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Bounded solutions for nonlinear delay differential equations of first order

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Abstract

In this work, we use Krasnoselskii's Theorem to prove an existence for the solutions for delay differentials equation under new sufficient conditions and bounded by decreasing functions.

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1. Introduction

A differential equation with a delayed argument in the state variable is called a delay differential equation (DDE). DDEs are used in many disciplines, including economics, biology, chemistry, engineering, and physics. In order to find approximate or numerical solutions. Numerous writers focused on analyzing qualitative traits such as oscillatory properties for a solution, asymptotic behaviors, and stabilities [1-4].

In 2019, Mohamad and Sharba [5] studied solutions to the NDEs of the following form.

$$(a(t)(\eta(t) - \mathcal{U}(t)\eta(\tau(t))')' - \mathcal{V}(t)\mathcal{F}(t, \eta(t), \eta(\sigma(t)), \eta'(t), \eta'(\sigma(t))) = 0.$$

In 2020, Santra et al. [6] consider a differentials equation

$$(\mathcal{W}(t)(\mathcal{Y}'(t))^{\alpha})' + \sum_{j=1}^{m} \mathcal{Z}_j(t)\mathcal{Y}^{c_j}\left(\mathcal{S}_j(t)\right) = 0, \text{ for } t \succ t_0$$

when α and c_j is integer & function $\mathcal{Z}_j$, $\mathcal{W}$ & $\mathcal{S}_j$ continuous.

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In 2023, Joddoa and Sharba [7] focused on finding an existing oscillatory solution to DDEs. :

$$\frac{d^2}{ds^2}\eta(s) = -\sum_{\xi=1}^{\eta}\mathcal{A}_\xi(s)g_\xi\left(\eta\left(F_\xi(s)\right)\right) + \frac{d}{ds}\sum_{\xi=1}^{\eta}\mathcal{B}_\xi(s)G_\xi(s,\eta\left(F_\xi(s)\right))$$

Investigating existences for bounded solution to a follows first-order non-linear DDE is the aim of this study:

$$r(\gamma)\frac{d}{d\gamma}\eta(\gamma) = -b(\gamma)\eta^{w}(\tau(\gamma)) + a(\gamma)\eta^{\epsilon}\left(M\left(\tau(\gamma)\right)\right) \tag{1}$$

where

1) ϵ and $0 \prec w \preccurlyeq 1$, is rationals prime number.
2) $a, b \in C\left([\gamma_0, \infty), (0, \infty)\right), r \in C^1\left([\gamma_0, \infty), (0, \infty)\right), \quad \forall \gamma \succcurlyeq \gamma_0$
3) $\tau \in C^1([\gamma_0, \infty), \mathcal{R}),\ M \in C^1([\gamma_0, \infty), \mathcal{R}), \tau'(\gamma) \succ 0,\ M'(\gamma) \succ 0,$

$$\lim_{\gamma\to\infty} \tau(\gamma) = \lim_{\gamma\to\infty} M(\gamma) = \infty$$

h1. $C(H_1, H_2)$ represent a sets for each continuous function; f: $H_1 \to H_2$ with a supermom norm $\|.\|$.

h2. $\mu_1 r(\gamma) \preccurlyeq a(\gamma) \preccurlyeq \mu_2 r(\gamma) \preccurlyeq b(\gamma) \preccurlyeq \mu_3 r(\gamma),\ \mu_1, \mu_2, \mu_3 \neq 0,$ are positive integer numbers.

Lemma 1: [8] *If $\mathcal{B}$ be closed, and it is bounded, convex sets in X & S_1, S_2, & X the Banach spaces, $\forall x, \gamma \in \mathcal{B}: \longrightarrow X, \ni S_1 x + S_2 \gamma \in \mathcal{B}$, A solutions in $\mathcal{B}$ be the $S_1 x + S_2 \gamma = x$. If S_1 be the contractions & S_2 be the completely continuous.*

Theorem 1 [9, 10]: *Assume that $\mathcal{Y}$ be an integrals functions on the E with dominate $\{\hbar_n\}$ on the E , which means $|\hbar_n(\chi)| \preccurlyeq \mathcal{Y}(x)$ on he E, $\forall$ n, & $\{\hbar_n\}$ the measurables function on E. If $\{\hbar_n\} \to \{\hbar\}$ the point - wise almost everywhere on E, so E be the measurables finite sets & $\hbar$ be the integrables on E such that $\lim_{n\to\infty} \int_E \hbar_n = \int_E \hbar$.*

2. Main Results

Here provide the novel sufficient condition to prove Lemma 1 of the solution of the equation (1).

