

The complex dynamics of rational maps having no Herman ring

Iftichar Mudhar Al-Shara`a *

Department of Mathematics

College of Education for Pure Sciences

University of Babylon

Babylon

Iraq

Hassanein Q. Al-Salami †

Department of Biology

College of Sciences

University of Babylon

Babylon

Iraq

Abstract

This work aims to study the complex dynamics of rational maps, which, according to the Sullivan component, include sets of forward cyclic iterations, which include both equivalent and doubly connected components, where according to study the family of complex rational maps given by $Q_a(z) = 2a^{n+1}z^{-n} + \frac{z^{2n} - a^{2n+1}}{z^n(z^{2n} - a^{n-1})}$ where $n \geq 2$ and $a \in \Lambda := \mathbb{C} \setminus \{0\} - \{a : a^{n+1} = 1 \text{ or } a^{2n+2} = 1 \text{ or } a^{2n+2} = 1\}$. So, the family would be considered a perturbation of a polynomial map. Therefore, we get the results. So Q_a has neither infinitely connected attracting components nor equivalent (equivalent) components if the orbits of all free critical points are not attracted to the origin or infinity. Also, Q_a does not contain Herman's ring. So Q_a has no Siegel disc.

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1. Introduction

One of the important applications of complex dynamics is the Julia set. In fact, we see similarities between the topography of a mountain range and some parts of this set. They are early examples of mathematical chaos, but understanding chaos is important. One of the things we were interested in in our research on classifications of Fatou components was giving some of these

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* E-mail: pure.iftichar.talb@uobabylon.edu.iq (Corresponding Author)

† E-mail: haszno732@gmail.com

classifications in our previous work. By [1], the authors set is connected, so each Fatou component of $\text{cap } Q$ sub a is simply connected. First of the components of the Fatou periodic is the parabolic basin or infinitely connected attracting, and the second is also the Herman ring [2]. The connectivity of the Julia sets of rational maps was studied by authors as in [3]. We can apply Shishikura's [3] results to the rational maps arising from Newton's method for polynomials. Also, this concept was studied by other author, see [4]. The family of renormalization transformation maps was studied by authors, see [5], and other author have been able to generalize to all complex parameters in [6]. The McMullen function forms a Julia set which is a connected was studied in [3], [7]. For constructing Herman rings there are two methods. In [5], [8], the original method is based on a careful analysis of real analytic diffeomorphisms of the circle. authors proved that the Julia set of the $\text{Cap } Q$ sub a , the Escape Quartet Theorem, see [9-10].

2. Main results

If the rational map R with $n \geq 2$ on $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. Let $\mathbb{F}(R)$, the Fatou set is the complement of the Julia set. A forward invariant component U of $\mathbb{F}(R)$ is an attracting component if it contains an attracting fixed point ζ of R ; a super-attracting component if it contains a super-attracting fixed point ζ of R ; a parabolic component if there is a rationally indifferent fixed point ζ of R on the boundary of W_0 , and if $R^n \rightarrow \zeta$ on W_0 ; a Siegel disc if $R : W_0 \rightarrow W_0$ is analytically conjugate to a Euclidean rotation of the unit disc onto itself, a Herman ring if $R : W_0 \rightarrow W_0$ is analytically conjugate to a Euclidean rotation of some annulus onto itself. Now, we offer the main results as:

Theorem A: *The multiply connected attractor or parabolic basins do not come true for Q_a .*

Theorem B: *Q_a has no Herman rings.*

Theorem C: *Q_a has no Siegel disc.*

3. Preliminaries

In this section, we give some of the theories that are an introduction, and we need them for our work.

Lemma 3.1 [10]

For any Fatou (or Julia) component $W \subset \mathbb{C}_\infty$ of Q_a . Then

1. If $\omega^{2n} = 1$, then $Q_a^m(\omega z) = \omega^{-nm} Q_a^m(z)$ for all $\omega \in \mathbb{C}$ and for some $m \geq 1$.
2. The map Q_a satisfy $Q_a \circ \mu(z) = \mu \circ Q_a(z)$, for each $\mu(z) = \frac{a^2}{z}$ and $z \in \mathbb{C}_\infty$.
3. Assume that a Fatou component V of Q_a , therefore $\mu(V) = V$. Especially, $\mu(\beta_0) = \beta_\infty$ also $\mu(\beta_\infty) = \beta_0$.

