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A new coupled natural transform and Adomian decomposition technique to solve some models of PDEs

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Abstract

Coupled NT and Decomposition technique (NTADM), is a powerful methodology for dealing with PDEs. The technique used NT to eliminate the time of the driving equation, whereas ADM tackles nonlinear variables methodically without resorting to linearization or numerical discretization. This work employs NTADM to solve a fourth-order NPDE and obtains approximate solutions up to the fifth iteration. The features of the solutions emerging from this show effortless, steady behavior in both space and time. The numerical results show that accuracy improves as the approximation order increases. The absolute error also remains below 1 across most of the computational domain, indicating very strong convergence. These results show that NTADM is a strong and reliable technique for analyzing complex high-order nonlinear PDE models.

Subject Classification: *35R11, 35R60.*

Keywords: *Natural transform, Semi-analytical method, Approximate solutions, Fourth-order PDE.*

1. Introduction

High-order partial differential equations naturally arise when traditional second-order models inadequately represent the full intricacies of physical and engineering systems. Examples include elastic beam and plate oscillations, thin-film flow dynamics, and nonlinear transport phenomena in which higher-order effects predominate.

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Although more advanced formulations subsequently arose to address increasingly sophisticated modeling requirements. From an analytical standpoint, these equations are notoriously challenging to analyze due to the interplay among high-order derivatives, nonlinear terms, and spatially variable coefficients. Consequently, precise closed-form solutions are infrequently obtainable. This constraint has driven the development of alternative solution methods that use integral transforms and decomposition techniques.

The Natural Transform offers an efficient method for simplifying time-dependent operators, thus alleviating the computing load of the problem. The Adomian Decomposition Method facilitates the systematic and transparent management of nonlinearities without requiring linearization or discretization. This study employs the integrated NTADM framework for a nonlinear fourth-order partial differential equation. The resulting approximations exhibit stable behavior and enhanced accuracy with elevated approximation orders, underscoring the practical significance of the suggested methodology [1-10].

2. Natural transform

In this paper, we introduced the Natural Transform, a modern integral transform, as an effective tool for solving differential, integral, and fractional equations. The Natural Transform is a natural transformation. By combining aspects of the Laplace transform with the Sumudu transform, it offers an exceptionally flexible framework for operations. Complex derivatives are simplified by the use of the Natural Transform, which converts them into algebraic equations that are more understandable. Here, the main properties of the Natural Transform.

1. Linearity $\boldsymbol{b\mathrm{N}\{g(t)\} + a\mathrm{N}\{f(t)\} = \{bg(t) + \mathrm{N}\{af(t)\}}$
2. Differentiation Property $\boldsymbol{f(0) - F(s,u)\frac{s}{u} = \mathrm{N}\{f'(t)\}}$
3. Integration Property $\boldsymbol{F(s,u)\frac{u}{s} = \mathrm{N}\{\int_0^t f(\tau)d\tau\}}$

3. Applications and Results

Example 3.1: Consider the following nonlinear fourth–order PDE with variable coefficients in both $\mathcal{E}$ and ω:

$$\frac{\partial \upsilon}{\partial \omega}(\mathcal{E},\omega) = \mathcal{E}^2\ \frac{\partial^4 \upsilon}{\partial \mathcal{E}^4}(\mathcal{E},\omega) + \omega\,(\upsilon(\mathcal{E},\omega))^2, \qquad \mathcal{E} \in \mathbb{R},\ \omega \geq 0.$$

With the Initial condition:

$$\upsilon(\mathcal{E},0) = g(\mathcal{E}) = e^{-\mathcal{E}^2} \tag{1}$$

Solution: We will obtain an approximate analytical solution using the Natural Transform (NT) combined with the Adomian Decomposition Method (ADM).

Natural Transform is defined by

$$\mathrm{N}\{f(\omega)\}(s,\upsilon) = F(s,\upsilon) = \frac{1}{\upsilon}\int_0^\infty f(\omega)\, e^{-\frac{s}{\upsilon}\omega}\, d\omega.$$

For the time derivative of $\text{ч}(\mathcal{E},\omega)$, we use: $\text{N}\left\{\frac{\partial \text{ч}}{\partial \omega}(\mathcal{E},\omega)\right\} = \frac{s}{\text{ч}}\check{\text{Ü}}(\mathcal{E},s,\text{ч}) - \frac{1}{\text{ч}}\text{ч}(\mathcal{E},0)$, where $\check{\text{Ü}}(\mathcal{E},s,\text{ч}) = \text{N}\{\text{ч}(\mathcal{E},\omega)\}(s,\text{ч})$.