Theorem 2: *If h2 hold. & there is bounded function $\mathcal{H}, \mathcal{K} \in \left([\gamma_0, \infty), (0, \infty)\right)$, s.t*

$$\ln\left(\frac{\mathcal{H}(\gamma_1) - \mathcal{H}(\gamma)}{\mathcal{K}(\gamma_1) - \mathcal{K}(\gamma)}\right) \succcurlyeq 0\ ,\ \gamma_0 \preccurlyeq \gamma \preccurlyeq \gamma_1 \tag{2}$$

$$\int_\gamma^\infty \mathcal{K}^\epsilon(M(\tau(t)))\, dt \preccurlyeq \frac{1}{\mu_2}\left(\mu_2 \int_\gamma^\infty \mathcal{H}^w\left(F(t)\right) dt + \mathcal{K}(\gamma)\right),$$

$$\frac{1}{\mu_1}\left(\mu_3 \int_\gamma^\infty \mathcal{K}^w\left(F(t)\right) dt + \mathcal{H}(\gamma)\right) \preccurlyeq \int_\gamma^\infty \mathcal{H}^\epsilon\left(M\left(\tau(t)\right)\right) dt, \gamma \succcurlyeq \gamma_1, \tag{3}$$

$$\int_\gamma^\infty \mathcal{K}^w(\tau(t))\, dt \prec \infty \tag{4}$$

so, eq. (1). Have solution bounded by a function $\mathcal{H}$ & $\mathcal{K}$.

Proof: If $I(t) = \int_{\gamma}^{\infty} \mathcal{K}^{w}(\tau(t))\, dt$. Afterward, eq. (4), suggests that

$$\lim_{t\to\infty} I(t) = \lim_{t\to\infty} \int_{\gamma}^{\infty} \mathcal{K}^{w}(\tau(t))\, dt = 0. \tag{5}$$

Let $(C([\gamma_0,\infty),\mathcal{R}), \|.\|)$ such that $\|\eta\| = \sup_{\gamma \geqslant \gamma_0} |\eta(\gamma)|$,then $C([\gamma_0,\infty),\mathcal{R})$ be Banach, and let $\psi \subset C([\gamma_0,\infty), \mathcal{R})$ define as:

$$\psi = \{\eta(\gamma) : \eta(\gamma) \in C([\gamma_0,\infty),\mathcal{R}) \text{ with } \mathcal{H}(\gamma) \leqslant \eta(\gamma) \leqslant \mathcal{K}(\gamma), \gamma \geqslant \gamma_0\} \tag{6}$$

here, ψ be the convex & closed. A definition for φ_1 & $\varphi_2 : \psi \to C([\gamma_0,\infty),\mathcal{R})$ by:

$$(\varphi_1\eta)(\gamma) = \begin{cases} \int_{\gamma}^{\infty} \frac{a(t)}{r(t)} \eta^{\epsilon}\left(\mathrm{M}(\mathrm{F}(t))\right) dt, & \gamma \geqslant \gamma_1, \\ (\varphi_1\eta)(\gamma_1), & \gamma_0 \leqslant \gamma \leqslant \gamma_1, \end{cases}$$

$$(\varphi_2\eta)(\gamma) = \begin{cases} -\int_{\gamma}^{\infty} \frac{b(t)}{r(t)} \eta^{w}(\tau(t))\, dt & ,\gamma \geqslant \gamma_1, \\ (\varphi_2\eta)(\gamma_1) - \mathcal{K}(\gamma_1) + \mathcal{K}(\gamma) & ,\gamma_0 \leqslant \gamma \leqslant \gamma_1, \end{cases} \tag{7}$$

