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Sufficient conditions for the oscillation of solutions of third-order neutral difference equations with alternating coefficients

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Abstract

This study deals with 3rd-order difference equations with alternating coefficients (NDEAC) with a focus on studying the oscillatory properties of their solutions. An additional sufficient condition can be derived to ensure continuity for the oscillatory solution in equations. An applied example is presented to illustrate the theoretical result obtained.

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1. Introduction

The remarkable development was observed in study for oscillatory behavior of solutions of neural difference equations with delay, especially when there are positive and negative coefficients in the 1st-and 2nd order equations (see [1-3]). Some authors studied the oscillation and behavior asymptotic of different types of delay equations, including [4-9]. In [9], the authors study oscillatory behaviors for the solution for neutral-type linear's difference equations, which includes Alternating coefficients.

$$\Delta^2(q_v - t_v q_{v-a}) + y_v \mathcal{F}(q_{v-f}) - k_v \mathcal{F}(q_{v-z}) = \mathfrak{C}_v \tag{1}$$

where

- $\Delta = q_v = q_{v+1} - q_v$
- y_v, k_v , $\mathfrak{C}_v$ are infinite sequences with $\mathbb{R}$ numbers.
- $y_v, k_v \in \mathbb{R}^+$, $a, f, z \in \mathbb{Z}^+$. $\mathcal{F}: \mathbb{R} \to \mathbb{R}$, satisfy $q_v \mathcal{F}(q_v) > 0$.

It is worth noting that equation (1) belongs to the class of 2nd-order NDE.

In this paper, the oscillations in all finite solutions of 3rd-order (NDEAC)are studied:

$$\Delta^3(q_v - t_v q_{v-a}) + y_v \mathcal{F}(q_{v-f}) - k_v \mathcal{F}(q_{v-z}) = \mathfrak{C}_v \tag{3}$$

This research aims to derive novel sufficient condition that guarantee oscillations for any solution for the equations (2). This study be based on following assumptions:

(H_1) $\sum_{a=v}^{\infty} \sum_{j=a}^{\infty} \sum_{i=j+z-f}^{j-1} k_i < \infty$

(H_2) $\exists$ sequences $\{Q_v\}$ s.t $\Delta^3 Q_v = \mathfrak{C}_v$ & $lim_{v \to \infty} Q_v = 0$;

(H_3) $j_1 q \le \mathcal{F}(q) \le j_2 q$, where j_1 and j_2 positive constant.

2. Sufficient Conditions for Oscillations of Equation (2)

Let

$$\mathcal{K}_v = q_v - t_v q_{v-a}, \tag{3}$$

Let the sequence $\wp_v$ be defined as

$$\wp_v = q_v - t_v q_{v-a} - \sum_{a=v}^{\infty} \sum_{j=a}^{\infty} \sum_{i=j+z-f}^{j-1} k_i \mathcal{F}(q_{i-z}) - Q_v \text{ , } f > z \tag{4}$$

Theorem 2.1: *Suppose that,* $0 \le t_v \le 1$, $k_{v+z-f} > y_v$ *and expectations* $(H_1 - H_3)$ *hold on, moreover,*

$$\liminf_{v \to \infty} j_2 \sum_{r=\varepsilon}^{v-1} \sum_{j=r}^{r+\omega} \sum_{i=j}^{j+\rho} (k_{v+z-f} - y_i) > \left(\frac{\varepsilon}{\varepsilon+1}\right)^{\varepsilon+1} \tag{5}$$

$f > \rho, \omega, \varepsilon = v - (f - \rho - \omega - a)$ *then all solutions to Eq (2) oscillate.*

Proof: Let Eq (2) admit non oscillatory solution, let $q_v > 0$, $v \ge v_0$.Then, Eqs. (2), (3) and (4) we obtain $\Delta^3 \wp_v = (k_{v+z-f} - y_v)\mathcal{F}(q_{v-f}) - Q_v$

$\Delta^3 \wp_\upsilon = (k_{\upsilon+z-f} - y_\upsilon)\mathcal{F}(q_{\upsilon-f}) - Q_\upsilon$, Since $\lim_{\upsilon\to\infty} Q_\upsilon = 0$, then for υ_1 large enough such that

$$\Delta^3 \wp_\upsilon = (k_{\upsilon+z-f} - y_\upsilon)\mathcal{F}(q_{\upsilon-f}) \geq 0 \tag{6}$$

