

The order of close-to-convexity and integral operators of analytic functions of complex order

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Abstract

This paper investigates the orders of close-to-convexity for analytic functions of complex order defined in the open unit disc. Sufficient conditions are established under which certain integral operators transform analytic functions into the class $\kappa(\gamma, \delta)$, representing a generalized form of the classical close-to-convex function class. Several new extensions of previously established results are obtained, together with sharp bounds for the related differential inequalities. The study further enhances the understanding of the geometric behavior of analytic functions under the action of complex-order integral operators.

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1. Introduction

Take $\mathcal{A}$ denoted a family for the analytic function within open unit disc. A normalize criteria $f(0) = f'(0) - 1 = 0$ are satisfied by $\mathfrak{D}$. If functions $g(z) \in \mathcal{S}(Y)$ exists s.t, then a function $f(z)$ of type $\mathcal{A}$ and complex order γ ($\gamma \in \mathbb{C} - \{0\}$) is said to be close to convex.

$$Re \left(1 + \frac{1}{\gamma} \left(\frac{zf'(z)}{g(z)} - 1 \right) \right) > \delta, (z \in \mathfrak{D}). \quad (1)$$

We define the subclass consisting of virtually convex function for the complex orders and category in $\mathfrak{D}$. We underscore that the category of close-to-convex functions delineated by authors in [1] and [2] constitutes the definitive class.

The subclass of $\mathcal{A}$ comprising a practically convex map for the complex orders and type within the open unit disc $\mathfrak{D}$ is represented by $\kappa(\gamma, \delta)$. A family

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for the close-to-convexity map, as proposed by [1, 2], is identical to $\kappa(1, 0)$. The evidence suggests that $f(z)$ in $\mathcal{A}$ meets the criterion:

$$\operatorname{Re}\left\{1 + \frac{1}{\gamma} \frac{zf''(z)}{f'(z)}\right\} > a \quad \left(\frac{1}{2} \leq a < 1\right),$$

in the class $\mathcal{S}$ than $f(z)$. Additionally, the authors in [3] have demonstrated that if the criterion is met by $f(z)$ in $\mathcal{A}$, then:

$$\operatorname{Re}\left\{1 + \frac{1}{\gamma} \frac{zf''(z)}{f'(z)}\right\} > -\frac{1}{2} \quad \left(\frac{1}{2} \leq a < 1\right),$$

then

$$\operatorname{Re}\left(\frac{zf'(z)}{g(z)}\right) > 0 \quad (z \in \mathcal{D}).$$

Recently, the authors in [4] demonstrated that if the following condition is met by $f(z)$ in $\mathcal{A}$:

$$\operatorname{Re}\left\{1 + \frac{1}{\gamma} \frac{zf''(z)}{f'(z)}\right\} > 0 \quad (z \in \mathcal{D}),$$

then

$$\operatorname{Re}\left(\frac{zf'(z)}{g(z)}\right) > \frac{3}{5}, \quad (z \in \mathcal{D}),$$

where $g(z) \in S^*\left(\frac{\alpha}{\alpha+1}\right)$, $\alpha \geq 0$.

Moreover, it has been shown in [5] that the following result holds for any function $f(z)$ in class $\mathcal{A}$ satisfying the condition:

$$\operatorname{Re}\left\{\frac{zf''(z)}{f'(z)} - \beta\right\} > 0 \quad (z \in \mathcal{D}),$$

$$\operatorname{Re}\left(\frac{zf'(z)}{g(z)}\right) > \frac{1}{2\beta-1} \quad (z \in \mathcal{D}),$$

where $g(z) \in S^*$ and $1 < \beta \leq \frac{3}{2}$, where S^* class of Starlike functions.

We demonstrate ordering for the nearly convexity of a function in the class $\kappa(\gamma, \delta)$ found by [6].

2. Preliminaries

We will require the following lemma from [7] in order to demonstrate our findings.

