigation from summability theory was conducted by [3, 19-24].

2. Preliminaries

If $(x,y) = (x_{k,l}\,,y_{k,l})$, the infinite array for elements $(x_{k,l}\,,y_{k,l})$, and ב denotes the family of all $\mathbb{C}^2$ sequences.

In this section, we defined positive functions $M(x,y)$ In term for the sequences space, as follows.

Definition 2.1: The positive functions M be the functions
$M\colon [0,\infty)\times[0,\infty)\to[0,\infty)$ s.t M be continuous, nondecreasing, even, convex, and satisfied follows condition:
1) $M(0, 0) = 0$.
2) $M(\nu,\kappa) > 0$
3) $If\ \nu,\kappa \to \infty$, then $M(\nu,\kappa) \to \infty$

Definition 2.2: A sequences $(\nu,\kappa) = (\nu_{k,l},\kappa_{k,l})$ have limits ℓ denote as $lim(\nu,\kappa) = \ell$, provide if $\epsilon > 0$, $\exists N \in \mathbb{N}$ s.t $|\nu_{k,l} - \ell, \kappa_{k,l} - \ell| < \epsilon, \forall k,l > N$.

Definitions 2.3: Doubles sequences $(x,y) = (x_{k,l},y_{k,l})$ named $P-$ sequences if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ s.t $|(x_{i,j} - x_{k,l}, y_{i,j} - y_{k,l})| < \epsilon\ \forall i,k,j,l \geq N$.

Definition 2.4: The sequences $(x,y) = (x_{k,l},y_{k,l})$named bounded, if there is positive numbers $\mathcal{B}$ s.t $|(x_{k,l},y_{k,l})| \leq \mathcal{B}$, $\forall k,l$.

We constructs the sequences spaces as:

$$ד = \left\{ \begin{matrix} (x_{k,l},y_{k,l}) \in ב\colon \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} inf\ sup\left\{M\left(\frac{|x_{k,l}|}{\rho},\frac{|y_{k,l}|}{\rho}\right)\right\} < \infty, \\ \text{for some } \rho > 0 \end{matrix} \right\},$$

A spaces ד with norms

$$\| (x_{k,l},y_{k,l}) \| = inf\left\{\rho > 0\colon \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} inf\ sup\left\{M\left(\frac{|x_{k,l}|}{\rho},\frac{|y_{k,l}|}{\rho}\right)\right\} \leq 1\right\}.$$

Definition 2.5: If M the functions & $p = (p_{k,l})$ denote any factorable sequences for the strict positive real number.

We defined followed:

$$\daleth(p)=\left\{\begin{matrix}(x,y)\in\beth:\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} inf sup\left\{\left(M\left(\frac{|x_{k,l}|}{\rho},\frac{|y_{k,l}|}{\rho}\right)\right)^{p_{k,l}}\right\}<\infty,\\ \text{where } \rho>0\end{matrix}\right\}.$$

$W^2(M,p)=$

$$\left\{\begin{matrix}(x,y)\in\beth: lim_{m,n}\frac{1}{mn}\sum_{k=1}^{m}\sum_{l=1}^{n} inf sup\left\{\left(M\left(\frac{|x_{k,l}-\ell|}{\rho},\frac{|y_{k,l}-\ell|}{\rho}\right)\right)^{p_{k,l}}\right\}=0,\\ \text{when } \rho>0\end{matrix}\right\}.$$

$W_0^2(M,p)=$

$$\left\{\begin{matrix}(x,y)\in\beth: lim_{m,n}\frac{1}{mn}\sum_{k=1}^{m}\sum_{l=1}^{n} inf sup\left\{\left(M\left(\frac{|x_{k,l}|}{\rho},\frac{|y_{k,l}|}{\rho}\right)\right)^{p_{k,l}}\right\}=0,\\ \text{where } \rho>0\end{matrix}\right\}.$$