Hence, from (6), We conclude that $\Delta^2\wp_\upsilon$ is nondecreasing therefor $\Delta\wp_\upsilon$, $\wp_\upsilon$ are monotone sequences, So either $\Delta^2\wp_\upsilon < 0$ or $\Delta^2\wp_\upsilon > 0$ for $\upsilon \geq \upsilon_1 \geq \upsilon_0$, Assume that $\Delta^2\wp_\upsilon > 0$ it follows that $\Delta\wp_\upsilon > 0$ and $\wp_\upsilon > 0$ and $\lim_{\upsilon\to\infty} \wp_\upsilon = \infty$, which implies that $q_\upsilon \to \infty$ a contradiction. Hence, $\Delta^2\wp_\upsilon < 0$, So either $\Delta\wp_\upsilon <$ or $\Delta\wp_\upsilon > 0$ for $\upsilon \geq \upsilon_2 \geq \upsilon_1$.

1. $\wp_\upsilon < 0$ and $\wp_\upsilon \to -\infty$. Since $q_\upsilon$is bounded, and $\limsup_{\upsilon\to\infty} q_\upsilon = h^*, h^* \geq 0$, so it's there a subsequence $\{\upsilon_j\}$ so as to $\lim_{j\to\infty}\upsilon_j = \infty$, and $\lim_{j\to\infty} q_{\upsilon_j} = h^*$. From (4) we get

$$q_{\upsilon_j} \leq \wp_{\upsilon_j} + t_{\upsilon_j} q_{\upsilon_j - a} + j_2 \sum_{a=\upsilon}^{\infty}\sum_{j=a}^{\infty}\sum_{i=j+z-f}^{j-1} k_i q_{i-z} + Q_{\upsilon_j}$$

Since q_υ is bounded and by lemma (1.5.5) in [10] ($\sum_{a=\upsilon}^{\infty}\sum_{j=a}^{\infty}\sum_{i=j+z-f}^{j-1} k_i < \infty$) and $lim_{\upsilon\to\infty} Q_\upsilon = 0$ then we obtain $q_{\upsilon_j} - q_{\upsilon_j - a} \leq \wp_{\upsilon_j} + Q_{\upsilon_j}$

so $-q_{\upsilon_j - a} \leq \wp_{\upsilon_j} + Q_{\upsilon_j}$, for large j, $q_{\upsilon_j} \to \infty$ a contradiction.

2. For $\Delta\wp_\upsilon > 0, \Delta^2\wp_\upsilon < 0\ \Delta^3\wp_\upsilon \geq 0$,we have two cases:

2.1: For $\Delta^3\wp_\upsilon \geq 0, \Delta^2\wp_\upsilon < 0\,; \Delta\wp_\upsilon, \wp_\upsilon > 0, \upsilon \geq \upsilon_2 \geq \upsilon_1$ then it's there $\gamma > 0$ so as to $\wp_\upsilon \geq \gamma > 0$ for $\upsilon \geq \upsilon_2 \geq \upsilon_1$. Since q_υ is bounded, and $\liminf_{\upsilon\to\infty} q_\upsilon = h_* \geq 0$, then it's $\{\upsilon_j\} \subseteq \{\upsilon\}$ so as to $\lim_{j\to\infty}\upsilon_j = \infty$, $\lim_{j\to\infty} q_{\upsilon_j} = h_*$. From (3) we get,

$$q_{\upsilon_j} \geq \gamma + t_{\upsilon_j} q_{\upsilon_j - a} + j_2 \sum_{a=\upsilon}^{\infty}\sum_{j=a}^{\infty}\sum_{i=j+z-f}^{j-1} k_i q_{i-z} + Q_{\upsilon_j}\ , q_{\upsilon_j} > \gamma + q_{\upsilon_j - a} - \varepsilon$$

Since ε is arbitrary, for every $j > \mathbb{N}$ (where $\mathbb{N}$ is a sufficiently large natural number), we can conclude that when: $q_{\upsilon_j} > \gamma + q_{\upsilon_j - a}$, when considering the limit when $j \to \infty$, we get $h_* \geq \gamma + h_*$ which contradicts itself.