Where φ_1 and φ_2 satisfies eq. (1). $\forall \eta, \lambda \in \psi$ & $\gamma \geqslant \gamma_1$, so:

$$\begin{aligned}(\varphi_1\eta)(\gamma) + (\varphi_2\lambda)(\gamma) &= \int_{\gamma}^{\infty} \frac{a(t)}{r(t)} \eta^{\epsilon}\left(\mathrm{M}(\mathrm{F}(t))\right) dt - \int_{\gamma}^{\infty} \frac{b(t)}{r(t)} \lambda^{w}(\tau(t))\, dt \\ &\leqslant \mu_2 \int_{\gamma}^{\infty} \eta^{\epsilon}\left(\mathrm{M}(\mathrm{F}(t))\right) dt - \mu_2 \int_{\gamma}^{\infty} \lambda^{w}(\tau(t))\, dt \\ &\leqslant \mu_2 \int_{\gamma}^{\infty} \mathcal{K}^{\epsilon}\left(\mathrm{M}(\mathrm{F}(t))\right) dt - \mu_2 \int_{\gamma}^{\infty} \mathcal{H}^{w}(\tau(t))\, dt.\end{aligned}$$

By eq. (3), we have

$$\leqslant \mu_2 \int_{\gamma}^{\infty} \mathcal{H}^{w}(\mathrm{F}(t))\, dt + \mathcal{K}(\gamma) - \mu_2 \int_{\gamma}^{\infty} \mathcal{H}^{w}(\tau(t))\, dt = \mathcal{K}(\gamma)$$

$\forall \gamma \in [\gamma_0, \gamma_1]$, we have

$$\begin{aligned}(\varphi_1\eta)(\gamma) + (\varphi_2\lambda)(\gamma) &= (\varphi_1\eta)(\gamma_1) + (\varphi_2\lambda)(\gamma_1) - \mathcal{K}(\gamma_1) + \mathcal{K}(\gamma) \\ &\leqslant \mathcal{K}(\gamma_1) - \mathcal{K}(\gamma_1) + \mathcal{K}(\gamma) = \mathcal{K}(\gamma).\end{aligned}$$

So, $\forall \gamma \geqslant \gamma_1$, yield:

$$\begin{aligned}(\varphi_1\eta)(\gamma) + (\varphi_2\lambda)(\gamma) &= \int_{\gamma}^{\infty} \frac{a(t)}{r(t)} \eta^{\epsilon}\left(\mathrm{M}(\mathrm{F}(t))\right) dt - \int_{\gamma}^{\infty} \frac{b(t)}{r(t)} \lambda^{w}(\tau(t))\, dt \\ &\geqslant \mu_1 \int_{\gamma}^{\infty} \eta^{\epsilon}\left(\mathrm{M}(\mathrm{F}(t))\right) dt - \mu_3 \int_{\gamma}^{\infty} \lambda^{w}(\tau(t))\, dt \\ &\geqslant \mu_1 \int_{\gamma}^{\infty} \mathcal{H}^{\epsilon}\left(\mathrm{M}(\mathrm{F}(t))\right) dt - \mu_3 \int_{\gamma}^{\infty} \mathcal{K}^{w}(\tau(t))\, dt\,.\end{aligned}$$

By eq. (3), we have

$$\geqslant \mu_3 \int_{\gamma}^{\infty} \mathcal{K}^{w}(\tau(t)\, dt + \mathcal{H}(\gamma) - \mu_3 \int_{\gamma}^{\infty} \mathcal{K}^{w}(\tau(t))\, dt = \mathcal{H}(\gamma)$$

$\forall \gamma \in [\gamma_0, \gamma_1]$, from eq. (2), we secure:

$$\begin{aligned}(\varphi_1\eta)(\gamma) + (\varphi_2\lambda)(\gamma) &= (\varphi_1\eta)(\gamma_1) + (\varphi_2\lambda)(\gamma) - \mathcal{K}(\gamma_1) + \mathcal{K}(\gamma) \\ &\geqslant \mathcal{H}(\gamma_1) - \mathcal{K}(\gamma_1) + \mathcal{K}(\gamma) \geqslant \mathcal{H}(\gamma)\end{aligned} \tag{8}$$