Lemma 1: Assume that f belong to $\mathcal{S}(b)$, $b \in \mathbb{C} - \{0\}$, along with consideration $a \in \mathbb{C} - \{0\}$ such that $0 < 2ab \leq 1$. Subsequently

$$\operatorname{Re}\left(\left(\frac{f(z)}{z}\right)^\alpha\right) > 2^{-2ab} \quad (z \in \mathcal{D}).$$

Theorem 1: In the event that $\mathcal{A}$ and $f(z)$, $g(z)$ the present with criteria are met:

$$\operatorname{Re}\left\{1 + \frac{1}{\gamma} \left(\frac{zf''(z)}{f'(z)}\right)\right\} > 0, \quad (z \in \mathcal{D}), \quad (2)$$

where

$$\operatorname{Im} \left(\frac{a+1}{b} \right) \leq 0 \text{ or } \operatorname{Im} \left(\frac{zf''(z)}{f'(z)} \right) \leq 0,$$

So $f(z) \in \kappa(\gamma, \delta)$, where

$$\delta = 1 + \left(2^{\frac{-2ab}{a+1}} - 1 \right) \operatorname{Re} \left(\frac{a+1}{b} \right).$$

Proof: Let us define the function $g(z)$ as

$$1 + \frac{a+1}{b} \left(\frac{zg'(z)}{g(z)} - 1 \right) = 1 + \frac{1}{b} \left(\frac{zf''(z)}{f'(z)} \right), \quad (3)$$

Then, by conditions (2) and (3), we have $g(z) \in \delta(\gamma)$, with $\gamma = \frac{b}{a+1}$. We can see that (3) implies

$$f'(z) = \left(\frac{g(z)}{z} \right)^{a+1}$$

or

$$\frac{zf'(z)}{g(z)} = \left(\frac{g(z)}{z} \right)^a.$$

$$\begin{aligned} \text{So, } \operatorname{Re} \left\{ 1 + \frac{a+1}{b} \left(\frac{zg'(z)}{g(z)} - 1 \right) \right\} &= \operatorname{Re} \left\{ 1 + \frac{a+1}{b} \left(\left(\frac{g(z)}{z} \right)^a - 1 \right) \right\} \\ &= 1 + \operatorname{Re} \left(\frac{a+1}{b} \right) \operatorname{Re} \left\{ \left(\frac{g(z)}{z} \right)^a - 1 \right\} - \operatorname{Im} \left(\frac{a+1}{b} \right) \operatorname{Im} \left\{ \left(\frac{g(z)}{z} \right)^a - 1 \right\} \\ &\geq 1 + \operatorname{Re} \left(\frac{a+1}{b} \right) \operatorname{Re} \left\{ \left(\frac{g(z)}{z} \right)^a - 1 \right\} \\ &> 1 + (2^{-2a\gamma} - 1) \operatorname{Re} \left(\frac{a+1}{b} \right) \\ &= 1 + \left(2^{\frac{-2ab}{a+1}} - 1 \right) \operatorname{Re} \left(\frac{a+1}{b} \right). \end{aligned}$$

Now, if we set $a = 1$ within Theorem 1, we get the subsequent conclusion:

Corollary 1: If $f \in C(b)$ and we have a positive $b \leq 2$, so $f \in \kappa\left(\frac{b}{2}, \delta\right)$, when

$$\delta = 1 + \frac{2^{1-b}-2}{b}.$$

Assuming that $b = 1$ in Theorem 1, there are

Corollary 2: If the criteria are met & function $f(z)$, $g(z)$ in the set $\mathcal{A}$:

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0 \quad (z \in \mathcal{D}),$$

with $0 < 2a\gamma \leq \gamma = \frac{1}{a+1}$; $\operatorname{Im} \left(\frac{zf'(z)}{g(z)} \right) \leq 0$,

so, $f(z) \in \kappa(\gamma, \delta)$, when

$$\delta = 1 + \left(2^{\frac{2a}{a+1}} - 1 \right) \operatorname{Re}(a+1).$$

3. Analytical functions of complex order with integral operators

Definition 1: According to [8], the integral operator $\mathcal{I}_{\nu,\delta}^n$ on $\mathcal{A}$ in the following manner:

For $n \in N_0 = N \cup \{0\}$, $\delta \geq 0$ with a real value ν and $\nu + \delta > 0$.