2.2: For $\Delta^3\wp_\upsilon \geq 0, \Delta^2\wp_\upsilon, \wp_\upsilon < 0\,; \Delta\wp_\upsilon > 0, \upsilon \geq \upsilon_2 \geq \upsilon_1$

Taking the summation of both sides of (6), from υ to $\upsilon + \rho, f > \rho$ we get, $\sum_{i=\upsilon}^{\upsilon+\rho} \Delta^3\wp_{\upsilon_i} = \sum_{i=\upsilon}^{\upsilon+\rho}(k_{i+z-f} - y_i)\mathcal{F}(q_{i-f})$ then $-\Delta^2\wp_\upsilon \geq j_2 \sum_{i=\upsilon}^{\upsilon+\rho}(k_{i+z-f}, -y_i) q_{i-f}$ Taking the summation of both sides of above inequality, from υ to $\upsilon + \omega$, $f > \omega$ (7)

From (4) we get $\wp_\upsilon \geq -t_\upsilon q_{\upsilon-a} - \sum_{a=\upsilon}^{\infty}\sum_{j=a}^{\infty}\sum_{i=j+z-f}^{j-1} k_i \mathcal{F}(q_{i-z}) - Q_\upsilon$, $\wp_\upsilon > -q_{\upsilon-a} - \varepsilon$, Since ε is arbitrary, for every $j > \mathbb{N}$, we can conclude that when: $\wp_\upsilon > -q_{\upsilon-a}\ , q_{\upsilon-f} \geq -\wp_{\upsilon+a-f}$

Substituting the last inequality in (7) leads to

$$\Delta\wp_\upsilon + j_2\left(\sum_{i=\upsilon}^{\upsilon+\omega}\sum_{i=\upsilon}^{\upsilon+\rho}(k_{i+z-f} - y_i)\right)\wp_{\upsilon-(f-a-\omega-\rho)} \geq 0$$

Based on theorem 2.1 ii in [10] and on relation (5), so lasted inequality cannot has eventual negatives solutions, leads to contradictions.

Theorem 2.2: *Take* $0 \leq t_v < 1, y_v > k_{v+z-f}$ *and* $(H_1 - H_3)$, *holds, in addition*

$$\lim\inf_{v\to\infty} j_1 \sum_{r=v-\varepsilon}^{v-1} \sum_{j=r}^{r+\omega} \sum_{i=j}^{j+\rho} \frac{|k_{i+z-f} - y_i|}{t_{i-f+a}} > \left(\frac{\varepsilon}{\varepsilon+1}\right)^{\varepsilon+1} \quad (8)$$

$\varepsilon = f - \rho - \omega - a, f > \rho + \omega + a$, Then all bounded solutions to Eq. (2) are either oscillating or tend to zero.

Proof: Assume that Eq. (2) has a non oscillatory solution, let $y_v > 0, v \geq v_0$. Then, from Eqs. (2), (3) and (4) we obtain $\Delta^3 \wp_v = -(y_v - k_{v+z-f})\mathcal{F}(q_{v-f}) \leq 0$ (9) Hence, from (9) we conclude that $\Delta^2 \wp_v$ is no increasing therefor, $\Delta\wp_v$, $\wp_v$ are repetitive sequences, we assert that $\Delta^2\wp_v > 0$ for$v \geq v_1 \geq v_0$, nonetheless, $\Delta^2 \wp_v < 0$ for $n \geq n_1$, thus, $\Delta\wp_v < 0$ $\wp_v < 0$ *and* $\wp_v \to -\infty$ *as* $v \to \infty$. From (4) we get $-\wp_v \leq t_v q_{v-a} + \sum_{a=v}^{\infty} \sum_{j=a}^{\infty} \sum_{i=j+z-f}^{j-1} k_i \mathcal{F}(q_{i-z}) + Q_v$.

Since q_v is bounded and by lemma (1.5.5) in [11]. ($\sum_{a=v}^{\infty} \sum_{j=a}^{\infty} \sum_{i=j+z-f}^{j-1} k_i < \infty$), Q_v is bounded then form the last inequality we conclude that $t_v q_{v-a} + \sum_{a=v}^{\infty} \sum_{j=a}^{\infty} \sum_{i=j+z-f}^{j-1} k_i \mathcal{F}(q_{i-z}) + Q_v$ is bounded which is a contradiction. So $\Delta^2 \wp_v > 0$, hence either $\Delta\wp_v > 0$ or $\Delta\wp_v < 0$

1: $\Delta\wp_v, \Delta^2\wp_v > 0, \Delta^3\wp_v \leq 0$;

2: $\Delta^2\wp_v > 0, \Delta\wp_v < 0, \Delta^3\wp_v \leq 0$.

1: For $\Delta\wp_v > 0, v \geq v_1, \wp_v > 0 \lim_{v\to\infty} \wp_v = \infty, y_n \to \infty$ contradiction.

