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A novel integral rule that uses the *Nk*1 rule to numerically compute the values of double integrals

Nada A. M. Al-Karamy *
Rana Hasan Hilal †
Zainab Imran Mousa §
Department of Mathematics
Faculty of Education for Women
University of Kufa
Kufa
Najaf 54001
Iraq

Abstract

The aims for the work are to derive a numerical method to find a value for the double integral, which is integrable either continuously or continuously, but its derivative is singular or singular at one or both ends, for an integration using the author rule to find the error formulas associated with it, improving the results using Romberg's acceleration. It delivers high-accuracy results quickly, even with a few partial periods. and comparing a result for a new rule and a result for rule (SS, TT, MM, ST, SM, MT) of accuracy and speed of approach.

Subject Classification: *Primary 93A30, Secondary 49K15.*

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1. Introduction

Double numerical integrals are essential for solving complex problems in various fields where analytical solutions are not possible. It provides accurate approximations of integrals over areas and irregular domains, enabling calculations of areas, volumes, flows, and other quantities in engineering and physics. For example, finding the surface area, evaluating the center of mass and moments of inertia for a plane surface, and calculating volumes under double integrals. Understanding and applying these numerical methods allows for effective analysis and modeling of real-world phenomena.

Alkiffai, Shaaban, and Aljassas [1], presented the composite SUSU method, which uses the SU method with Romberg acceleration to calculate

* *E-mail:* **nadaa.kharmi@uokufa.edu.iq (Corresponding Author)**
† *E-mail:* **ranah.salman@uokufa.edu.iq**
§ *E-mail:* **Zainabu.ghrban@uokufa.edu.iq**

defective double integrals. The method yielded great accuracy in terms of the number of integer orders and the rate at which the real value approached. Hilal [2] also covered two numerical methods for computing numerically defective double integrals when the number of subintervals is unequal. These methods combined the midpoint rule, and Simpson's rule SIM & M and M & SIM, and the results were excellent and highly accurate when it came to approaching the true value quickly with a small number of subintervals. In 2020, a numerical method for calculating single integrals called NK1 was presented. In this area, the other Authors [3-8] also sought to develop numerical techniques for computing double integrals, and all of the techniques were sound and dependable. To compute continuous double numerical integrals (integrals, improper integrals, or continuous integrals with partial derivatives that are improper) in one or both limits of integration, we have examined a complex numerical approach based on the Nk1 rule with Romberg acceleration in this study. Additionally, we have been working to determine the correction limits for the rule in each continuity scenario.

2. Priorities and Basics

Suppose the integral J is defined as $\int_{x_0}^{x_n} f(x)\, dx$ are aware that single integral $\int_{x_0}^{x_n} f(x)\, dx$ value as per the rule $Nk1$

$$\int_{x_0}^{x_n} f(x)dx \simeq \frac{2t}{3}\left[2\sum_{i=1}^{n} f_{\frac{2k-1}{2}} - \sum_{i=1}^{\frac{n}{2}} f_{2i-1}\right] + E_{NK1}(t)$$

Because the rule $(Nk1)$ demands even periods when applied, the number n is even in this case. Using the rule $Nk1$. The error formula for single integrals with continuous integrals is:

$$E_{NK1}(t) = -at^4(f_n^{(3)} - f_0^{(3)}) + bt^6(f_n^{(5)} - f_0^{(5)}) - \cdots \quad (1)$$

Where $a, b, \cdots$ are constant by applying the mean-value theorem to formula (3)'s derivatives, we obtain:

$$E_{NK1}(t) = a(x_n - x_0)t^4 f^{(4)}(\mu_1) + b(x_n - x_0)t^6 f^{(6)}(\mu_2) + \ldots \quad (2)$$

Where $i = 1,2,3,\ldots \quad , \quad \mu_i, \in (x_0, x_n)$.