$W_\infty^2(M,p)=$

$$\left\{\begin{matrix}(x,y)\in\beth: sup_{m,n}\frac{1}{mn}\sum_{k=1}^{m}\sum_{l=1}^{n} inf sup\left\{\left(M\left(\frac{|x_{k,l}|}{\rho},\frac{|y_{k,l}|}{\rho}\right)\right)^{p_{k,l}}\right\}<\infty,\\ \text{where } \rho>0\end{matrix}\right\}.$$

For a function $M(x,y), A=(\mathbb{C}^2,1,1)$ and $p_{k,l}=1$, we now introduce the following sequence spaces:

$W^2(M)=$

$$\left\{\begin{matrix}(x,y)\in\beth: lim_{m,n}\frac{1}{mn}\sum_{k=1}^{m}\sum_{l=1}^{n} inf sup\left\{M\left(\frac{|x_{k,l}-\ell|}{\rho},\frac{|y_{k,l}-\ell|}{\rho}\right)\right\}=0,\\ \text{when } \rho>0\end{matrix}\right\}.$$

$$W_0^2(M)=\left\{\begin{matrix}(x,y)\in\beth: lim_{m,n}\frac{1}{mn}\sum_{k=1}^{m}\sum_{l=1}^{n} inf sup\left\{M\left(\frac{|x_{k,l}|}{\rho},\frac{|y_{k,l}|}{\rho}\right)\right\}=0,\\ \text{for some } \rho>0\end{matrix}\right\}.$$

$$=\left\{\begin{matrix}(x,y)\in\beth: sup_{m,n}\frac{1}{mn}\sum_{k=1}^{m}\sum_{l=1}^{n} inf sup\left\{M\left(\frac{|x_{k,l}|}{\rho},\frac{|y_{k,l}|}{\rho}\right)\right\}<\infty,\\ \text{when } \rho>0\end{matrix}\right\}.$$

3. Some Important Result

Theorem 3.1: *If* $H=sup_{k,l}p_{k,l}$, *then* $\daleth(p)$ *be linear sets over* $\mathbb{C}^2$.

Proof: If $(x,y)=(x_{k,l},y_{k,l})$, $(a,b)=(a_{k,l},b_{k,l})$ be elements for $\daleth(p)$ and let $(\alpha,\alpha),(\beta,\beta)\in\mathbb{C}^2$ are complex numbers. We must funded some $\rho_3>0$ s.t

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} inf sup\left\{\left(M\left(\frac{|\alpha x_{k,l}+\beta a_{k,l}|}{\rho_3},\frac{|\alpha y_{k,l}+\beta b_{k,l}|}{\rho_3}\right)\right)^{p_{k,l}}\right\}<\infty.$$

Since $(x,y),(a,b)$ are in $ℸ(p)$, $M(x,y)$ be a function, there is some positives ρ_1 & ρ_2 s.t

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} inf sup\left\{\left(M\left(\frac{|x_{k,l}|}{\rho_1},\frac{|y_{k,l}|}{\rho_1}\right)\right)^{p_{k,l}}\right\} < \infty,$$

and

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} inf sup\left\{\left(M\left(\frac{|a_{k,l}|}{\rho_2},\frac{|b_{k,l}|}{\rho_2}\right)\right)^{p_{k,l}}\right\} < \infty.$$

We defined $\rho_3 = max\{2|\alpha|\rho_1, 2|\beta|\rho_2\}$. so

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} inf sup\left\{\left(M\left(\frac{|\alpha x_{k,l}+\beta a_{k,l}|}{\rho_3},\frac{|\alpha y_{k,l}+\beta b_{k,l}|}{\rho_3}\right)\right)^{p_{k,l}}\right\} \leq$$

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} inf sup\left\{\left(M\left(\frac{|\alpha x_{k,l}|}{\rho_3}+\frac{|\beta a_{k,l}|}{\rho_3},\frac{|\alpha y_{k,l}|}{\rho_3}+\frac{|\beta b_{k,l}|}{\rho_3}\right)\right)^{p_{k,l}}\right\} <$$

$$\Im\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} inf sup\left\{\left(M\left(\frac{|x_{k,l}|}{\rho_1},\frac{|y_{k,l}|}{\rho_1}\right)\right)^{p_{k,l}}\right\}$$

$$+\Im\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} inf sup\left\{\left(M\left(\frac{|a_{k,l}|}{\rho_2},\frac{|b_{k,l}|}{\rho_2}\right)\right)^{p_{k,l}}\right\} \leq \infty,$$

where $\Im = max\{1, 2^{H-1}\}$. Thus $ℸ(p)$ be linear spaces.

