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Some results of locally finite lie and associative algebras

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Abstract

We examine locally finite Lie and associative algebras over a field $\mathbb{F}$ of characteristic $p \geq 0$ in this paper. If any finite set of elements is an infinite-dimensional associative algebra A is included in a subalgebra of A in finite-dimensional, then A is said to be locally finite. The local system of locally finite Lie and associative algebras is also considered, and some results are presented. We described a conical local system of simple, locally finite Lie algebras and studied the connections among them. Inner ideals of simple locally finite-dimensional Lie algebras are described in this paper.

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Keywords: *Associative algebra, Lie algebra, Inner ideals.*

1. Introduction

Throughout the paper we fix $\mathbb{F}$ is an algebraically closed field of characteristic $p \geq 0$ and A is a locally finite associative algebra over a field $\mathbb{F}$ with infinite countable dimensions. Remember that if any finite collection of elements in an algebra A is included in a finite-dimensions subalgebra of that algebra, then that algebra is said to be locally finite [1, 7]. A system of finite-dimensional subalgebra $\{A_\zeta\}_{\zeta\in\Gamma}$ of A over a field $\mathbb{F}$ known as local system for A if $A = \cup_{\zeta\in\Gamma} A_\zeta$ there exist $\delta \in \Gamma$ and all $\zeta, \eta \in \Gamma$, such that $\mathrm{A}_\zeta, \mathrm{A}_\eta \subseteq \mathrm{A}_\delta$. Baranov and Rowley [2] gave a few examples of locally finite Algebras of Lie. In [3] and [4], simple classification Lie algebras over $\mathbb{F}$ that are locally finite with characteristic zero. They proved existence two type of Lie algebras that are locally finite possess several means of description.

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The study of simple associative algebras over $\mathbb{F}$ that are locally finite of characteristic zero was conducted by The authors [5]. They emphasized their connection to locally finite Lie over $\mathbb{F}$ with $p = 0$. Other authors [6] examined the Lie sub-algebras of locally finite associative algebras. Shlaka [5] studied local systems of A over fields $\mathbb{F}$ of free characteristic in A subset start equation cap G of start equation cap L in which open bracket cap G ,open bracket cap G ,cap L close bracket close bracket subset or equals cap G is said to be an inner ideal (or simply, $I-$ideal). The authors [6] was the first to thoroughly study inner ideals, and they were found to be helpful in categorizing fundamental Lie algebras in both finite and infinite dimensions.

In this article, we present some results for associative algebras over $\mathbb{F}$ of $p \geq 0$ and locally finite Lie. In the second section, we start by going over some background information. We discuss some results about plain Lie algebras and their inner ideals in section three.

2. Preliminaries

In this part, some definitions are recalled and basics about locally finite Lie and associative algebra.

Definition 2.1: [6] A system of finite dimension subalgebra of $\{A_\zeta\}_{\zeta\in\Gamma}$ of A is considered $local\ system$(or simply, L.system) for A if $A = \cup_{\zeta\in\Gamma} A_\zeta$ there exist $\delta \in \Gamma$ and any $\zeta, \eta \in \Gamma$, such that $A_\zeta, A_\eta \subseteq A_\delta$.

Put $\zeta \leqslant \eta$ if $A_\zeta \subseteq A_\eta$. In that case, $A = \varinjlim A_\zeta$ is a direct limit of the algebra A_ζ, and Γ is a direct set. Remember that if $AA = A$. Then A is a *perfect algebra.*

Lemma 2.2: [6] *Let $\{A_\zeta\}_{\zeta\in\Gamma}$ be a L.system of A over a field $\mathbb{F}$ and let J be a subspace of A with finite dimensional. Therefore, $\{A_\eta\}_{\eta\in\Gamma}$ is a L.system of A, where $\Gamma_J = \{\eta \in \Gamma \mid J \subseteq A_\eta\}$.*

Definition 2.3: [5] If there are no ideals of codimension 1 *in* A over $\mathbb{F}$, it is claimed to be *1-perfect*.

