Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 29 (2026), No. 5-B, pp. 1515–1518
DOI : 10.47974/JIM-2541

Convergence of fractional differential equations using efficient technique

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Abstract

Differential equations play a role in Applied Physics; it's not always feasible to find analytical solutions for nonlinear partial differential equations when dealing with certain phenomena. In this instance, we provide series solutions using a semi-analytical approach. These approaches' solutions are looked for as series. The basic principle of semi-analytical approaches is to determine the series' other terms from specified initial conditions. Some semi-analytic techniques can achieve extremely good convergence with only a few series terms, but other issues may require more terms to improve convergence to the analytical solution. This study uses the Variational Iteration Adomian Decomposition Method (VIADM) to investigate the convergence of approximate-analytical solutions of certain kinds of nonlinear differential equations.

Subject Classification: *35R11, 35R60.*

Keywords: *Laplace transform, Semi analytical method, Ordinary differential equations, Fractional order.*

1. Introduction

Many researchers have extensively investigated the Variational Iteration Method (VIM) for solving a wide range of scientific and engineering problems. VIM is an efficient analytical technique that provides approximate solutions to nonlinear equations without requiring linearization or small perturbation assumptions. Owing to its simplicity, flexibility, and high accuracy, the method has been successfully applied to various nonlinear differential equations arising in mathematics, physics, and engineering. The VIM is used to solve and manage certain kinds of nonlinear differential and integral equations. The VIM has since been adapted to solve additional problems, including integro-differential

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equations, linear and nonlinear boundary value problems, integro-differential equations (both linear and nonlinear), non-linear fractional differential equations, and non-linear fractional integro-differential equations. Additionally, various studies have been conducted to test the convergence of the VIM by determining a number of prerequisites for solving a variety of problems, including fractional differential equations, ODEs, PDEs, integral equations, and integro-differential equations. If a solution is found, the VIM provides convergent successive approximations of it relatively quickly. If not, only a few approximations are suitable for numerical applications [1-11].

2. An Analysis of the Semi-Analytical Methods

The approximate solution using Modified ADM given by when $\lambda = -1$:

$$\mathcal{Z}_{n+1}(x,t) = \mathcal{Z}_n(x,t) - J^{\mathfrak{h}}\{{}^cD_t^{\mathfrak{h}}\mathcal{Z}_n(x,r) + L(\mathcal{Z}_n(x,r)) + N(\mathcal{Z}_n(x,r)) - g(x,r)\}$$

Since $\mathcal{Z}$ is the exact solution of the differential equation then

$$\mathcal{Z}(x,t) = \mathcal{Z}(x,t) - J^{\mathfrak{h}}\{{}^cD_t^{\mathfrak{h}}\mathcal{Z}(x,r) + L\big(\mathcal{Z}(x,r)\big) + N\big(\mathcal{Z}(x,r)\big) - g(x,r)\}$$

Let, $\mathfrak{J}_n = \mathcal{Z}_n - \mathcal{Z}$. Subtracting $\mathcal{Z}$ to $\mathcal{Z}_{n+1}$ yields to:

$$\mathfrak{J}_{n+1}(x,t) = \mathfrak{J}_n(x,t) - J^{\mathfrak{h}}\{{}^cD_t^{\mathfrak{h}}(\mathcal{Z}_n - \mathcal{Z}) + L(\mathcal{Z}_n - \mathcal{Z}) + N(\mathcal{Z}_n - \mathcal{Z})\}$$

Since C is constant and $\mathfrak{J}_n(0) = \mathcal{Z}_n(0) - \mathcal{Z}(0)$, and $\mathcal{Z}_n(0) = \mathcal{Z}(0)$, Then,

$$\mathfrak{J}_{n+1}(x,t) = J^{\mathfrak{h}}\{L(\mathcal{Z}_n - \mathcal{Z}) + N(\mathcal{Z}_n - \mathcal{Z})\}$$
$$= \frac{1}{\Gamma(\mathfrak{h})}\int_0^t (t-r)^{\mathfrak{h}-1}\{L(\mathcal{Z}_n - \mathcal{Z}) + N(\mathcal{Z}_n - \mathcal{Z})\}$$