Theorem 3.2: ℸ(p) be total paranormal spaces with para norms

$$g_M(x,y) = inf\left\{\rho^{p_{k,l}/H}: \left(\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} inf sup\left\{\left(M\left(\frac{|x_{k,l}|}{\rho},\frac{|y_{k,l}|}{\rho}\right)\right)^{p_{k,l}}\right\}\right)^{1/H} \leq 1, \atop k,l = 1,2,\ldots\right\},$$

and $H = max\left\{1, sup_{k,l} p_{k,l}\right\}$.

Proof: It clear, $g_M(x,y) = g_M(-x,-y)$. use Th. 1 for $(\alpha,\alpha) = (\beta,\beta) = (1,1)$, so:

$g_M\big((x,y)+(a,b)\big) = g_M\big((x+a),(y+b)\big) = g_M\big((x,y)+(a,b)\big) \leq g_M(x,y) + g_M(a,b)$, then

Since $inf\left\{p^{p_{k,l}/H}\right\} = 0$ for $x = 0, y = 0$, so, $M(0,0) = 0$.

Conversely, if $g_M(x,y) = 0$, so

$$inf\left\{\rho^{p_{k,l}/H}: \left(\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} inf\ sup\left\{\left(M\left(\frac{|x_{k,l}|}{\rho},\frac{|y_{k,l}|}{\rho}\right)\right)^{p_{k,l}}\right\}\right)^{1/H} \leq 1\right\} = 0.$$

Hence, for $\epsilon > 0$, $\exists\rho_\epsilon$ $(0 < \rho_\epsilon < \epsilon)$ s.t

$$\left(\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} inf sup\left\{\left(M\left(\frac{|x_{k,l}|}{\rho_\epsilon},\frac{|y_{k,l}|}{\rho_\epsilon}\right)\right)^{p_{k,l}}\right\}\right)^{1/H} \leq 1.$$

Thus,

$$\left(\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} inf sup\left\{\left(M\left(\frac{|x_{k,l}|}{\epsilon},\frac{|y_{k,l}|}{\epsilon}\right)\right)^{p_{k,l}}\right\}\right)^{1/H} \leq$$

$$\left(\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} inf sup\left\{\left(M\left(\frac{|x_{k,l}|}{\rho_\epsilon},\frac{|y_{k,l}|}{\rho_\epsilon}\right)\right)^{p_{k,l}}\right\}\right)^{1/H} \leq 1.$$

Suppose $x_{n_m,s_t} \neq 0, y_{n_m,s_t} \neq 0$ for some m, t.

If $\epsilon \to 0$, therefore $\left(\frac{|x_{n_m,s_t}|}{\epsilon},\frac{|y_{n_m,s_t}|}{\epsilon}\right) \to \infty$,

So,

$$\left(\sum_{m=1}^{\infty}\sum_{t=1}^{\infty} inf sup\left\{\left(M\left(\frac{|x_{n_m,s_t}|}{\epsilon},\frac{|y_{n_m,s_t}|}{\epsilon}\right)\right)^{p_{m,t}}\right\}\right)^{1/H} \to \infty,$$

this contradiction. so, $(x_{n_m,s_t}, y_{n_m,s_t}) = (0,0)$, $\forall m, t$.

To shows the scalar multiplications be continuous.