Afterward, we define the operator $\mathcal{I}_{\nu,\delta}^n$ for $f \in \mathcal{A}$ by
for $n = 0$,

$$\mathcal{I}_{\nu,\delta}^0 f(z) = f(z)$$

for $n = 1$,

$$\mathcal{I}_{\nu,\delta}^1 f(z) = \left(\frac{\nu+\delta}{\delta}\right) z^{1-\left(\frac{\nu+\delta}{\delta}\right)} \int_0^z t^{\left(\frac{\nu+\delta}{\delta}\right)-2} f(t) dt, z \in \mathcal{D}$$

for $n \geq 2$,

$$\begin{aligned} \mathcal{I}_{\nu,\delta}^n f(z) &= \left(\frac{\nu+\delta}{\delta}\right) z^{1-\left(\frac{\nu+\delta}{\delta}\right)} \int_0^z t^{\left(\frac{\nu+\delta}{\delta}\right)-2} \mathcal{I}_{\nu,\delta}^{n-1} f(t) dt, z \in \mathcal{D} \\ &= \mathcal{I}_{\nu,\delta}^1 \left(\frac{z}{1-z}\right) * \mathcal{I}_{\nu,\delta}^2 \left(\frac{z}{1-z}\right) * \dots * \mathcal{I}_{\nu,\delta}^n \left(\frac{z}{1-z}\right) * f(z). \end{aligned} \quad (4)$$

We note that the integral operator $\mathcal{I}_{\nu,\delta}^n: \mathcal{A} \rightarrow \mathcal{A}$ is

$$\mathcal{I}_{\nu,\delta}^n f(z) = \sum_{i=1}^n \left(\frac{\nu+\delta}{\nu+\alpha_i\delta}\right)^n a_i z^{\alpha_i} \alpha_i > 0. \quad (5)$$

In this part, we established a few prerequisites that must be met of aforementioned integral operator to belong to the class, each of $\mathcal{S}_\alpha^*(b)$ & $\mathcal{C}_\alpha^*(b)$ established by [9].

Theorem 2: Let $\alpha_i, i \in \{1, \dots, n\}$ be real values that include the attributes $\alpha_i > 0$ for $i \in \{1, \dots, n\}$, and

$$0 \leq 1 - \sum_{i=1}^n \alpha_i < 1.$$

We suppose that the functions $f_i \in \mathcal{S}(b)$, if $i \in \{1, \dots, n\}$, & $b \in \mathbb{C} - \{0\}$. So we have the integral operators $\mathcal{I}_{\nu,\delta}^n f(z) \in \mathcal{C}_\gamma(b)$, where $\gamma = 1 - \sum_{i=1}^n \alpha_i$.

Proof: We compute the first and second order derivatives for $\mathcal{I}_{\nu,\delta}^n f(z)$. From (4), we obtain:

$$\begin{aligned} \left(\mathcal{I}_{\nu,\delta}^n f(z)\right)' &= \sum_{i=1}^n \left(\frac{\nu+\delta}{\nu+\alpha_i\delta}\right)^n a_i (\alpha_i) z^{\alpha_i-1}, \\ \left(\mathcal{I}_{\nu,\delta}^n f(z)\right)'' &= \sum_{i=1}^n \left(\frac{\nu+\delta}{\nu+\alpha_i\delta}\right)^n a_i (\alpha_i - 1) z^{\alpha_i-2}. \end{aligned}$$

Then, we have

$$\frac{\left(\mathcal{I}_{\nu,\delta}^n f(z)\right)''}{\left(\mathcal{I}_{\nu,\delta}^n f(z)\right)'} = \frac{\sum_{i=1}^n \left(\frac{\nu+\delta}{\nu+\alpha_i\delta}\right)^n a_i (\alpha_i - 1) z^{\alpha_i-2}}{\sum_{i=1}^n \left(\frac{\nu+\delta}{\nu+\alpha_i\delta}\right)^n a_i (\alpha_i) z^{\alpha_i-1}}. \quad (6)$$