By taking the maximum norm of two sides of $\mathfrak{J}_{n+1}(x,t)$, will give:

$$\|\mathfrak{J}_{n+1}(x,t)\| = \left\| \frac{1}{\Gamma(\mathfrak{h})}\int_0^t (t-r)^{\mathfrak{h}-1}\{L(\mathcal{Z}_n - \mathcal{Z}) + N(\mathcal{Z}_n - \mathcal{Z})\} \right\| dr$$

$$\leq \frac{1}{\Gamma(\mathfrak{h})}\int_0^t \|t-r\|^{\mathfrak{h}-1}\{\|L(\mathcal{Z}_n - \mathcal{Z})\| + \|N(\mathcal{Z}_n - \mathcal{Z})\|\}\,dr \leq \frac{1}{\Gamma(\mathfrak{h})}\int_0^t \max_{r\in[0,t]}|t - r|^{\mathfrak{h}-1}L_1\,\|\mathcal{Z}_n - \mathcal{Z}\| + L_2\|\mathcal{Z}_n - \mathcal{Z}\|\,dr$$

$= \frac{1}{\Gamma(\mathfrak{h})}\int_0^t t^{\mathfrak{h}-1}\,(L_1 + L_2)\|\mathfrak{J}_n\|$ Where, $L_1 + L_2 = S$

Hence, $\|\mathfrak{J}_{n+1}(x,t)\| \leq \frac{S}{\Gamma(\mathfrak{h})}\, t^{\mathfrak{h}-1}\int_0^t \|\mathfrak{J}_n\| dr\ \ \forall\, n = 0, 1, \ldots$

For$n = 0$, $\|\mathfrak{J}_1\| \leq \frac{S}{\Gamma(\mathfrak{h})}\, t^{\mathfrak{h}-1}\int_0^t dr.\ \max\limits_{r\in[0,t]}|\mathfrak{J}_0(r)| = \frac{S}{\Gamma(\mathfrak{h})}\, t^{\mathfrak{h}}\ \max_{r\in[0.t]}\|\mathfrak{J}_0\|$

For $n = 1$, $\|\mathfrak{J}_2\| \leq \frac{S}{\Gamma(\mathfrak{h})}\, t^{\mathfrak{h}-1}\int_0^t \|\mathfrak{J}_1\|\, dr$

$$\leq \frac{S}{\Gamma(\mathfrak{h})}\, t^{\mathfrak{h}-1}\int_0^t \left[\frac{S}{\Gamma(\mathfrak{h})}\, t^{\mathfrak{h}} \max\ \|\mathfrak{J}_0\|\right] dr$$

$$\leq \frac{S^2}{(\Gamma(\mathfrak{h}))^2}\, t^{2\alpha-1}\ \max_{r\in[0,t]}|\mathfrak{J}_0(r)|\int_0^t dr = \frac{S^2}{(\Gamma(\mathfrak{h}))^2}\frac{t^2}{(\mathfrak{h}+1)}\max_{r\in[0,t]}|\mathfrak{J}_0(r)|$$

By the same way, we can get:

$$\|\mathfrak{J}_{n+1}(x,t)\| \leq \left(\frac{S}{\Gamma(\mathfrak{h})}\right)^n \frac{t^{n\alpha}}{(\alpha+1)(2\alpha+1)\ldots((n-1)\,(\alpha+1))}\ \max_{r\in[0,t]}|\mathfrak{J}_0(r)|$$

Since $S < \Gamma(\mathfrak{h}\ \mathrm{a} + \beta)$, so $n \to \infty$, $\|\mathfrak{J}_n(x,t)\| \to 0$, then $\mathcal{Z}_n \to \mathcal{Z}$.