$$\text{N}^{-1}\left\{\frac{\text{ч}}{s}F(s,\text{ч})\right\}(\omega) = \int_0^{\omega} f(\tau)\,d\tau,$$

if $\text{N}\{f(\omega)\} = F(s,\text{ч})$. Apply $\text{N}\{\cdot\}$ in ω to both sides of (1). For the left-hand side:

$$\text{N}\{\text{ч}_{\omega}\} = \frac{s}{\text{ч}}\check{\text{Ü}}(\mathcal{E},s,\text{ч}) - \frac{1}{\text{ч}}\text{ч}(\mathcal{E},0) = \frac{s}{\text{ч}}\check{\text{Ü}}(\mathcal{E},s,\text{ч}) - \frac{1}{\text{ч}}g(\mathcal{E}).$$

For the right-hand side: $\text{N}\{\mathcal{E}^2\text{ч}_{\mathcal{E}\mathcal{E}\mathcal{E}\mathcal{E}} + \omega\,\text{ч}^2\} = \mathcal{E}^2\,\text{N}\{\text{ч}_{\mathcal{E}\mathcal{E}\mathcal{E}\mathcal{E}}\} + \text{N}\{\omega\,\text{ч}^2\}$.

So, we get $\frac{s}{\text{ч}}\check{\text{Ü}}(\mathcal{E},s,\text{ч}) - \frac{1}{\text{ч}}g(\mathcal{E}) = \mathcal{E}^2\,\text{N}\{\text{ч}_{\mathcal{E}\mathcal{E}\mathcal{E}\mathcal{E}}\} + \text{N}\{\omega\,\text{ч}^2\}$.

$$\check{\text{Ü}}(\mathcal{E},s,\text{ч}) = \frac{g(\mathcal{E})}{s} + \frac{\text{ч}}{s}\left[\mathcal{E}^2\,\text{N}\{\text{ч}_{\mathcal{E}\mathcal{E}\mathcal{E}\mathcal{E}}\} + \text{N}\{\omega\,\text{ч}^2\}\right].$$

$$\text{ч}(\mathcal{E},\omega) = g(\mathcal{E}) + \int_0^{\omega}\left[\mathcal{E}^2\frac{\partial^4\text{ч}}{\partial\mathcal{E}^4}(\mathcal{E},\tau) + \tau\left(\text{ч}(\mathcal{E},\tau)\right)^2\right]d\tau.$$

$$\text{ч}(\mathcal{E},\omega) = \sum_{n=0}^{\infty}\text{ч}_n(\mathcal{E},\omega).$$

The nonlinear term is $N(\text{ч}) = \left(\text{ч}(\mathcal{E},\omega)\right)^2$, ADM rewrites it as a series of Adomian polynomials: $\left(\text{ч}(\mathcal{E},\omega)\right)^2 = \sum_{n=0}^{\infty}\breve{\text{A}}_n(\mathcal{E},\omega)$,

$$\sum_{n=0}^{\infty}\text{ч}_n(\mathcal{E},\omega) = g(\mathcal{E}) + \int_0^{\omega}\left[\mathcal{E}^2\frac{\partial^4}{\partial\mathcal{E}^4}\left(\sum_{n=0}^{\infty}\text{ч}_n(\mathcal{E},\tau)\right) + \tau\left(\sum_{n=0}^{\infty}\breve{\text{A}}_n(\mathcal{E},\tau)\right)\right]d\tau.$$

Then $\text{ч}_0(\mathcal{E},\omega) = g(\mathcal{E}) = e^{-\mathcal{E}^2}$.

$$\text{ч}_{n+1}(\mathcal{E},\omega) = \int_0^{\omega}\left[\mathcal{E}^2\frac{\partial^4\text{ч}_n}{\partial\mathcal{E}^4}(\mathcal{E},\tau) + \tau\,\breve{\text{A}}_n(\mathcal{E},\tau)\right]d\tau.$$

$$\text{ч}_1(\mathcal{E},\omega) = \omega\left(16\mathcal{E}^6 - 48\mathcal{E}^4 + 12\mathcal{E}^2\right)e^{-\mathcal{E}^2} + \frac{\omega^2}{2}e^{-2\mathcal{E}^2}.$$

$$\text{ч}_2(\mathcal{E},\omega) = 8\omega^2(16\mathcal{E}^{12} - 288\mathcal{E}^{10} + 1560\mathcal{E}^8 - 2880\mathcal{E}^6 + 1485\mathcal{E}^4 - 90\mathcal{E}^2)e^{-\mathcal{E}^2}$$
$$+\frac{\omega^3}{3}\left(160\mathcal{E}^6 - 288\mathcal{E}^4 + 48\mathcal{E}^2\right)e^{-2\mathcal{E}^2} + \frac{\omega^4}{4}e^{-3\mathcal{E}^2}.$$