Let (γ,γ) is the any numbers. from definitions of paranorm,

$$g_M(\gamma x,\gamma y) = inf\left\{\rho^{p_{k,l}/H}: \left(\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} inf sup\left\{\left(M\left(\frac{|\gamma x_{k,l}|}{\rho},\frac{|\gamma y_{k,l}|}{\rho}\right)\right)^{p_{k,l}}\right\}\right)^{1/H} \leq 1 \\ k,l = 1,2,\ldots\right\}.$$

Then

$$g_M(\gamma x,\gamma y) = \inf\left\{(\gamma r)^{p_{k,l}/H}: \left(\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} inf sup\left\{\left(M\left(\frac{|x_{k,l}|}{r},\frac{|y_{k,l}|}{r}\right)\right)^{p_{k,l}}\right\}\right)^{1/H} \leq 1 \\ k,l = 1,2,\ldots\right\}$$

where $r = \rho/\gamma$.

Since $|\gamma|^{p_{k,l}} \leq max(1,|\gamma|^H)$, therefore $|\gamma|^{p_{k,l}/H} \leq max(1,|\gamma|^H)^{1/H}$.

Hence,

$$g_M(\gamma x,\gamma y) \leq max(1,|\gamma|^H)^{1/H} inf \left\{(r)^{p_{k,l}/H}: \left(\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} inf sup\left\{\left(M\left(\frac{|x_{k,l}|}{r},\frac{|y_{k,l}|}{r}\right)\right)^{p_{k,l}}\right\}\right)^{1/H} \leq 1 \\ k,l = 1,2,3,\ldots\right\},$$

which converge to the zero as $g_M(x,y)$ converge to the zero in ᄀ(p).
if $(x,y) \to 0$ in ᄀ(p).

For an arbitrary $\epsilon > 0$, if N positive integers s.t

$\sum_{k=N+1}^{\infty}\sum_{l=N+1}^{\infty} inf sup\left\{\left(M\left(\frac{|x_{k,l}|}{\rho},\frac{|y_{k,l}|}{\rho}\right)\right)^{p_{k,l}}\right\} < \frac{\epsilon}{2}$ where $\rho > 0$.

This implies that

$$\left(\sum_{k=N+1}^{\infty}\sum_{l=N+1}^{\infty} inf sup\left\{\left(M\left(\frac{|x_{k,l}|}{\rho},\frac{|y_{k,l}|}{\rho}\right)\right)^{p_{k,l}}\right\}\right)^{1/H} \le \frac{\epsilon}{2},$$

let $0 < |\gamma| < 1$, by use convexity for the M we obtain

$$\sum_{k=N+1}^{\infty}\sum_{l=N+1}^{\infty} inf sup\left\{\left(M\left(\frac{|x_{k,l}|}{\rho},\frac{|y_{k,l}|}{\rho}\right)\right)^{p_{k,l}}\right\} <$$

$$\sum_{k=N+1}^{\infty}\sum_{l=N+1}^{\infty} inf sup\left\{\left(|\gamma| M\left(\frac{|x_{k,l}|}{\rho},\frac{|y_{k,l}|}{\rho}\right)\right)^{p_{k,l}}\right\} < (\frac{\epsilon}{2})^{H}.$$

but M the continuous in $[0,\infty)\times[0,\infty)$, so $F(j_1,j_2)$ be continuous at 0. So there exist $1 > \delta > 0$ s.t $|F(j_1,j_2)| < \frac{\epsilon}{2}$ for $(0,0) < (j_1,j_2) < (\delta,\delta)$.

If K the positive integers s.t $|\gamma_n| < \delta$ for $n > K$, then when $n > K$,

$$\left(\sum_{k=1}^{N}\sum_{l=1}^{N} inf sup\left\{\left(M\left(\frac{|\gamma_n x_{k,l}|}{\rho},\frac{|\gamma_n y_{k,l}|}{\rho}\right)\right)^{p_{k,l}}\right\}\right)^{1/H} < \frac{\epsilon}{2}.$$

Thus,

$$\left(\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} inf sup\left\{\left(M\left(\frac{|\gamma_n x_{k,l}|}{\rho},\frac{|\gamma_n y_{k,l}|}{\rho}\right)\right)^{p_{k,l}}\right\}\right)^{1/H} < \epsilon\text{, for } n > K.$$

A functions M is represent in followed integrals forms

$$M(x,y) = \int_0^x\int_0^y (f(t),h(s))dtds,$$

A kernel for $M(x,y)$ are right-differentiable for $t,s \ge 0$,
$f(0)=0, h(0)=0, f(t)>0, h(s)>0, f,h$ are non-decreasing
$f(t)\to\infty, h(s)\to\infty$ as $t,s\to\infty$.