3. The rule (NN) for computing continuous double integrals:

Suppose the integral J is:

$$J = \int_{y_0}^{y_m} \int_{x_0}^{x_n} f(x,y)\ dxdy \quad (3)$$

Where the integration region $[x_0, x_n] \times [y_0, y_m]$ contains all points where integrals $f(x,y)$ be continuous & differentiable functions. Now, present a numerical method for evaluating double integrals, and then apply the Romberg acceleration method to a value obtained by applying a complex rule $(Nk1)$ to both the outer dimension (y) and the inner dimension (x). In this method, we divide the outer integration period $([y_0, y_m])$ into m subintervals and the inner integration period (x) into n subintervals, resulting in $(t = \overline{t})$.

$$J = \int_{y_0}^{y_m}\int_{x_0}^{x_n} f(x,y)\ dxdy = (NN)(t) + E_{NN}(t) \tag{4}$$

Where $t = \bar{t}$. The inner integral $\int_{x_0}^{x_n} f(x,y)dx$ may be computed numerically use a following formula by applying the $(Nk1)$ rule on the x dimension (treating y as a constant):

$$Nk1 = \int_{x_0}^{x_n} f(x,y)dx = \frac{2\,t}{3}\left[2\sum_{i=1}^{n} f_{\frac{2i-1}{2},\,y} - \sum_{i=1}^{\frac{n}{2}} f_{2i-1\,,\,y}\right]$$
$$+a(x_n - x_0)t^4\frac{\partial^4 f(\mu_2,y)}{\partial x^4} + b(x_n - x_0)t^6\frac{\partial^6 f(\mu_3,y)}{\partial x^6}$$
$$+c(x_n - x_0)t^8\frac{\partial^8 f(\mu_1,y)}{\partial x^8} + \cdots \tag{5}$$

Where $\mu_1, \mu_2, \mu_3, \cdots \in (x_0, x_n)$. Next, we integrate formula (5) with respect to y using rule $(Nk1)$, and the result is:

$$(NN) = \int_{y_0}^{y_n}\int_{x_0}^{x_n} f(x,y)\ dxdy =$$
$$\frac{4t^2}{9}\left[4\sum_{i=1}^{n}\left(\sum_{j=1}^{n} f_{\frac{2i-1}{2},\frac{2j-1}{2}} - \frac{1}{2}\sum_{j=1}^{\frac{n}{2}} f_{\frac{2i-1}{2},\,2j-1}\right) - \sum_{i=1}^{\frac{n}{2}} 2\left(\sum_{j=1}^{n} f_{2i-1\,,\,\frac{2j-1}{2}} - \frac{1}{2}\sum_{j=1}^{\frac{n}{2}} f_{2i-1\,,\,2j-1}\right)\right] +$$
$$\int_{y_0}^{y_n}\left(a(x_n - x_0)t^4\frac{\partial^4 f(\mu_1,y)}{\partial x^4} + b(x_n - x_0)t^6\frac{\partial^6 f(\mu_2,y)}{\partial x^6} + c(x_n - x_0)t^8\frac{\partial^8 f(\mu_3,y)}{\partial x^8} + \cdots\right) dy +$$
$$\frac{2t}{3}\left[\sum_{i=1}^{n} a_i(y_n - y_0)t^4\frac{\partial^4 f\left(x_{\frac{2i-1}{2}},\theta_{1i}\right)}{\partial x^4} + b_i(y_n - y_0)t^6\frac{\partial^6 f\left(x_{\frac{2i-1}{2}},\theta_{2i}\right)}{\partial x^6} + \cdots\right] +$$
$$\frac{2t}{3}\left[\sum_{i=1}^{\frac{n}{2}} a_{i+n}(y_n - y_0)t^4\frac{\partial^4 f(x_{2i-1},\theta_{1i+n})}{\partial x^4} + b_{i+n}(y_n - y_0)t^6\frac{\partial^6 f(x_{2i-1},\theta_{2i+n})}{\partial x^6} + \cdots\right] \tag{6}$$

Where $\theta_{kl} \in (y_0, y_n)$, and $l = 1,2,3,\cdots\frac{3n}{2}$ $k = 1,2,3,\cdots$ and $a, b, \cdots$, $a_r, b_r, \cdots$ are constant wheres $r = 1,2,\cdots\frac{3n}{2}$

By utilizing integration mean value theorem, we obtain:

$$(NN) = \int_{y_0}^{y_n}\int_{x_0}^{x_n} f(x,y)\ dxdy =$$
$$\frac{4t^2}{9}\left[2\sum_{i=1}^{n} 2\left(\sum_{j=1}^{n} f_{\frac{2i-1}{2},\frac{2j-1}{2}} - \frac{1}{2}\sum_{j=1}^{\frac{n}{2}} f_{\frac{2i-1}{2},\,2j-1}\right) - \sum_{i=1}^{\frac{n}{2}} 2\left(\sum_{j=1}^{n} f_{2i-1\,,\frac{2j-1}{2}} - \frac{1}{2}\sum_{j=1}^{\frac{n}{2}} f_{2i-1\,,\,2j-1}\right)\right] +$$
$$(y_n - y_0)(x_n - x_0)\left(at^4\frac{\partial^4 f(\mu_1,\omega_1)}{\partial x^4} + bt^6\frac{\partial^6 f(\mu_2,\omega_2)}{\partial x^6} + ct^8\frac{\partial^8 f(\mu_3,\omega_3)}{\partial x^8} + \cdots\right) +$$
$$t^4\left[\frac{2t}{3}\sum_{i=1}^{n} a_i(y_n - y_0)\frac{\partial^4 f(x_{\frac{2i-1}{2}},\theta_{1i})}{\partial x^4} + \frac{2t}{3}\sum_{i=1}^{\frac{n}{2}} a_{i+n}(y_n - y_0)\frac{\partial^4 f(x_{2i-1},\theta_{1i+n})}{\partial x^4}\right] +$$

$$t^6\left[\frac{2t}{3}\sum_{i=1}^{n} b_i(y_n - y_0)\frac{\partial^6 f(x_{\frac{2i-1}{2}},\theta_{2i})}{\partial x^6} + \frac{2t}{3}\sum_{i=1}^{\frac{n}{2}} b_{i+n}(y_n - y_0)t\frac{\partial^6 f(x_{2i-1},\theta_{2i+n})}{\partial x^6}\right] + \cdots \tag{7}$$

Where $\omega_i \in (y_0, y_n)$ Accordingly, employing the rule

Where $A, B, C, \cdots$Constants whose values depended on a partial derivative for a function f of a variable x & y never rely on t.

4. (NN) for calculating double integrals for Continuous Integrands With Singular Partial Derivatives:

This section deals with how to numerically evaluate double integrals (with partial or improper derivatives that are singular or singular on one or both sides of the region of integration) and find their correction limits. When computing the integral's value numerically and getting close to its exact value, the correction limitations play an important part in showing this. Assume we have integration.

$\int_0^1\int_0^1 y\sqrt{1-xy}dxdy$, where its analytical value, which is equal to the volume of a cube of side length (0.73680629972808), is (0.4), rounded to fourteen decimal places.

The numerical integration value, obtained by applying the rule (NN), is (0.40000000996859). This value corresponds to the volume of a cube with a side length of (0.73680630584884). It is observed that there is a notable disparity in the volumes of the two cubes.

It can be observed that the difference in volume between the two cubes decreased when the correction limits affecting the impairment were added. However, when the correction limits affecting the impairment were added, the value of integration was (0.400000000000000), which is equivalent to the volume of a cube whose side length is (0.73680629972808) as shown in Figure 1. These results were obtained by adding correction limits while ignoring the impairment.

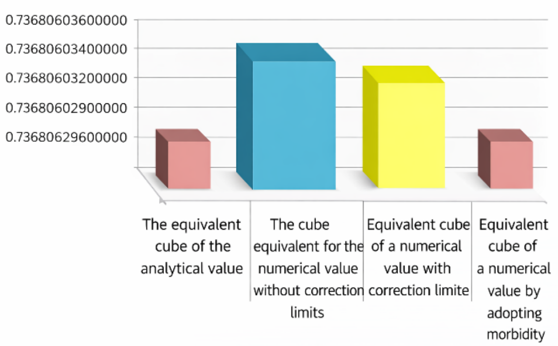

Figure 1

Numerical and analytical cube values obtained with and without correction limits.

The first case: - Suppose we have integration

$$J = \int_{y_0}^{y_n} \int_{x_0}^{x_n} f(x\ ,\ y)\ dx\ dy \tag{9}$$

The aforementioned integration can be written like this:

$$J = \int_{y_0}^{y_n} \int_{x_0}^{x_n} f\ (x\ ,\ y)\ dx\ dy = \int_{y_0}^{y_2} \int_{x_0}^{x_2} f(x\ ,\ y\ )\ dx\ dy$$
$$+ \int_{y_0}^{y_2} \sum_{i=2,4,6,\cdots}^{n-2} \int_{x_i}^{x_{i+2}} f(x\ ,\ y)\ dx\ dy$$
$$+ \sum_{i=2,4,6,\cdots}^{n-2} \int_{y_i}^{y_{i+2}} \int_{x_0}^{x_2} f(x\ ,\ y)\ dx\ dy + \int_{y_2}^{y_n} \int_{x_2}^{x_n} f(x\ ,\ y)\ dx\ dy \tag{10}$$