4. The β_∞ and β_0 have $2n$ -fold symmetry if $z \in \beta_\infty$ or β_0 , so $\omega z \in \beta_\infty$ or β_0 such that $\omega^{2n} = 1$.
5. If $z_0, \omega^i z_0 \in U$, where $\omega^i \neq 1$ for integer i . Then $\omega^k z_0 \in U$ for all k . Especially, U has $2n$ -fold symmetry around 0, where all Fatou components U contain in Q_a .
6. Q_a has a Fatou (or Julia) component V . Then either V does not around 0 and there are $2n$ for components $\omega_0^m V$ where $1 \leq m \leq 2n$ such that $\omega_0^m V \cap \omega_0^i V = \emptyset, \forall m \neq i \pmod{2n}$, or V component of Q_a around 0 and satisfies $\omega_0^m V = V$ for any $m \in \mathbb{Z}$.

4. Proof of the Main Theorems

Now, we prove the first result:

Theorem 4.1: *The multiply connected attractor or parabolic basins do not exist for Q_a .*

Proof: We know that β_∞ and β_0 are simply connected, also all the preimages of $Q_a^{-\kappa}(\beta_\infty)$ and $Q_a^{-\kappa}(\beta_0)$ are simply connected for any $\kappa \in \mathbb{N}$. Assume Q_a has parabolic basins or a cycle periodic attracting other than β_∞ and β_0 with period $s \geq 1$. Set B , connecting a periodic point $v_0 (\neq 0, \infty)$. By [5], suppose that B has a critical point c_a at least. The connectivity of B has the same number of $Q_a^p(B)$ ($1 \leq p \leq s-1$) for any of the components of s - multiply connected attractor or parabolic basin, $Q_a^s(B) = B$. Let C_a (the finite set contains all c_a of Q_a , where $p \geq 1$ be the number of points in $B \cap C_a$. We discuss the cases q satisfies $1 \leq q \leq 2$. Otherwise, $2 < q \leq 4n$. By Lemma 3.1, and if $2 < q < 2n+2$, then $\omega^p B = B$ for any $p \in \mathbb{N}$ also B contain $2n$ free critical points $\omega^p c_a$ ($0 \leq p \leq 2n-1$). On the other hand, we put $\zeta \subset B$ is a Jordan curve that surrounds 0, also $\omega^p \zeta = \zeta$. By Lemma 3.1, $Q_a^{np}(\zeta)$ must surround 0 for any $n \in \mathbb{N}$. Moreover, we note that B is different from β_∞ and β_0 . Thus $Q_a^{np}(\zeta) \rightarrow v_0$ as $n \rightarrow \infty$ for any $\zeta \subset B$. Also, it follows that $Q_a^{np}(\zeta)$ cannot wrap around 0. This is contradicted. Similarly, if $2n+2 \leq q \leq 4n$, we have B contain $4n$ free critical points $\omega^p c_a, \omega^p a^2/c_a$ ($0 \leq p \leq 2n-1$), and we get a contradiction. Now, we have two cases:

Case one:

If $q = 1$, then B is simply connected by the Riemann – Hurwitz formula. There are two subcases, if B is a multiply connected attractor, for a periodic attracting point $v_0 \in B$ with period s also fix $v_p = Q_a^p(v_0) \in Q_a^p(B)$ for $1 \leq p < s$. For any small disk centered at v_0 , say D_0 such that $Q_a^s(D_0) \subset D_0$ also, all points in the critical orbits of Q_a does not belong in the boundary of D_0 . We use that D_{is+p} the connected component of $Q_a^{-(is+p)}(D_0)$ containing v_{s-p} ($v_s = v_0$) for $i \geq 0$ and $0 \leq p < s$. Thus, we have: $D_p \subset D_{s+p} \subset D_{2s+p} \subset \dots \subset D_{is+p} \subset \dots$. (1)