Definition 2.4: [4] A system $\{L_\zeta\}_{\zeta\in\Gamma}$ of subalgebras of L over $\mathbb{F}$ is said to be L.system if the following requirements are satisfied:

1- $\cup_{\zeta\in\Gamma} \mathrm{L}_\zeta = \mathrm{L}$

2- For any $\zeta, \eta \in \left(\mathrm{L}_\zeta\right)_{\in\Gamma}$ there exist $\delta \in \left(\mathrm{L}_\zeta\right)_{\zeta\in\Gamma}$ so that $\mathrm{L}_\zeta, \mathrm{L}_\eta \subseteq \mathrm{L}_\delta$.

Definition 2.5: [6] If Γ has the least element 1, Then A perfect L.system $(L_\zeta)_{\zeta\in\Gamma}$ is said to be conical where

(1) $L_1 \subseteq L_\zeta$ for any $\zeta \in \Gamma$.

(2) L_1 is simple.

(3) for all $\zeta \in \Gamma$ any constraint on a standard L_ζ −module to L_1 has anon-trivial composition factor.

Proposition 2.6: [6] Any L.system $\{A_\zeta\}_{\zeta\in\Gamma}$ contains an algebra that is not nilpotent.

Definition 2.7: [5] Consider L as a finite dimension and let G is I −ideal of L. Then, an integer n exist so that $\mathrm{G_n} = \mathrm{G_{n+1}}$. We call that $\mathrm{G_n}$ is the core of G, indicated by core $_{\mathrm{L}}(\mathrm{G})$.

3. The Main Results

In this part, we proved some result about locally finite associative algebra over $\mathbb{F}$ and prove that if G be an I −ideal of $L = [A, A]$. Then there exist a system of I −ideals $\{G_\alpha\}_{\alpha\in\Gamma}$ such that $G_\alpha \in [A_\alpha, A_\alpha]$. Also, we show that if A is a locally over $\mathbb{F}$, then for every I −ideal G of $A^{(1)}$ we have $core_L(G) \neq 0$.

Proposition 3.1: Suppose that A is locally over $\mathbb{F}$ and let $\{A_\zeta\}_{\zeta\in\Gamma}$ be a L.system of A. Let I_ζ be the ideal in A_ζ such that $I_\zeta \subset I_\eta$ whenever $A_\zeta \subset A_\eta$. Then $\{I_\zeta | \zeta \in \Gamma\}$ is a L.system.

Proof: Let $\{\mathrm{A_\zeta}\}_{\zeta\in\Gamma}$ be a *L.system* of A and $I_\zeta = I \cap A_\zeta$. Consider $\zeta, \eta \in \Gamma$, we need to show that $I_\zeta, I_\eta \subseteq I_\delta$ for some $\delta \in \Gamma$. Since $\{\mathrm{A_\zeta}\}_{\zeta\in\Gamma}$ is a *L.system*, there exist δ such that $\mathrm{A_\zeta} \subseteq \mathrm{A_\delta}$ and $\mathrm{A_\eta} \subseteq \mathrm{A_\delta}$, so $\mathrm{I_\zeta} \subseteq \mathrm{I_\delta}$ and $\mathrm{I_\eta} \subseteq \mathrm{I_\delta}$. Next, we need to show $I = \bigcup_{\zeta\in\Gamma} I_{\zeta.}$. We have $A = \bigcup_{\zeta\in\Gamma} A_\zeta$ and $\mathrm{I_\zeta} \subseteq \mathrm{A_\zeta}$ for all $\zeta \in \Gamma$. If $x \in I_\zeta$, then $x \in A_\zeta$ that is $x \in \bigcup_{\zeta\in\Gamma} A_\zeta$ for some ζ, this means that $x \in I \cap \left(\bigcup_{\zeta\in\Gamma} A_\zeta\right)$, that is there exist ζ such that $x \in I \cap A_\zeta = I_\zeta$. We have $x \in \mathrm{A_\zeta} \subseteq \mathrm{A_\eta}$. Consequently, $x \in I_\zeta \subset I_\eta$ for some $\zeta, \eta \in \Gamma$, so $x \in \bigcup_{\zeta\in\Gamma} I_{\zeta.}$ then $I \subseteq \bigcup_{\zeta\in\Gamma} I_{\zeta.}$, but $\bigcup_{\zeta\in\Gamma} I_{\zeta.} \subseteq I$, therefore $\{I_\zeta\}_{\zeta\in\Gamma}$ is a *L.system* of I.

The lemma that follos is well-known.