So, $\varphi_1\eta + \varphi_2\lambda \in \psi, \forall \eta, \lambda \in \psi, \eta \succ \lambda$. demonstrate φ_1 the contractions map on ψ. $\forall \eta, \lambda \in \psi$ for $\gamma \geqslant \gamma_1$:

$$\|\varphi_1\eta - \varphi_1\lambda\| = \sup_{t \geqslant t_1} |(\varphi_1\eta)(\gamma) - (\varphi_1\lambda)(\gamma)|$$

$$= \sup_{\gamma \geqslant \gamma_1} \left| \int_\gamma^\infty \frac{a(t)}{r(t)} \eta^\epsilon\Big(M\big(F(t)\big)\Big)\, dt - \int_\gamma^\infty \frac{a(t)}{r(t)} \lambda^\epsilon\Big(M\big(F(t)\big)\Big)\, dt \right|$$

$$\leqslant \sup_{\gamma \geqslant \gamma_1} \left| \mu_2 \int_\gamma^\infty \eta^\epsilon\Big(M\big(F(t)\big)\Big)\, dt - \mu_1 \int_\gamma^\infty \lambda^\epsilon\Big(M\big(F(t)\big)\Big)\, dt \right|$$

$$\leqslant \sup_{\gamma \geqslant \gamma_1} \left| \mu_2 \int_\gamma^\infty \mathcal{K}^\epsilon\Big(M\big(F(t)\big)\Big)\, dt - \mu_1 \int_\gamma^\infty \mathcal{H}^\epsilon\Big(M\big(F(t)\big)\Big)\, dt \right|$$

$$\leqslant \sup_{\gamma \geqslant \gamma_1} \mu_2 \left| \int_\gamma^\infty \mathcal{K}^\epsilon\Big(M\big(F(t)\big)\Big)\, dt - \int_\gamma^\infty \mathcal{H}^\epsilon\Big(M\big(F(t)\big)\Big)\, dt \right|$$

By eq. (3), we have

$$\leqslant \sup_{\gamma \geqslant \gamma_1} \frac{\mu_2}{\mu_3} |\mathcal{K}(\gamma) - \mathcal{H}(\gamma)|$$

$$\leqslant M\|\eta - \lambda\| \tag{9}$$

Where, $M = \frac{\mu_2}{\mu_3}$. Also for $\gamma \in [\gamma_0,\ \gamma_1]$.

$$\|\varphi_1\eta - \varphi_1\lambda\| = \sup_{\gamma_0 \leqslant \gamma \leqslant \gamma_1} |(\varphi_1\eta)(\gamma_1) - (\varphi_1\lambda)(\gamma_1)|$$

$$= \sup_{\gamma \geqslant \gamma_1} \left| \int_{\gamma_1}^\infty \frac{a(t)}{r(t)} \eta^\epsilon\Big(M\big(F(t)\big)\Big)\, dt - \int_{\gamma_1}^\infty \frac{a(t)}{r(t)} \lambda^\epsilon\Big(M\big(F(t)\big)\Big)\, dt \right|$$

$$\leqslant \left| \mu_2 \int_{\gamma_1}^\infty \eta^\epsilon\Big(M\big(F(t)\big)\Big)\, dt - \mu_1 \int_{\gamma_1}^\infty \lambda^\epsilon\Big(M\big(F(t)\big)\Big)\, dt \right|$$

$$\leqslant \left| \mu_2 \int_{\gamma_1}^\infty \mathcal{K}^\epsilon\Big(M\big(F(t)\big)\Big)\, dt - \mu_1 \int_{\gamma_1}^\infty \mathcal{H}^\epsilon\Big(M\big(F(t)\big)\Big)\, dt \right|$$

$$\leqslant \mu_2 \left| \int_{\gamma_1}^\infty \mathcal{K}^\epsilon\Big(M\big(F(t)\big)\Big)\, dt - \int_{\gamma_1}^\infty \mathcal{H}^\epsilon\Big(M\big(F(t)\big)\Big)\, dt \right|$$