So, we obtain the following for the first integral in the subregion$[x_0,\ x_2] \times [y_0,\ y_2]$ … (11):

$$\int_{y_0}^{y_2} \int_{x_0}^{x_2} f(x\ ,\ y)\ dx\ dy = \Big[4t^2 - 4t^3 De_y - 4t^3 De_x + 16t^4 \frac{De_x^2}{3!} + 16t^4 \frac{De_y^2}{3!} + 4t^4 De_{xy} - 32t^5 \frac{De_x^3}{4!} - 32t^5 \frac{De_y^3}{4!} - 8t^5 \frac{De_x^2 De_y}{3} - 8t^5 \frac{De_y^2 De_x}{3} + \cdots \Big] f(x_2\ ,\ y_2) \tag{12}$$

Equation (11) is changed with the following point $4f_{\frac{1}{2},\frac{1}{2}}, 4f_{\frac{1}{2},\frac{3}{2}}, -2f_{\frac{1}{2},1}, 4f_{\frac{3}{2},\frac{1}{2}}, 4f_{\frac{3}{2},\frac{3}{2}}, -2f_{\frac{3}{2},1}, -2f_{1,\frac{1}{2}}, -2f_{1,\frac{3}{2}}, 2f_{1,1},$ … (13), (14) and the result is added using equation (12) after being multiplied by $\frac{4t^2}{9}$ So we get:

$$\int_{y_0}^{y_2} \int_{x_0}^{x_2} f(x\ ,\ y)\ dx\ dy =$$
$$\frac{4t^2}{9}\Big[4 \sum_{i=1}^{n} \Big(\sum_{j=1}^{n} f_{\frac{2\,i-1}{2},\frac{2\,j-1}{2}} - \frac{1}{2} \sum_{j=1}^{\frac{n}{2}} f_{\frac{2\,i-1}{2},\,2\,j-1} \Big) -$$
$$\sum_{i=1}^{\frac{n}{2}} 2 \Big(\sum_{j=1}^{n} f_{2\,i-1,\,\frac{2\,j-1}{2}} - \frac{1}{2} \sum_{j=1}^{\frac{n}{2}} f_{2\,i-1,\,2\,j-1} \Big) \Big] +$$
$$[at^6 (De_x^4 + De_y^4) + ct^8 (De_x^6 + De_y^6 + De_x^4 De_y^2 + De_y^4 De_x^2 + De_x^5 De_y +$$
$$De_y^5 De_x + De_x^3 De_y^3 + De_y^3 De_x^3)$$
$$+dt^9 (De_x^7 + De_y^7 + De_x^5 De_y^2 + De_y^5 De_x^2 + De_x^6 De_y^1 + De_y^6 De_x^1 + \cdots) + \cdots]$$
$$f(x_1\ ,\ y_1) \tag{15}$$

The values for a second, third, fourth integral can be found using formula (8) by adding the equations that result from the integration of the second, third, and fourth integrals using formula (8) with equation (15). As for an other integral, we note an integrations functions are continuous at each points for its integrals area the result is

$$(NN) = \int_{y_0}^{y_n} \int_{x_0}^{x_n} f(x,\ y)\ dx\ dy =$$
$$\frac{4t^2}{9}\Big[4 \sum_{i=1}^{n} (\sum_{j=1}^{n} f_{\frac{2\,i-1}{2},\frac{2\,j-1}{2}} - \frac{1}{2} \sum_{j=1}^{\frac{n}{2}} f_{\frac{2\,i-1}{2},\,2\,j-1}) - \sum_{i=1}^{\frac{n}{2}} 2 (\sum_{j=1}^{n} f_{2\,i-1,\,\frac{2\,j-1}{2}} -$$
$$\frac{1}{2} \sum_{j=1}^{\frac{n}{2}} f_{2\,i-1,\,2\,j-1}) \Big] + At^4 + Bt^6 + Ct^8 + \cdots$$

$$\Big[at^6(De_x^4+De_y^4)+ct^8(De_x^6+De_y^6+De_x^4De_y^2+De_y^4De_x^2+De_x^5De_y + De_y^5De_x+De_x^3De_y^3+De_y^3De_x^3)+ dt^9\big(De_x^7+De_y^7+De_x^5De_y^2+De_y^5De_x^2+De_x^6De_y^1+De_y^6De_x^1+\cdots\big)+\cdots\Big] f(x_1,\ y_1) \tag{19}$$