At most one critical point contained in all D_{is+p} and all points in the critical orbits of Q_a does not belong in the boundary of D_0 . By the Riemann – Hurwitz formula, then D_{is+p} is simply connected. From (1), we get a simply connected $B = \bigcup_{i \geq 0} D_{is}$. Assume that B is a multiply connected attractor. For any the parabolic periodic point v_0 belong in the boundary of D_0 such that $Q_a^{is}(v)$ is attracted by v_0 for all $v \in B$. Let $v_p = Q_a^p(v_0) \in Q_a^p(\partial B)$ for $1 \leq p < s$. We take $D_0 \subset B$ be a small disk, since the dynamics of Q_a on the parabolic basins such that $v_0 \in \partial D_0$, $\overline{D_0} \subset B \cup \{v_0\}$, so $Q_a^s(\overline{D_0}) \subset D_0 \cup \{v_0\}$ also all points in the critical orbits of Q_a does not belong in the boundary of D_0 . Thus B is simply connected.

Case two

If $q = 1$, then B have u_1 and u_2 are distinct free critical points. Let $u_2 = \omega^{p_0}u_1$ and from Lemma 3.1, then B contain $2n$ free critical points $\omega^p u_1$ ($0 \leq p \leq 2n - 1$). This is a contradiction, we get $u_2 = \omega^{p_0}\mu(u_1)$ for some $p_0 \in N$. We claim the degree of $Q_a|B$ is equal to three. If the degree of $Q_a|B$ is greater than of three, then by Lemma 3.1, Q_a has more than $3n$ preimages for any $v \in Q_a(B)$ because $\deg(Q_a) = 3$. On the other hand, we remember that $v_0 \in B$ is a periodic point with period s , we get B have at least three preimages of $Q_a(v_0)$ also the boundary of B have at least three preimages of $Q_a(v_0) \in Q_a(\partial B)$ for the parabolic case.

Indeed, it follows that $\omega^p B$ ($0 \leq p \leq 2n - 1$) have $4n$ free critical points. Thus $\omega^p \mu(B)$ ($0 \leq p \leq 2n - 1$) have are $2n$ various components of Fatou consist of $4n$ critical points by Lemma 3.1. Then $\omega^{p_0}\mu(B) = B$ also $\omega^{p_0}\mu(v_0) = v_0$ for some p_0 . Assume that $v_1 \in B$ satisfies $Q_a(v_1) = Q_a(v_0)$, it follows that

$$\begin{aligned} Q_a(\omega^{p_0}\mu(v_1)) &= \omega^{-np_0}\mu(Q_a(v_1)) = \omega^{-np_0}\mu(Q_a(v_0)) \\ &= \omega^{-np_0}\mu(Q_a(\omega^{p_0}\mu(v_0))) \\ &= \omega^{-np_0}\mu(\omega^{-np_0}\mu(Q_a(v_0))) = \omega^{-np_0} \frac{a^2}{\omega^{-np_0}\mu(Q_a(v_0))} = a^2(Q_a(v_0)) = Q_a(v_0). \end{aligned}$$

We fix $\omega^{p_0}\mu(v_1) = v_2$. Then $v_1 = \pm v_0$ if $v_1 = v_2$, from Lemma 2.1 ($2n$ – fold symmetry), we exclude $(v_1 = -v_0)$, since if $v_1 = -v_0$, thus B surrounds 0 also $\omega^m B = B$ for any $m \in N$, so it is a contradiction, since B is various from β_0 . Therefore $v_0 = v_1 = v_2$ also which is $c_a = v_0$, so it local degree consists of three at least three and $\deg Q_a|_{v_0}$ equal to three. Now, if $v_1 \neq v_0$, implies that $v_1 \neq v_2$. Thus v_0, v_1 and v_2 are three distinct points because $\omega^{p_0}\mu(B) = B$, so $v_2 \in B$ and therefore the degree of $Q_a|B$ is 3. If $\omega^{p_0}\mu(D_0) = D_0$, then D_1 contain two critical points c_1 and $c_2 = \omega^{p_0}c_1$ of Q_a , where D_1 the preimage of D_0 under Q_a lie in the s - parabolic basin or periodic attracting. Therefore, the parabolic basin or the periodic attracting B is simply connected from the Riemann-Hurwitz formula.