By eq. (3), we have

$$\leqslant \sup_{\gamma \geqslant \gamma_1} \frac{\mu_2}{\mu_3} |\mathcal{K}(\gamma_1) - \mathcal{H}(\gamma_1)|$$

$$\leqslant \sup_{t \geqslant t_1} \frac{\mu_2}{\mu_3} |\eta(\gamma) - \lambda(\gamma)|$$

$$\leqslant M\|\eta - \lambda\| \tag{10}$$

Where, $M = \frac{\mu_2}{\mu_3}$. This implies that

$$\|\varphi_1\eta - \varphi_1\lambda\| \leqslant M\|\eta - \lambda\| \tag{11}$$

Hence, φ_1 be the contractions functions on ψ. To shows φ_2 be completely cont. function, shown φ_2 be the continuous map. If $\eta_i = \eta_i(\gamma) \in \psi$. but ψ closed, so $\eta_i(\gamma) \to \eta(\gamma)$ as $\gamma \to \infty, \eta(\gamma) \in \psi$. For $\gamma \geqslant \gamma_1$, yield:

$$\|(\varphi_2\eta_i)(\gamma) - (\varphi_2\eta)(\gamma)\| = \sup_{\gamma \geqslant \gamma_1} |(\varphi_2\eta_i)(\gamma) - (\varphi_2\eta)(\gamma)|$$

$$\leqslant \sup_{\gamma \geqslant \gamma_1} \left| -\int_\gamma^\infty \frac{b(t)}{r(t)} \eta_i^w\big(\tau(t)\big)\ dt + \int_\gamma^\infty \frac{b(t)}{r(t)} \eta^w\big(\tau(t)\big)\ dt \right|$$

$$\leqslant \sup_{\gamma \geqslant \gamma_1} \left|\mu_2 \int_\gamma^\infty \eta_i^w\big(\tau(\text{ŧ})\big)\, d\text{ŧ} - \mu_3 \int_\gamma^\infty \eta^w(\tau(\text{ŧ}))\, d\text{ŧ}\right|$$

$$\leqslant \sup_{\gamma \geqslant \gamma_1} \left|\mu_3 \int_\gamma^\infty \mathcal{K}_i^w\big(\tau(\text{ŧ})\big)\, d\text{ŧ} - \mu_3 \int_\gamma^\infty \mathcal{H}^w(\tau(\text{ŧ}))\, d\text{ŧ}\right|$$

$$\leqslant \sup_{\gamma \geqslant \gamma_1} \mu_3 \left|\int_\gamma^\infty [\mathcal{K}_i^w\big(\tau(\text{ŧ})\big) - \mathcal{K}^w(\tau(\text{ŧ}))]\, d\text{ŧ}\right| \tag{12}$$

so, we have

$$\int_\gamma^\infty \mathcal{K}^w(\tau(\text{ŧ}))\, d\text{ŧ} \prec \infty \tag{13}$$

Since $\left|\mathcal{K}_i^w\big(\tau(\text{ŧ})\big) - \mathcal{K}^w(\tau(\text{ŧ}))\right| \to 0$, as i tend to ∞, By dominative convergence theorem to Lebesgue, yield:

$$\lim_{i \to \infty} \|(\varphi_2 \eta_i)(\gamma) - (\varphi_2 \eta)(\gamma)\| = 0 \tag{14}$$

so φ_2 is cont.

To shows φ_2 be relatively compacts, using theorems for the Arzelã-Ascoli [2]. To shows $\{\varphi_2 \eta : \eta \in \psi\}$ be uniformly bound & equi-cont. on $[\gamma_0, \infty]$, From eq. (6), we have $\{\varphi_2 \eta : \eta \in \psi\}$ be the uniformly bound. To show $\{\varphi_2 \eta : \eta \in \psi\}$ the equicontinuous on $[\gamma_0, \infty)$,

let $\eta \in \psi$ and for all $\varepsilon \succ 0$, by eq. (13), so $\exists\, \gamma_* \succcurlyeq \gamma_1$ large enough:

