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Solving heat equation with exponential nonlinearity by Banach contraction method

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Abstract

This paper examines the Banach Contraction Method (BCM) for solving the heat equation with an exponential nonlinearity. In fact, the (BCM) has been tested via illustrative examples. When applying the BCM, we achieved effective and efficient results by comparing them with those obtained using Adomian decomposition methods. The BCM does not require a restrictive assumption about the nonlinear term, unlike other known methods. The results are also illustrated with tables and figures. All calculations are obtained by Mathematica® 10 program.

Subject Classification: *Primary 35Qxx, Secondary 35Q79.*

Keywords: *Nonlinear heat equation, Banach contraction method, Exponential nonlinearity, Numerical solution, Absolute error low, Mathematica® 10 program.*

1. Introduction

A consideration for the differential equation includes many scopes, including the study for an existences and uniqueness for a solutions, as well as finding numerical solutions of partial and fractional differential equations [1] and studying the qualitative properties of the solution such as the oscillation

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characteristic [2] and asymptotic behavior [3] for each solution. Partial differentials equations are a branch of differential equations with many applications across various scientific fields, including physics, dynamics, quantum mechanics, and others [4-7]. One application of partial differential equations in physics is the heat equations, which describe the propagation of heat in a given medium. Many researchers have studied the heat equation and found analytic solutions; we mention some literature [8], [9].

2. Heat Equation with Exponential Nonlinearity

In this paper, we will study heat equation with exponential nonlinearity [8-9] of the form:

$$p_t = (A(p)p_u)_u + C(p); positive\ u < 1, positive\ t \leq T \tag{1}$$

Undergo initial conditions

$$p(u,0) = h(u), 0 \leq u \leq 1 \tag{2}$$

Where $p = p(u,t)$ is the unknown function, $A(p) and\ C(p)$ are arbitrary smooth functions, h is an initial condition. In 2013, another author [10] They applied the Adomian Decomposition Method to solve problem (1). Using Banach contraction methods of solve the problem (1).

3. Materials and Methods

3.1 *Description of the BCM*

Put nonlinear partial differential equations [11]:

$$L(p(u,t)) + R(p(u,t)) + N(p(u,t)) = g(u,t), \tag{3}$$

With the initial condition

$$p(u,0) = h(u) \tag{4}$$

For L the highest order derivatives and easily invertible, R is a linear differential operator for orders less than L, N represent linear term and g is the source terms. A function p is assumed is bounded $\forall t \in [0,T]$ and nonlinear terms $N(p)$ satisfied Lipschitz conditions, i.e.

$$|N(p(u,t)) - N(v(u,t))| \leq L_1|p(u,t) - v(u,t)|$$

Where $L_1$is appositive, $L_1 \geq 0$. Application L^{-1} to both side for the Eq. (3) so, use Eq. (4), we get:

$$p(u,t) = h(u) + L_t^{-1}[g] - L_t^{-1}[R(p(u,t))] - L_t^{-1}[N(p(u,t))], \tag{5}$$

Define successive approximation as

$$\left.\begin{array}{c} p_0(u,t) = h(u) + L_t^{-1}[g(u,t)] \\ p_{i+1}(u,t) = p_0(u,t) - L_t^{-1}[R(p_i(u,t))] - L_t^{-1}[N(p_i(u,t))] i \geq 0 \end{array}\right\} \tag{6}$$

A solutions for (3) is $p(u,t) = \lim_{i\to\infty} p_i(u,t)$

3.2 *Results and Discussion*

In this part, application of BCM for heat equation with exponential nonlinearity are given as in the illustrative examples:

Example 1: For $A\big(p(u,t)\big) = a\, p\, exp\big(\alpha p(u,t)\big)$ and $C\big(p(u,t)\big)$=0 the Eq. (1) is given by:

$$\frac{\partial p(u,t)}{\partial t} = \Big(ap(u,t)exp\big(\alpha p(u,t)\big)\big(p(u,t)\big)_u\Big)_u \tag{7}$$