Multiply the relation (6) by z , we have:

$$\frac{z \left(\mathcal{I}_{\nu,\delta}^n f(z)\right)''}{\left(\mathcal{I}_{\nu,\delta}^n f(z)\right)'} = \sum_{i=1}^n \alpha_i \left(\frac{\left(\mathcal{I}_{\nu,\delta}^n f(z)\right)'}{\mathcal{I}_{\nu,\delta}^n f(z)} - 1 \right). \quad (7)$$

Multiply the relation (7) by $\frac{1}{b}$, we have:

$$\begin{aligned} \frac{1}{b} \frac{z(j_{v,\delta}^n f(z))''}{(j_{v,\delta}^n f(z))'} &= \frac{1}{b} \sum_{i=1}^n \alpha_i \left(\frac{(j_{v,\delta}^n f(z))'}{j_{v,\delta}^n f(z)} - 1 \right) \\ &= \sum_{i=1}^n \alpha_i \left(1 + \frac{1}{b} \left(\frac{(j_{v,\delta}^n f(z))'}{j_{v,\delta}^n f(z)} - 1 \right) \right) - \sum_{i=1}^n \alpha_i. \end{aligned} \quad (8)$$

The relation (5) is equivalent to:

$$1 + \frac{1}{b} \frac{z(j_{v,\delta}^n f(z))''}{(j_{v,\delta}^n f(z))'} = \sum_{i=1}^n \alpha_i \left[1 + \frac{1}{b} \left(\frac{(j_{v,\delta}^n f(z))'}{j_{v,\delta}^n f(z)} - 1 \right) \right] - \sum_{i=1}^n \alpha_i + 1.$$

When we take the real component out of the aforementioned inequality, we get:

$$\operatorname{Re} \left\{ 1 + \frac{1}{b} \frac{z(j_{v,\delta}^n f(z))''}{(j_{v,\delta}^n f(z))'} \right\} = \sum_{i=1}^n \alpha_i \operatorname{Re} \left[1 + \frac{1}{b} \left(\frac{(j_{v,\delta}^n f(z))'}{j_{v,\delta}^n f(z)} - 1 \right) \right] - \sum_{i=1}^n \alpha_i + 1.$$

Due we use the inequality (1) with $\delta = 0$ in the relation $f_i \in \mathcal{S}(b)$, above, we get the following results for $i \in \{1, \dots, n\}$:

$$\operatorname{Re} \left\{ 1 + \frac{1}{b} \frac{z(j_{v,\delta}^n f(z))''}{(j_{v,\delta}^n f(z))'} \right\} > 1 - \sum_{i=1}^n \alpha_i.$$

Since $0 \leq 1 - \sum_{i=1}^n \alpha_i < 1$, we have that $j_{v,\delta}^n f(z) \in C_\gamma(b)$, where $\gamma = 1 - \sum_{i=1}^n \alpha_i$.

Corollary 3: Let $\alpha_1 > 0$. If $0 \leq 1 - \alpha_1 < 1$ and the function $f_1 \in \mathcal{S}(b)$, then the integral operator $j_{v,\delta}^1 f(z) \in C_{\mathcal{P}}(b)$, where $\mathcal{P} = 1 - \alpha_1$.

Proof: We get the outcome by setting $n=1$ in Theorem 2.

Corollary 4: Let the function $f_1 \in \mathcal{S}_\lambda^*(b)$. Then the integral operator $j_{v,\delta}^n f(z) \in C_\sigma(b)$, where $\sigma = 1 - \alpha_1$ and $0 \leq 1 - \alpha_1 < 1$.

Proof: We get the outcome by setting $n=1$ in Theorem 2.

Conclusion

In conclusion, this study established sufficient conditions for close-to-convexity of analytic functions of complex order through certain integral operators. Several new extensions and sharp bounds for the associated differential inequalities were obtained, contributing to a deeper understanding of the geometric properties of analytic functions in the open unit disc.

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