2) When $\lambda = (t - s)$. Since the Modified ADM for the approximate solution is given by:

$$Ƨ_{n+1} = Ƨ_n + J_\alpha[(t - s)\, {}^cD_t^\alpha\, Ƨ_n + L(Ƨ_n) + N(Ƨ_n) - g(x, s)]$$

And y is the exact solution, hence $Ƨ = Ƨ + J_\alpha\big[(t - s)[{}^cD_t^\alpha Ƨ + L(Ƨ) + N(Ƨ) - g(x, s)]\big]$. Subtraction $Ƨ$ to $Ƨ_{n+1}$ will yield to:

$$Ƨ_{n+1} - Ƨ = Ƨ_n - Ƨ + J_\alpha\,[(t - s)({}^cD_t^\alpha\, Ƨ_n - {}^cD_t^\alpha\, Ƨ) + L(Ƨ_n - Ƨ) + N(Ƨ_n - Ƨ)]$$

And since $\Im_n(x, t) = Ƨ_n(x, t) - Ƨ(x, t)$, then:

$$\Im_{n+1}(x, t) = \Im_n(x, t) + J_\alpha\big[(t - s)\big({}^cD_t^\alpha\, \Im_n(x, t)\big) + J^\alpha\big((t - s) + L(Ƨ_n - Ƨ) + N(Ƨ_n - Ƨ_n)\big)\big]$$

Suppose that the interval $[a, b]$ may be scaled to the interval $[0,1]$ and $\sup|s - t| < 1$, then: $\Im_{n+1}(x, t) \leq \Im_n(x, t) - \Im_n(x, t) + \Im_n(0)$

Then,

$$\Im_{n+1}(x, t) = \frac{-1}{\Gamma(\mathfrak{h})} \int_0^t (t - s)^{\alpha - 1}[{}^cD_t^\alpha\, \Im_n(x) + L_1\, \Im_n(x) + L_2\, \Im_n(x)]$$

Now, applying the *supremum norm to the both sides to get:*

$$\|\Im_{n+1}\| \leq \frac{1}{\Gamma(\mathfrak{h})} \int_0^t \|t - s\|^{\alpha - 1}\, [\|\, {}^cD_t^\alpha\, \Im_n(s)\| + (L_1 + L_2)\, \|\Im_n(s)\|\, ds]$$

Let $M = L_1 + L_2$, $0 < M < 1$

$$\leq \frac{S}{\Gamma(\mathfrak{h})} \int_0^t C\, \|\Im_n(s)\| + \|\Im_n(s)\|\, ds \leq \frac{S}{\Gamma(\mathfrak{h})} \int_0^t (C + M)\, \|\Im_n(s)\|\, ds$$

$$\leq \frac{S(M + C)}{\Gamma(\mathfrak{h})} \int_0^t ds\, \|\Im_n(s)\|$$

For $n = 0 \ldots \|\Im_1(s)\| \leq \frac{(M + C)S}{\Gamma(\mathfrak{h})} \int_0^t \|\Im_0(s)\|\, ds = \frac{(M + C)S}{\Gamma(\mathfrak{h})}\, t\, \|\Im_0\|$

If $n = 1 .. \|\Im_1(x)\| \leq \frac{S(M + C)}{\Gamma(\alpha)} \int_0^t \|\Im_1(s)\|\, ds$

$$\leq \frac{S(M + C)}{\Gamma(\mathfrak{h})} \int_0^t \frac{S(M + C)}{\Gamma(\mathfrak{h})}\, \|\Im_0\|\, ds = \frac{S^2(M + C)^2}{(\Gamma(\mathfrak{h}))^2}\, \|\Im_0\|\, \frac{t^2}{2}$$

In general, by using mathematical induction, we have:

$$\|\Im_n(x)\| \leq \frac{(S(M + C))^n}{(\Gamma(\mathfrak{h}))^n}\, \frac{t^n}{n!}\, \|\Im_0\|$$

Since, $S < \frac{\Gamma(\mathfrak{h})}{M + C}$, then $n \to \infty$, we have: $\frac{(S(M + C))^n}{(\Gamma(\mathfrak{h}))^n} \to 0$ and $\frac{1}{n!} \to 0$, then $\Im_n \to 0$.

3. Conclusion

To get an approximate analytical solution for a number of DEs with fractional order. We have successfully used VIADM. The converging series results are rapidly produced, and the suggested technique is easy to apply, appropriate, and successful in achieving the desired outcomes for these problems. Fast and efficient work was done to derive the approximate analytical solution for DEs.

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Received November, 2025