The approximate solution obtained by truncating after ч_2 is

$$\text{ч}^{(2)}(\mathcal{E},\omega) = \text{ч}_0(\mathcal{E},\omega) + \text{ч}_1(\mathcal{E},\omega) + \text{ч}_2(\mathcal{E},\omega).$$

$$\text{ч}^{(2)}(\mathcal{E},\omega) = e^{-\mathcal{E}^2} + \omega\left(16\mathcal{E}^6 - 48\mathcal{E}^4 + 12\mathcal{E}^2\right)e^{-\mathcal{E}^2} + \frac{\omega^2}{2}e^{-2\mathcal{E}^2}$$
$$+8\omega^2\left(16\mathcal{E}^{12} - 288\mathcal{E}^{10} + 1560\mathcal{E}^8 - 2880\mathcal{E}^6 + 1485\mathcal{E}^4 - 90\mathcal{E}^2\right)e^{-\mathcal{E}^2} +$$
$$\frac{\omega^3}{3}\left(160\mathcal{E}^6 - 288\mathcal{E}^4 + 48\mathcal{E}^2\right)e^{-2\mathcal{E}^2} + \frac{\omega^4}{4}e^{-3\mathcal{E}^2}.$$

This $\text{ч}^{(2)}(\mathcal{E},\omega)$ is an explicit analytic approximation to the solution of the nonlinear variable-coefficient fourth-order PDE (1), constructed by the Natural Transform + Adomian Decomposition Method. Higher-order terms $\text{ч}_3, \text{ч}_4, \ldots$ can be generated in exactly the same way:

4. Results and Discussion

The following Figures 1-2 express the results of consecutive error at various values of ω For example (3.1). Also, the figures show that the numerical simulations were performed in order to determine whether the technique described would lead to increased accuracy. Here, we use few iteration terms to get the approximate solutions which is closed to the analytical solution. Results show that the proposed method NTADM achieves high convergence and higher accuracy.

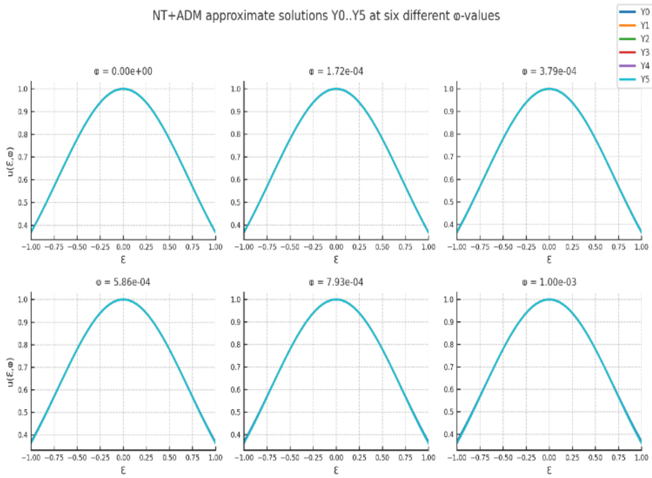

Figure 1
Show the approximate solutions using NT+ADM

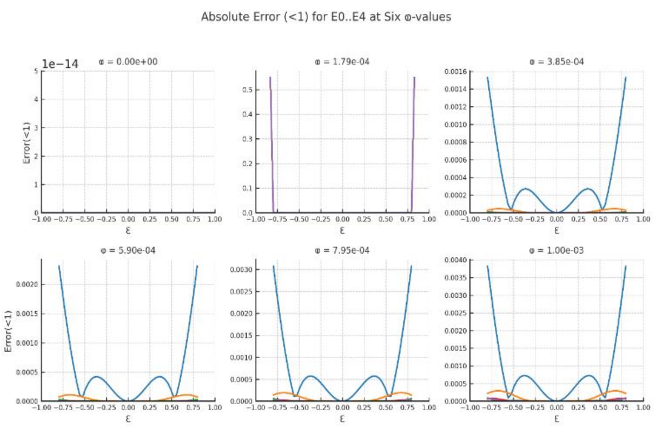

Figure 2
Show the consecutive errors between the current approximate solution and the previous approximate solution

5. Conclusion

The NT+ADM method demonstrated strong capability in accurately approximating the nonlinear fourth-order PDE under study. The solution series Y0 through Y5 showed clear improvement with increasing order, producing smooth, stable behavior across both space and time. Consecutive error plots confirmed that most points satisfy the error <1 criterion, indicating the reliability of the approach in capturing the nonlinear dynamics. Higher-order approximations, particularly Y4 and Y5, closely matched the reference solution, while lower-order approximations still exhibited meaningful qualitative behavior.

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