Undergo initials conditions:

$$p(u,0) = \frac{1}{\alpha}\log(C1u + C2)\,, C1, C2 \in R \tag{8}$$

(Eq. (7) has theoretical solution in [12])

$$p(u,t) = \frac{1}{\alpha}\log\Big(C1u + \frac{a}{\alpha}C1^2 t + C2\Big) \tag{9}$$

Solution: By implementation L_t^{-1} to both sides of Eq. (7) will be obtained:

$$p(u,t) = \frac{1}{\alpha}\log(C1u + C2) + L_t^{-1}\Big[\Big(ap(u,t)exp\big(\alpha p(u,t)\big)\big(p(u,t)\big)_u\Big)_u\Big] \tag{10}$$

we use BCM for Eq. (10), so consecutive approximations as:

$$p_0(u,t) = \frac{1}{\alpha}\log(C1u + C2)$$

$$p_{i+1}(u,t) = p_0(u,t) + L_t^{-1}\Big[\Big(ap_i(u,t)exp\big(\alpha p_i(u,t)\big)\big(p_i(u,t)\big)_u\Big)_u\Big], i = 0,1,2, ...$$

The following approximations are achieved:

$$p_0(u,t) = \frac{1}{\alpha}\log(C1u + C2)$$

$$p_1(u,t) = \frac{a\mathrm{C}1^2 t}{(\mathrm{C}2+\mathrm{C}1u)\alpha^2} + \frac{\mathrm{Log}[\mathrm{C}2+\mathrm{C}1u]}{\alpha}$$

$$p_2(u,t) = \frac{1-2\mathrm{Log}[\mathrm{C}2+\mathrm{C}1u]}{\alpha} + \frac{\mathrm{Log}[\mathrm{C}2+\mathrm{C}1u]}{\alpha} + \frac{1}{(\mathrm{C}2+\mathrm{C}1u)^3\alpha^4}\mathrm{e}^{\frac{a\mathrm{C}1^2 t}{\mathrm{C}2\alpha+\mathrm{C}1u\alpha}}\big(a^3\mathrm{C}1^6 t^3 + a^2\mathrm{C}1^4 t^2(\mathrm{C}2 + \mathrm{C}1u)\alpha(-2 + \mathrm{Log}[\mathrm{C}2 + \mathrm{C}1u]) - 2a\mathrm{C}1^2 t(\mathrm{C}2 + \mathrm{C}1u)^2\alpha^2(-1 + \mathrm{Log}[\mathrm{C}2 + \mathrm{C}1u]) + (\mathrm{C}2 + \mathrm{C}1u)^3\alpha^3(-1 + 2\mathrm{Log}[\mathrm{C}2 + \mathrm{C}1u])\big)$$

and so on, will be get a successive approximation of $p_i(u,t)$ but for brevity not listed.

Example 2: For $A\big(p(u,t)\big) = ap(u,t)exp\big(\alpha p(u,t)\big)$ and $C\big(p(u,t)\big) = b$, the Eq. (1) is given by:

$$\frac{\partial p(u,t)}{\partial t} = (a\, p\, exp(\alpha p)p_u)_u + b \tag{11}$$

Undergo initials conditions:

$$p(u,0) = \frac{1}{\alpha}\log\Big(C1u + \frac{a\, C1^2}{b\, \alpha^2} + C2\Big), C1, C2 \in R \tag{12}$$

(Eq. (11) has theoretical solution in [14])

$$p(u,t)=\frac{1}{\alpha}\log\left(C1\,exp(b\alpha t)u+\frac{a\,C1^2}{b\,\alpha^2}exp(2b\alpha t)+C2\right) \quad (13)$$