Proposition 4.2: [10] A rational map F with n greater than one have stable Herman rings $A_0, \dots, A_{n-1}$. Call the Y_i is the subset of A_i is the core curve such that the union divides the Riemann sphere $\mathbb{C}_\infty$ into $n + 1$ connected components $V_0, V_1, \dots, V_n$ for $0 \leq i \leq n - 1$. Then F have disjoint infinite critical orbits $O_F(c_{a_i})$ in $J(F)$ at least $n + 1$ such that $O_F(c_{a_i}) \subseteq V_i \cap J(F)$, for c_{a_i} ,

$0 \leq i \leq n$ is the critical point of F . Proving that $J(Q_a)$ is connected, we need also to prove second result:

Theorem 4.3: Q_a does not include Herman rings.

Proof: From the contradicting, we assume that Q_a has a cycle of Herman rings. Assume that Q_a include s - periodic Herman ring, say, $\{W_0, W_1, \dots, W_s = W_0\}$ where $s \geq 1$. For each component W_0 of the cycle of these Herman rings, so Q_a^s associated with irrational rotation $v \rightarrow \tau v$ in W_0 , where $\tau = e^{2\pi i \theta}$ for θ is an irrational number. So $Q_a: W_p \rightarrow W_{p+1}$ is conformal. From the semiconjugacy we suppose a new rational map $\gamma(v) = v^{2n}$ because Q_a include $2n$ - symmetric also $J(Q_a)$. We define $h_a(v) = \frac{1}{v^n} \left[2a^{n+1} + \frac{v-a^{3n+1}}{v-a^{n-1}} \right]^{2n}$. Hence γ is continuous and onto, so γ is a semiconjugacy between Q_a and h_a such that $\gamma \circ Q_a = h_a \circ \gamma$. Because Q_a include a cycle of Herman rings iff h_a has. From Lemma (2.1) and because Q_a is one to one in the component W_0 , this component is bounded and around 0. Maybe one of W_p is bounded but not around 0. So all cycle of Herman rings of h_a are bounded but does not surround 0. Therefore there are $2n$ - Fatou components $\omega^j W_0$ satisfying $\omega^j W \cap \omega^p W = \emptyset$, for $1 \leq j \leq 2n$ also for each $j \neq p \pmod{2n}$. Assume that s is the period of $\gamma(W_0)$, we remark that s is a divisor of $\dot{s}$ or maybe not equal to $\dot{s}$. So we get an impossible from Proposition (3.2). Fix $U_0 = \gamma(W_0)$ also $U_j = h_a^j(U_0)$ for $0 \leq j \leq s - 1$. Special case, $h_a(U_{s-1}) = U_0$ also $\{U_0, U_1, \dots, U_s = U_0\}$ is a s - periodic Herman rings of h_a . Set $\varphi_0(v) = \frac{a^{4n}}{v}$, we obtain $\varphi_0 \circ h_a(v) = h_a \circ \varphi_0(v)$. (2)

We remark $\deg(h_a)$ is equal to $3n$ and h_a contains $(6n - 2)$ critical points. So the number of local degrees of infinity and the origin is n . Thus, the number of local degrees for the root at a^{n+1} is $2n$ also the local degree of the pole at a^{3n-1} of h_a is $2n$. So this leaves only two free critical points, also from (2), we can denote by c'_a and $\varphi_0(c'_a) = \frac{a^{4n}}{c'_a}$ and hence the local degrees of c'_a is two and the local degrees of $\varphi_0(c'_a)$ is two. So U_j is bounded and does not around zero for $0 \leq j \leq s - 1$.