$$\int_{\gamma_*}^\infty \mathcal{K}^w(\tau(\text{ŧ}))\, d\text{ŧ} \prec \frac{\varepsilon}{2\mu_3}, \; , \; \gamma \succcurlyeq \gamma_* \succcurlyeq \gamma_1 , \tag{15}$$

if $\varepsilon \succ 0$ & $\eta \in \varphi$, $T_2 \succ T_1 \succcurlyeq \gamma_*$, so

$$\|(\varphi_2 \eta_i)(T_2) - (\varphi_2 \eta)(T_1)\| = \sup_{T_2 \succ T_1 \succcurlyeq \text{ŧ}_*} |(\varphi_2 \eta_i)(T_2) - (\varphi_2 \eta)(T_1)|$$

$$\leqslant |(\varphi_2 \eta_i)(T_2)| + |(\varphi_2 \eta)(T_1)|$$

$$\leqslant \int_{T_2}^\infty \frac{b(\text{ŧ})}{r(\text{ŧ})} \eta_i^w(\tau(\text{ŧ}))\, d\text{ŧ} + \int_{T_1}^\infty \frac{b(\text{ŧ})}{r(\text{ŧ})} \eta^w(\tau(\text{ŧ}))\, d\text{ŧ}$$

$$\leqslant \mu_3 \int_{T_2}^\infty \eta_i^w\big(\tau(\text{ŧ})\big)\, d\text{ŧ} + \mu_3 \int_{T_1}^\infty \eta^w\big(\tau(\text{ŧ})\big)\, d\text{ŧ}$$

$$\leqslant \mu_3 \int_{T_2}^\infty \mathcal{K}_i^w\big(\tau(\text{ŧ})\big)\, d\text{ŧ} + \mu_3 \int_{T_1}^\infty \mathcal{K}^w\big(\tau(\text{ŧ})\big)\, d\text{ŧ}$$

$$\leqslant \mu_3 \frac{\varepsilon}{2\mu_3} + \mu_3 \frac{\varepsilon}{2\mu_3} = \varepsilon \tag{16}$$

For $\eta \in \psi$ and $\gamma_1 \leqslant T_1 \prec T_2 \leqslant \gamma_*$, we get

$$\|(\varphi_2 \eta)(T_2) - (\varphi_2 \eta)(T_1)\| = \sup_{\gamma_1 \leqslant T_1 \prec T_2 \leqslant \gamma_*} |(\varphi_2 \eta)(T_2) - (\varphi_2 \eta)(T_1)|$$

$$= \sup_{\gamma_1 \leqslant T_1 \prec T_2 \leqslant \gamma_*} \left|\int_{T_2}^{\gamma_*} \frac{b(\text{ŧ})}{r(\text{ŧ})} \eta^w(\tau(\text{ŧ}))\, d\text{ŧ} - \int_{T_1}^{\gamma_*} \frac{b(\text{ŧ})}{r(\text{ŧ})} \eta^w(\tau(\text{ŧ}))\, d\text{ŧ}\right|$$

$$\leqslant \sup_{\gamma_1 \leqslant T_1 \prec T_2 \leqslant \gamma_*} \mu_3 \left|\int_{T_2}^{\gamma_*} \eta^w\big(\tau(\text{ŧ})\big)\, d\text{ŧ} - \int_{T_1}^{\gamma_*} \eta^w\big(\tau(\text{ŧ})\big)\, d\text{ŧ}\right|$$

$$\leqslant \mu_3 \int_{T_1}^{T_2} \eta^w\big(\tau(\text{ŧ})\big)\, d\text{ŧ} \qquad \leqslant \mu_3 \int_{T_1}^{T_2} \mathcal{K}^w\big(\tau(\text{ŧ})\big)\, d\text{ŧ}$$

$$\leqslant \mu_3 \frac{\varepsilon}{2\mu_3}(T_2 - T_1). \quad \text{so } \exists \delta_1 = \frac{2}{\varepsilon}, \text{ s.t}$$

$$|(\varphi_2 \eta)(T_2) - (\varphi_2 \eta)(T_1)| \prec \varepsilon, \text{if } \; 0 \prec T_2 - T_1 \prec \delta_1 \tag{17}$$