2: We have two subcases to consider:

2.1: Then it's there $\gamma < 0$ so as to $\wp_v \leq \gamma < 0$ for $v \geq v_2 \geq v_1$. Since q_v is bounded, and $\lim\sup_{v\to\infty} q_v = h_* \geq 0$ so $\{v_j\}$ so as to $\lim_{j\to\infty} v_j = \infty$, $\lim_{j\to\infty} q_{v_j} = h_*$. From (3) we get,

$$q_{v_j} \leq \gamma + t_{v_j} q_{v_j - a} + j_2 \sum_{a=v_j}^{\infty} \sum_{j=a}^{\infty} \sum_{i=j+z-f}^{j-1} k_i q_{i-z} + Q_{v_j}$$

For v_j large enough $\sum_{a=v_j}^{\infty} \sum_{j=a}^{\infty} \sum_{i=j+z-f}^{j-1} k_i q_{i-z} + Q_{v_j} < \varepsilon, q_{v_j} < \gamma + q_{v_j - a} + \varepsilon$, Since ε is arbitrary, for every $j > \mathbb{N}$, we can conclude that when: $q_{v_j} \leq \gamma + q_{v_j - a}$, as $j \to \infty$, consequently $h_* \leq \gamma + h_*$ which contradicts itself.

2.2 : $\wp_v > 0$: Taking Summation to (9) from v to $v + \rho$, $0 < \rho < f - a$, Taking the summation of both sides from v to $v + \omega$, $f - a > \omega$ we get,

$$\Delta\wp_v \leq j_2 \sum_{i=v}^{v+\omega} \sum_{i=v}^{v+\rho} (k_{i+z-f} - y_i) q_{i-f} \quad (10)$$

From (4) we get $q_v > \wp_v - \varepsilon$, Since ε is arbitrary, we have $q_v \geq \wp_v$
Substituting the last inequality in (10) leads to

Based on theorem 2.1 i in [10] and on relation (8), it cannot have eventual negative solutions, which leads to a contradiction.

3. Examples:

This section is devoted presenting a set of applied examples that aim to illustrate the effectiveness of the main theoretical results, highlight the practical applications of the study, and provide numerical support for presented proofs.

Example 1: Examine the difference equation:

$$\Delta^3\left(q_{v}-\left(\frac{1}{2}+\left(\frac{1}{2}\right)^{v}\right)q_{v-1}\right)+\frac{5}{32}\left(\frac{1}{2}\right)^{v}q_{v-4}-\frac{5}{8}q_{v-1}=\frac{15}{4}\left(\frac{1}{2}\right)^{v}\left(-\frac{1}{2}\right)^{v}$$

where $a = 1, f = 4, z = 1, \rho = 1, \omega = 1, \mathcal{F}(q_{v}) = q_{v}, j_2 = 2$

The following criteria of theorem 2.1 can be found to be true:

$$\lim\inf_{v\to\infty}\ 2\sum_{r=v-1}^{v-1}\sum_{j=r}^{r+1}\sum_{i=j}^{j+\rho}\left(\frac{5}{8}-\frac{5}{32}\left(\frac{1}{2}\right)^{i}\right)>\left(\frac{1}{2}\right)^{2}$$

So, by theorem 2.1 each bounded solution oscillates, solution is $q_{v} = \left(-\frac{1}{2}\right)^{v}$

Example 2: Examine the DE:

$$\Delta\left(q_{v}-\left(1+\left(\frac{1}{e}\right)^{v}\right)q_{v-1}\right)+\frac{(1+e)^4}{e^7}q_{v-4}-e^{-6}\left(\frac{1}{e}\right)^{v}q_{v-1}=\frac{(-1)^{v}(3+3e^2+e^4)}{e^{3+2v}}$$

$$\lim_{v\to\infty}\inf\ 0.2\sum_{r=v-1}^{v-1}\sum_{j=r}^{r+1}\sum_{i=j}^{j+1}\frac{e^{-(n-3+6)}-\frac{(1+e)^4}{e^7}}{1+\left(\frac{1}{e}\right)^{n-3}}>\left(\frac{1}{2}\right)^{2}$$

So, by Th 2.2 each bounded solution oscillates, $q_{v} = \left(-\frac{1}{e}\right)^{v}$ is such a solution.

4. Conclusion

This study presents a set of sufficient conditions governing oscillatory behavior for all finite solution of 3rd-order delayed neutrals difference equation with alternating coefficients. An analysis uses special mathematical sequences to achieve this goal.

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