Note: The sequence spaces named complete if each $P-$ sequences be convergent.

Theorem 3.3: *If* $1 \le p_{k,l} < \infty$. *Then* ٦(p) *be complete para normed spaces with*

$$g_M(x,y) = inf\left\{\begin{matrix}\rho^{p_{k,l}/H}: \left(\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} inf sup\left\{\left(M\left(\frac{|x_{k,l}|}{\rho},\frac{|y_{k,l}|}{\rho}\right)\right)^{p_{k,l}}\right\}\right)^{1/H} \le 1\\ k,l=1,2,\dots\end{matrix}\right\}$$

Proof: If $(x^i,\ y^i)$ the sequences in ٦(p).

Let r, η are fixed. so $\forall \frac{\epsilon}{r\eta} > 0$, $\exists N$ s.t $g_M(x^i - x^j, y^i - y^j) < \frac{\epsilon}{r\eta}, \forall i, j \geq N$.

Use definition for the paranorm, we obtain

$\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} inf sup \left\{ \left(M \left(\frac{|x_{k,l}^i - x_{k,l}^j|}{g_M(x^i - x^j)}, \frac{|y_{k,l}^i - y_{k,l}^j|}{g_M(y^i - y^j)} \right) \right)^{p_{k,l}} \right\} \leq 1, \forall i, j \geq N.$

Since $1 \leq p_{k,l} < \infty$, so

$$sup \left\{ M \left(\frac{|x_{k,l}^i - x_{k,l}^j|}{g_M(x^i - x^j)}, \frac{|y_{k,l}^i - y_{k,l}^j|}{g_M(y^i - y^j)} \right) \right\} \leq 1$$

Hence one can find $r > 0$ with $(\frac{\eta}{2}) r f (\frac{\eta}{2}) \geq 1$ and $(\frac{\eta}{2}) r h (\frac{\eta}{2}) \geq 1$, f, h are kernels associated with M, therefore,

$$\left(M \left(\frac{|x_{k,l}^i - x_{k,l}^j|}{g_M(x^i - x^j)}, \frac{|y_{k,l}^i - y_{k,l}^j|}{g_M(y^i - y^j)} \right) \right) \leq r \left(\frac{\eta}{2}\right) f \left(\frac{\eta}{2}\right).$$

so (x^i, y^i) be sequences in $\mathbb{C}^2$.

Thus, when $(0 < \epsilon < 1)$, $\exists\ N$ s.t

$$|(x^i - x, y^i - y)| < \epsilon$$

By use continuity for M, find that

$$\left(\sum_{k=1}^{N} \sum_{l=1}^{N} inf sup \left\{ \left(M \left(\frac{|x_{k,l}^i - x|}{\rho}, \frac{|y_{k,l}^i - y|}{\rho} \right) \right)^{p_{k,l} p_{k,l}} \right\} \right)^{1/H} \leq 1.$$

Taking infimum for such $\rho's$, so

$$inf \left\{ \rho^{p_{k,l}/H} : \left(\sum_{k=1}^{N} \sum_{l=1}^{N} inf sup \left\{ \left(M \left(\frac{|x_{k,l}^i - x|}{\rho}, \frac{|y_{k,l}^i - y|}{\rho} \right) \right)^{p_{k,l}} \right\} \right)^{1/H} \leq 1 \right\} < \epsilon$$

$\forall i \geq N\ \&\ j \to \infty$.

Now, $(x^i, y^i) \in ℸ(p)$ & Mbe continuous, so $(x, y) \in ℸ(p)$.

This is completed a prove for a theorem.

Theorem 3.4: *If* $p = (p_{k,l})$ *is bounded Then class for the sequence space* $W^2(M, p)$, *and the* $W_0^2(M, p)$ & $W_\infty^2(M, p)$ *be linears space on* $\mathbb{C}^2$.