Where $A, B, C, \cdots a, c, d, \cdots$ are constant

The second case: Suppose that a partials derivative for the functions $f(x,y)$ not define at the (x_n, y_n), which means that the Tyler series of functions with two variables is present at every points for a integrations regions except a points (x_n, y_n). For the last integral in the subregion $[x_{n-2}, x_n] \times [y_{n-2}, y_n]$, by using Tyler series of functions $f(x,y)$ about a points (x_{n-2}, y_{n-2}), and in the same way as before, the terms of the correction associated with the base NN can be found

$$(NN)=\int_{y_0}^{y_n}\int_{x_0}^{x_n} f(x\ ,\ y)\ dx\ dy =$$

$$\frac{4t^2}{9}\Big[4\sum_{i=1}^{n}(\sum_{j=1}^{n} f_{\frac{2\,i-1}{2},\frac{2\,j-1}{2}}-\frac{1}{2}\sum_{j=1}^{\frac{n}{2}} f_{\frac{2\,i-1}{2},\,2\,j-1})-\sum_{i=1}^{\frac{n}{2}}2(\sum_{j=1}^{n} f_{2\,i-1,\,\frac{2\,j-1}{2}}- \frac{1}{2}\sum_{j=1}^{\frac{n}{2}} f_{2\,i-1\,,\,2\,j-1})\Big]+At^4+Bt^6+Ct^8+\cdots$$

$$\Big[at^6(De_x^4+De_y^4)+ct^8(De_x^6+De_y^6+De_x^4De_y^2+De_y^4De_x^2+De_x^5De_y + De_y^5De_x+De_x^3De_y^3+De_y^3De_x^3)+ dt^9\big(De_x^7+De_y^7+De_x^5De_y^2+De_y^5De_x^2+De_x^6De_y^1+De_y^6De_x^1+\cdots\big)+\cdots\Big] f(x_{n-1}\ ,\ y_{n-1})(17)$$

Third case: For the first integral in the subregion, $[x_0, x_2] \times [y_0, y_2]$ we use the Tyler series for a function $f(x\ ,\ y)$ about a point $(x_2\ ,\ y_2)$, while the good integral in the partial region $[x_{n-2}, x_n] \times [y_{n-2}, y_n]$ we use the Tyler series for a function $f(x\ ,\ y)$about a point $[x_{n-2}, x_n] \times [y_{n-2}, y_n]$, and in the same way as before, the terms of correction associated with the base can be found and it will be

$$(NN)=\int_{y_0}^{y_n}\int_{x_0}^{x_n} f(x\ ,\ y)\ dx\ dy =$$

$$\frac{4t^2}{9}\Big[4\sum_{i=1}^{n}(\sum_{j=1}^{n} f_{\frac{2\,i-1}{2},\frac{2\,j-1}{2}}-\frac{1}{2}\sum_{j=1}^{\frac{n}{2}} f_{\frac{2\,i-1}{2},\,2\,j-1})-\sum_{i=1}^{\frac{n}{2}}2(\sum_{j=1}^{n} f_{2\,i-1,\,\frac{2\,j-1}{2}}- \frac{1}{2}\sum_{j=1}^{\frac{n}{2}} f_{2\,i-1\,,\,2\,j-1})\Big]$$

$$\Big[a_1t^6(De_x^4+De_y^4)+c_1t^8(De_x^6+De_y^6+De_x^4De_y^2+De_y^4De_x^2+De_x^5De_y + De_y^5De_x+\cdots)+\cdots\Big]f(x_1\ ,\ y_1)$$

$$\Big[a_2t^6(De_x^4+De_y^4)+c_2t^8(De_x^6+De_y^6+De_x^4De_y^2+De_y^4De_x^2+De_x^5De_y+ De_y^5De_x+\cdots)+\cdots\Big]f(x_{n-1}\ ,\ y_{n-1}) \tag{18}$$

Where $a_1, c_1, \cdots a_2, c_2, \cdots$is constant.

In the following Table 1, we compare the results of the rules SS, TT, MM, ST, SM, MT, and prove (the number of correct decimal places reached and the number of subintervals associated with them) with the results of the rule(NN) applied to several double integrals. We find that the number of correct decimal places in the values of the integrals when applyingNN It was more than its counterparts when applying the rules SS, TT, MM, ST, SM, MT and at the same partial periods.We have attempted to improve the results of the integrations by utilizing the Romberg acceleration to take advantage of the correction limits that come with the rule. The results of five of these integrations are documented in the tables that are linked to the study. In order to reduce time and effort, we have listed in the following table the summary of each integration in terms of the number of integer rankings we attained and the number of subintervals that go along with them.