Lemma 3.2: *Let S be a subalgebra of A. If G is I −ideal of $A^{(k)}$ $k \geq 0$. Then $G \cap S^{(k)}$ is I −ideal of $S^{(k)}$.*

Proof: Let G is I −ideal of $A^{(k)}$ $k \geq 0$. Then $\left[G \cap S^{(k)}, \left[G \cap S^{(k)}, S^{(k)}\right]\right] \subseteq G \cap S^{(k)}$. So $G \cap S^{(k)}$ is I −ideal of $S^{(k)}$.

Theorem 3.3: *Let $\{A_\zeta\}_{\zeta\in\Gamma}$ be a conical L.system and A be a simple associative algebra. Then $\{[A_\zeta, A_\zeta]\}$ is conical L.system of $[A, A]$.*

Proof: Let $\zeta, \eta \in \Gamma$, we need to show that $[A_\zeta, A_\zeta], [A_\eta, A_\eta] \subseteq [A_\delta, A_\delta]$. Since $\{A_\zeta\}_{\zeta\in\Gamma}$ is *L.system*, there exist $\delta \in \Gamma$ such that $A_\zeta, A_\eta \subseteq A_\delta$, for all $\zeta, \eta \in \Gamma$. Then

$$[A_\zeta, A_\zeta] \subseteq [A_\delta, A_\delta]$$

and

$$[A_\eta, A_\eta] \subseteq [A_\delta, A_\delta].$$

Now, Let $x \in [A, A]$ and write $x = \sum_{i,j\in\Gamma} [a_i, b_j]$, so there exist A_i such that $a_i \in A_i$ and A_j such that $a_j \in A_j$. Since $\{A_\zeta\}_{\zeta\in\Gamma}$ is *L.system*, there exist k such that $A_i, A_j \subseteq A_k$. Hence, $[a_i, b_j] \subseteq A_k$ for some k, and $x = [a_i, b_j] \in \cup_{\zeta\in\Gamma} [A_\zeta, A_\zeta]$. Therefore, $A = \cup_{\zeta\in\Gamma} [A_\zeta, A_\zeta]$.

Theorem 3.4: *If G be an I −ideal of $L = [A, A]$. Then there exist a system of I −ideals $\{G_\zeta\}_{\zeta\in\Gamma}$ such that $G_\zeta \in [A_\zeta, A_\zeta]$, where $G_\zeta = G \cap [A_\zeta, A_\zeta]$.*

Proof: Let G be a proper I −ideal of $L = [A, A]$ and suppose that $\{A_\zeta\}_{\zeta\in\Gamma}$ be a conical *L.system* of A. Then by Lemma 3.3, $\{[A_\zeta, A_\zeta]\}$ be a *L.system* of $[A, A]$, and by Lemma 3.2, $G_\zeta = G \cap [A_\zeta, A_\zeta]$ is an I −ideal of $[A_\zeta, A_\zeta]$. Now, we need to show that $\cup_{\zeta\in\Gamma} G_\zeta = G$, for all $\zeta, \eta \in \Gamma$ there exist $\delta \in \Gamma$, such that $A_\zeta^{(1)}, A_\eta^{(1)} \subseteq A_\delta^{(1)}$, that is, $G_\zeta, G_\eta \subseteq G_\delta$. And

$$G_\zeta = G \cap A_\zeta^{(1)} \subseteq G \cap A_\delta^{(1)}$$

$$G_\eta = G \cap A_\eta^{(1)} \subseteq G \cap A_\delta^{(1)}.$$

Hence,

$$\cup_{\zeta\in\Gamma} G_\zeta = \cup_{\zeta\in\Gamma} \left(G \cap A_\zeta^{(1)}\right) = G \cap A^{(1)} = G.$$

Therefore, $\{G_\zeta\}_{\zeta\in\Gamma}$ is a *L.system*.