Finally, let $F(\gamma)=\frac{\mathcal{K}(\gamma)}{\frac{a(\gamma)}{r(\gamma)}}$, then for any $\eta \in \varphi$, $\gamma_0 \preccurlyeq T_1 \prec T_2 \preccurlyeq \gamma_1$, by mean value theorem $\exists i_1 \in (T_1, T_2)$& $\delta_2=\frac{\varepsilon}{F'(i_1)} \succ 0$ s.t

$$|(\varphi_2\eta)(T_2)-(\varphi_2\eta)(T_1)| = \left|\left(\frac{\mathcal{K}}{\frac{a}{r}}\right)(T_2)-\left(\frac{\mathcal{K}}{\frac{a}{r}}\right)(T_1)\right|$$
$$= |F(T_2)-F(T_1)|$$
$$= |F'(i_1)|(T_2-T_1) \prec \varepsilon,$$

If
$$0 \prec T_2 - T_1 \prec \delta_2 \prec \delta_1. \tag{18}$$

Thus, $\{\varphi_2\eta : \eta \in \psi\}$ be equi – cont. & uniformly bounded sets.

By **lemma 1**, We has solutions to eq. (1) that be the bounded by a function $\mathcal{H}$ & $\mathcal{K}$.

3. Discussion and Conclusion

This work obtained the novel sufficient condition which proved a lemma 1, which in turn proved to us an existence for a solution. These solutions are bounded by decreasing function. The conditions that were obtained are ideal and effective for determining the no oscillatory property.

References

[1] D. W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations: An Introduction for Scientists and Engineers*, 4th ed. Oxford, U.K.; New York, USA: Oxford University Press, pp. 531 (2007).

[2] B. Balachandran, T. Kalmár-Nagy, and D. E. Gilsinn, Eds., *Delay Differential Equations: Recent Advances and New Directions.* New York, USA: Springer, pp. 350 (2009).

[3] A. H. Jasim, "Focus on the solutions of the delay differential equations," *Journal of Interdisciplinary Mathematics*, vol. 28, no. 3-A, pp. 703–708 (2025), doi: 10.47974/JIM-1965.

[4] J. V. Shaalini and S. E. Fadugba, "A proposed improved ratio method for the solution of IVPs in delay differential equations of various characteristics," *Journal of Interdisciplinary Mathematics*, vol. 28, no. 5, pp. 1795–1806 (2025), doi: 10.47974/JIM-2033.

[5] H. A. Mohamad and B. A. Sharba, "Existence of nonoscillatory relatively bounded solutions of second order neutral differential equations," *Journal of Al-Qadisiyah for Computer Science and Mathematics*, vol. 11, no. 2, pp. 63–71 (2019), doi: 10.29304/jqcm.2019.11.2.549.

[6] S. S. Santra, O. Bazighifan, H. Ahmad, and S. -W. Yao, "Second-Order Differential Equation with Multiple Delays: Oscillation Theorems and Applications," *Complexity*, vol. 8853745, pp. 1–6 (2020), doi: 10.1155/2020/8853745.

[7] A. F. Jaddoa and B. A. Sharba, "Existence and qualitative property of differential equation with delayed arguments," *Iraqi Journal of Science*, vol. 64, no. 9, pp. 4669–4686 (2023).

[8] E. M. Elabbasy, T. S. Hassan, and S. H. Saker, "Oscillation and Non-oscillation of Nonlinear Neutral Delay Differential Equations with Several Positive and Negative Coefficients," *Kyungpook Mathematical Journal*, vol. 47, no. 1, pp. 1-20 (2007).

[9] H. L. Royden and P. M. Fitzpatrick, *Real Analysis*, 4th ed. Upper Saddle River, NJ, USA: Prentice Hall, pp. 544 (2010).

[10] Sarah A. Al-Ameedee and Ahmed J. Obaid, "New results and application of differential quasi subordinations for higher-order derivatives of meromorphic multivalent functions," *Journal of Interdisciplinary Mathematics*, vol. 26, no. 4, pp. 643–650 (2023), doi: 10.47974/JIM-1483.

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