Whereas

1. The integrals and their precise values are listed in the first column.
2. The second column lists the number of right ranks that are acquired after applying the ruleNN along with the number of related sub-periods.
3. When using the rules SS, TT, MM, ST, SM, and MT, respectively, columns 2 through 7 show the number of right rankings attained along with the number of sub-periods linked to them.
4. The ninth column shows the number of integers ranks and the accompanying sub-intervals that arise from using the rule NN with Romberg acceleration.
5. The eighth column contains the integration status in terms of continuity and singularity.

Table 1

Integrals, exact values, numerical ranks, sub-intervals, and continuity status are obtained by different numerical integration rules.

T	Integration	The number of integer ranks obtained when using the rule (NN) and comparing it with the results of the rules (SS, TT, MM, ST, SM, MT) at the same number of sub-intervals							Integration status	IImproved results after taking advantage of the correction limits associated with the rule
		NN/n	SS/n	TT/n	MM/n	ST/n	SM/n	MT/n		RNN /n
1	$\int_1^2 \int_0^1 xe^{(-x-y)}\, dxdy$ 0.061447728197321	8/32	8/32	4/32	5/32	4/32	3/32	4/32	The integrator is Continuous in the integration region	16/32
2	$\int_{0.5}^1 \int_{0.5}^1 sin^{-1}(\frac{x+y}{2})$ $dxdy$ 0.2160911888398	7/128	7/128	5/128	6/128	5/128	6/128	6/128	The derivative of the integration is not defined at the maximum limit of the integration	13/128

3	$\int_0^1 \int_0^1 \frac{y}{\sqrt{x+y}} dxdy$ 0.48758056659898	8/256	6/256	6/256	6/256	5/256	5/256	5/256	Integration is not defined at minimum integration	14/256
4	$\int_0^1 \int_0^1 -\frac{ln(x+y)}{x+y} dxdy$ 0.90584134720169	2/256	1/256	1/256	1/256	1/256	0/256	2/256	The integrator is not defined and the integral is derived at the minimum integration	12/256

Table 2

Example of integration value

n	NN	k=4	k=6	k=8	k=10	k=12
2	2.08325051279700					
4	2.08320190202717	2.08319866130918				
8	2.08319779370608	2.08319751981800	2.08319750169910			
16	2.08319751427721	2.08319749564862	2.08319749526498	2.08319749523975		
32	2.08319749642523	2.08319749523510	2.08319749522854	2.08319749522840	2.08319749522839	
64	2.08319749530328	2.08319749522848	2.08319749522838	2.08319749522838	2.08319749522838	2.08319749522838
$\int_2^3 \int_2^3 \sqrt[y]{xy}\, dxdy$Unknown value analysis						

5. Conclusions

We conclude that the NN integral rule is regarded as a good rule for calculating continuous or improper binary integrals or improper derivatives at one or both ends of the integration, based on the results of the integrations, some of which we have mentioned in Table 1. The accuracy of the results we obtained in the case of continuous integrals within the integration region ranged between (7-8) decimal places in partial periods ranging between (32-128). By contrasting these outcomes with those that come from utilizing the same integrations and number of partial periods while following the rules (SS, TT, MM, ST, SM, MT). We conclude that the integral rule NN produced superior results than the rules TT, MM, ST, SM, and MT, but its results were equally accurate as the base (SS) results, utilizing the same integrals and the same number of subintervals. The NN integral rule produced better results than all the other rules when integrals were not defined in the integration region, and the integrator's difficulty was not defined at one or both ends of the integration. (SS, TT, MM, ST, SM, MT) where a number for the partial period is (256) partial periods, an accuracy of integral rules NN results ranges between (2-8) accurate decimal places, and these results are the best as indicated in Table 1. We have achieved greater accuracy with the same number of partial intervals within the integration region, with accuracy ranging from 12 to 16 decimal places. To improve the rule's results, a correction limit was applied using Romberg accelerations, whether to ensure the integral is continuous within the integration region or to handle cases where the integral or its derivative is not defined at one of the integration limits.

We observed that, regardless of how many partial periods of the integration area we increase, the results of the integration $\int_2^3 \int_2^3 \sqrt[y]{xy}\, dxdy$, whose analytical value is unknown, converged to a certain value and then fixed at that value of 2.08319749522838. This suggests that the sequence of correction boundaries is convergent, and, as a result, we conclude that the integration value in Table 2 is 2.08319749522838, which is one of the most significant advantages of using the correction limits.

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