Corollary 3.5: *If G be an I −ideal of $L = [A, A]$. Then there exist a system of L.system $\{C_\zeta\}_{\zeta\in\Gamma}$ of $core_L(G)$.*

Corollary 3.6: *If G be an I −ideal of $L = [A, A]$. Then there exist a system of L.system of L −perfect.*

Theorem 3.7: *If A is a locally over field $\mathbb{F}$. Then for every I −ideal G of $A^{(1)}$ we have $core_L(G) \neq 0$.*

Proof: Let $\{A_\zeta\}_{\zeta\in\Gamma}$ is a *L.system* of A, then by Theorem 3.3, $\{A_\zeta^{(1)}\}_{\zeta\in\Gamma}$ is *L.system* of $A^{(1)}$. Since G is I −ideal of $A^{(1)}$ and by using Corollary 3.5, there exist a *L.system* $\{C_\zeta\}_{\zeta\in\Gamma}$ of $core_L(G)$ and C_ζ is L −perfect such that $core_L(C_\zeta) = C_\zeta$. Now, by Proposition 2.6, $\{A_\zeta\}_{\zeta\in\Gamma}$ content a non nilpotent element say A_η. Then $A_\eta^{(1)}$ is non solvable. Therefore, $core_{A_\zeta^{(1)}}(C_\eta) \neq 0$, that is $C_\eta \neq 0$. Hence, $core_L(G) \neq 0$.

Corollary 3.8: *Let A be a locally over field $\mathbb{F}$. If G is $I-$ideal of $A^{(1)}$ such that $core_L(G) \neq 0$. Then there exist of $L-$perfect $I-$ideal.*

Corollary 3.9: *Let A be a locally over field $\mathbb{F}$. If G is $I-$ideal of $A^{(1)}$ such that $core_L(G) \neq 0$. Then there exist of bar-minimal.*

Proposition 3.10: Let $\{A_\zeta\}_{\zeta\in\Gamma}$ be a L.system of A over field $\mathbb{F}$. Let $\{G_\zeta\}_{\zeta\in\Gamma}$ be a system of $I-$ideals such that for every $\zeta \in \Gamma$, G_α is an $I-$ideal of $[A_\zeta, A_\zeta]$. Then either $\cap_{\zeta\in\Gamma} G_\zeta = 0$ or for each $k \in \Gamma$ or there is $\eta_k \in \Gamma$ so that $A_k \subseteq G_{\eta k}$.

Proof: Put $G = \cap_{\zeta\in\Gamma} G_\zeta$. Suppose that $G \neq 0$. Let $G'_\zeta = \cap \{X_\zeta | X_\zeta$ is an $I-$ideal of $[A_\zeta, A_\zeta]$ with $X_\zeta \supset G\}$.

Then G'_ζ is an $I-$ideal of $[A_\zeta, A_\zeta]$ with $G'_\zeta \subseteq G_\zeta$ for each $\zeta \in \Gamma$. Let X_ζ be a member in G'_ζ. Then X_ζ be an $I-$ideal of $[A_\zeta, A_\zeta]$ with $G \subseteq X_\zeta$. Note that for each $A_\gamma \in \{A_\zeta\}_{\zeta\in\Gamma}$ with $[A_\gamma, A_\gamma] \subset [A_\zeta, A_\zeta]$, we have $X_\zeta \cap [A_\zeta, A_\zeta]$ is an $I-$ideal of $[A_\zeta, A_\zeta]$ with $G \subseteq X_\zeta \cap [A_\zeta, A_\zeta]$, so $G'_\gamma \subseteq G'_\zeta$. Hence, $G' = \cup_{\zeta\in\Gamma} G'_\zeta$ is an $I-$ideal of $[A, A]$ with $G \subseteq G'$. The simplicity of $[A, A]$ implies that $[A, A] = G' = \cup_{\zeta\in\Gamma} G'_\zeta$. Thus, $\{G'_\zeta\}_{\zeta\in\Gamma}$ is a *L.system* of $[A, A]$. Let $k \in \Gamma$. Since $[A_k, A_k]$ is a finite dimensional, there exist $\eta_k \in \Gamma$ such that $[A_k, A_k] \subseteq G'_{\eta_k}$, but $G'_\zeta \subseteq G_\zeta$. Therefore, for each $k \in \Gamma$, there is $\eta_k \in \Gamma$ such that $[A_k, A_k] \subseteq G_{\eta_k}$.

5. Discussion and Conclusion

This paper investigated the structure of the locally finite Lie and *associative algebra over* $\mathbb{F}$. Show that *If G be an inner ideal of $L = [A, A]$, then there exist a system of inner ideals $\{G_\zeta\}_{\zeta\in\Gamma}$. Also proved that* every *inner ideal G of $A^{(1)}$ we have $core_L(G) \neq 0$.*

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