We study h_a^s and the critical points of h_a^s are $(\bigcup_{j=0}^{s-1} h_a^{-j}(c'_a)) \cup (\bigcup_{j=0}^{s-1} h_a^{-j}(\frac{a^{4n}}{c'_a}))$. Hence at most $\exists 2n$ separate critical orbits of h_a^s , the following model as:

$\{O_{h_a^s}(c_{a_0}), O_{h_a^s}(\varphi_0(c_{a_0})), \dots, O_{h_a^s}(c_{a_{s-1}}), O_{h_a^s}(\varphi_0(c_{a_{s-1}}))\}$. So they s - periodic Herman rings have the collection of core curves $\{\Gamma_0, \Gamma_1, \dots, \Gamma_{s-1}\}$ for h_a such that

there are $s + 1$ connected components $V_0, V_1, \dots, V_s$ separate $\mathbb{C}_\infty$. We have two cases.

Case one

We assume that $\varphi_0(U_0) = U_0$. Thus from (2), $\varphi_0(U_j) = U_j$ for any $0 \leq j \leq s - 1$.

We have the interior boundary K_j and exterior boundary l_j components of U_j . First claim that $\varphi_0(l_j) = l_j \forall j \in N$, and we see this because U_j is bounded also does not around zero.

Therefore l_j must have the points in the closure of U_j for the smallest and largest modulus. Note that the map φ_0 the point of the closure of U_j for largest modulus to the smallest one. So $\varphi_0(l_j) \neq K_j$, we conclude $\varphi_0(l_j) = l_j$ with $j \in N$, this the claim is true. Hence component V_j have $O_{h_a^s}(c_{a_j})$ and $O_{h_a^s}\varphi_0((c_{a_j}))$. From Proposition (3.2), the Julia of h_a^s contain at least

$s + 1$ disjoint infinite critical orbits $O_{h_a^s}(c_{a_j})$ such that $O_{h_a^s}(c_{a_j}) \subseteq V_j \cap J(h_a^s)$. Nevertheless for h_a^s , the critical orbits (2s) only belong to s of $s + 1$ component of V_j for $j = 0, \dots, s$. This is contradictory.

Case two

We assume that $\varphi_0(U_0) \neq U_0$. From (2), we conclude $\varphi_0(U_j) = U_j$ for any $0 \leq j \leq s - 1$. Hence the $2s$ disjoint fixed Herman rings belong in h_a^s . Otherwise, at most, only there are $2s$ disjoint critical orbits. From Proposition (3.2), this is impossible. So h_a and Q_a has no Herman rings.

Theorem 4.4: $J(Q_a)$ It is not a Cantor circle iff it is connected.

Proof: We assume that $Q_a^n \rightarrow 0$ and ∞ . From Theorem (4.1), the multiply connected attractor or parabolic basins do not exist for Q_a and Theorem (4.3), Q_a does not include Herman rings. So, any components periodic Fatou of Q_a is simply connected. Because each critical point does not lie in the preimages of the components of the periodic Fatou. Hence, all are simply connected. From [1], thus $J(Q_a)$ is connected. If $Q_a^n \rightarrow 0$ and ∞ , so $J(Q_a)$ is connected if it is not a Cantor circle from The Escape Quartation Theorem [9].

Theorem 4.5: [4] For all irrational no. θ , there exists a rational map with a Siegel disk of rotation no. θ . If there exists a rational map with a Herman ring of rotation no. θ .

We prove the third result:

Theorem 4.6: Q_a has no Siegel disk.

Proof: By using Theorem (4.3) and Theorem (4.5), thus Q_a has no Siegel disk.

Discussion and Conclusion

The following family of maps are defined as:

$Q_a(z) = 2a^{n+1}z^{-n} + \frac{z^{2n}-a^{3n+1}}{z^n(z^{2n}-a^{n-1})}$, and our get as follows:

- The map Q_a has no Herman rings.
- The multiply connected attractor or parabolic basins does not come true for Q_a .
- $J(Q_a)$ is not a Cantor circles iff it is connected.
- Q_a has no Siegel disk.

We have focused our work on classifications of Fatou components, some of which have been addressed in our previous work and now we are addressing these cases. One of the limitations of the current study is the important condition for giving parameter values to satisfying $a \in \Lambda := \mathbb{C} \setminus \{0\} - \{a: a^{n+1} = 1 \text{ or } a^{2n+2} = 1\}$. Also, the condition one of the free critical points has image for Q_a is diverge on the origin or infinity.

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