Definition 3.5: The function $M(x, y)$ satisfies Δ-conditions each of value for the x, y, if $\exists K > 0$ s.t $M(2x, 2y) \leq KM(x, y), \forall x \geq 0, y \geq 0$.

Lemma 3.6: *If* $M(x, y)$ *the functions that satisfied* Δ*-conditions,* & $0 < \delta <$ *Then* $\forall x \geq \delta, y \geq \delta$, *so* $M(x, y) = K \frac{1}{\delta}(x, y) M(2,2)$, *when constant* $K > 0$.

Theorem 3.7*: For positive functions* M *that satisfied* $\triangle_2$*-conditions, then*

1) $(\mathbb{C}^2, 1,1) \subset W^2(M)$.

2) $(\mathbb{C}^2, 1,1)_0 \subset W_0^2(M)$.
3) $(\mathbb{C}^2, 1,1)_\infty \subset W_\infty^2(M)$.

Theorem 3.8:
1) *If* $0 < inf p_{k,l} \leq p_{k,l} \leq 1$. *so*
$$W^2(M,p) \subseteq W^2(M).$$
2) *If* $1 \leq p_{k,l} \leq sup\, p_{k,l} < \infty$. *so,* $W^2(M) \subseteq W^2(M,p)$.

Proof:
1) If $(x,y) = (x_{k,l}, y_{k,l}) \in W^2(M,p)$, but $0 < inf p_{k,l} \leq 1$, so,
$$\frac{1}{mn}\sum_{k=1}^{m}\sum_{l=1}^{n} inf sup\left\{M\left(\left(\frac{|x_{k,l}-\ell_1|}{\rho}\right),\left(\frac{|y_{k,l}-\ell_2|}{\rho}\right)\right)\right\}$$
$$\leq \frac{1}{mn}\sum_{k=1}^{m}\sum_{l=1}^{n} inf sup\left\{M\left(\left(\frac{|x_{k,l}-\ell_1|}{\rho}\right)^{p_{k,l}},\left(\frac{|y_{k,l}-\ell_2|}{\rho}\right)^{p_{k,l}}\right)\right\}$$
Thus $(x,y) \in W^2(M)$.
2) Let $p_{k,l} \geq 1, \forall k$ & l, $sup\, p_{k,l} < \infty$.
If $(x,y) = (x_{k,l}, y_{k,l}) \in W^2(M)$. so, where $0 < \epsilon < 1$, we have $(0,0) < (\epsilon,\epsilon) < (1,1)$, $\exists N$ s.t
$$\frac{1}{mn}\sum_{k=1}^{m}\sum_{l=1}^{n} inf sup\left\{M\left(\left(\frac{|x_{k,l}-\ell_1|}{\rho}\right),\left(\frac{|y_{k,l}-\ell_2|}{\rho}\right)\right)\right\} \leq \epsilon < 1,$$
$\forall m,n \geq N$. So, we get
$$\frac{1}{mn}\sum_{k=1}^{m}\sum_{l=1}^{n} inf sup\left\{M\left(\left(\frac{|x_{k,l}-\ell_1|}{\rho}\right)^{p_{k,l}},\left(\frac{|y_{k,l}-\ell_2|}{\rho}\right)^{p_{k,l}}\right)\right\}$$
$$\leq \frac{1}{mn}\sum_{k=1}^{m}\sum_{l=1}^{n} inf sup\left\{M\left(\left(\frac{|x_{k,l}-\ell_1|}{\rho}\right),\left(\frac{|y_{k,l}-\ell_2|}{\rho}\right)\right)\right\}.$$
Hence, $(x,y) \in W^2(M,p)$.

4. Conclusion

In this work, a new function is defined with certain properties; it's called a positive function. Some results are presented that were obtained by studying many sequence spaces defined by this positive function. The convergence of these spaces is constructed and the space of this function is proved linear, total and complete paranormed. Many good results and properties are established and discussed in this study.

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