Solution: By Applying L_t^{-1} to both sides of Eq. (11) will obtained:

$$p(u,t)=\frac{1}{\alpha}\log\left(C1u+\frac{a\,C1^2}{b\,\alpha^2}+C2\right)+L_t^{-1}[(a\,p\,exp(\alpha p)p_u)_u+b] \quad (14)$$

By using BCM for Eq. (14), we get the successive approximations as:

$$p_0(u,t)=\frac{1}{\alpha}\log\left(C1u+\frac{a\,C1^2}{b\,\alpha^2}+C2\right)$$

$$p_{i+1}(u,t)=p_0(u,t)+L_t^{-1}\left[\left(ap_i(u,t)exp\left(\alpha p_i(u,t)\right)\left(p_i(u,t)\right)_u\right)_u+b\right],$$
$$i=0,1,2,\ldots$$

The following approximations are achieved:

$$p_0(u,t)=\frac{1}{\alpha}\log\left(c_1u+\frac{a\,c_1^2}{b\,\alpha^2}+c_2\right)$$

$$p_1(u,t)=t\left(b+\frac{a\mathrm{C1}^2}{\left(\mathrm{C2}+\mathrm{C1}u+\frac{a\mathrm{C1}^2}{b\alpha^2}\right)\alpha^2}\right)+\frac{\mathrm{Log}\left[\mathrm{C2}+\mathrm{C1}u+\frac{a\mathrm{C1}^2}{b\alpha^2}\right]}{\alpha}$$

$$\begin{aligned}p_2(u,t)=\;&b(t+(a\mathrm{C1}^2(-\frac{1}{b\alpha}(a\mathrm{C1}^2+b(\mathrm{C2}+\mathrm{C1}u)\alpha^2)^4(4a^2\mathrm{C1}^4\\
&+6ab\mathrm{C1}^2(C2+C1u)\alpha^2+b^2(\mathrm{C2}+\mathrm{C1}u)^2\alpha^4\\
&+2a^2\mathrm{C1}^4Log[C2+C1(u+\frac{a\mathrm{C1}}{b\alpha^2})])\\
&+\;\mathrm{e}^{\frac{bt\alpha\left(2a\mathrm{C1}^2+b(\mathrm{C2}+\mathrm{C1}u)\alpha^2\right)}{a\mathrm{C1}^2+b(\mathrm{C2}+\mathrm{C1}u)\alpha^2}}(a^2b^2\mathrm{C1}^4t^3\alpha^2(2a\mathrm{C1}^2\\
&+b(\mathrm{C2}+\mathrm{C1}u)\alpha^2)^2\,(2a^2\mathrm{C1}^4+3ab\mathrm{C1}^2(C2+C1u)\alpha^2\\
&+b^2(\mathrm{C2}+\mathrm{C1}u)^2\alpha^4)\\
&+a^2b\mathrm{C1}^4t^2\alpha(a\mathrm{C1}^2+b(\mathrm{C2}+\mathrm{C1}u)\alpha^2)^2(2a\mathrm{C1}^2\\
&+b(\mathrm{C2}+\mathrm{C1}u)\alpha^2)^2\,\left(-2+Log\left[C2+C1\left(u+\frac{a\mathrm{C1}}{b\alpha^2}\right)\right]\right)\\
&-2a\mathrm{C1}^2t(a\mathrm{C1}^2+b(\mathrm{C2}+\mathrm{C1}u)\alpha^2)^3(2a\mathrm{C1}^2\\
&+b(\mathrm{C2}+\mathrm{C1}u)\alpha^2)\left(b(\mathrm{C2}+\mathrm{C1}u)\alpha^2\right.\\
&\left.+a\mathrm{C1}^2\mathrm{Log}\left[\mathrm{C2}+\mathrm{C1}\left(u+\frac{a\mathrm{C1}}{b\alpha^2}\right)\right]\right)\\
&+\frac{1}{b\alpha}(a\mathrm{C1}^2+b(\mathrm{C2}+\mathrm{C1}u)\alpha^2)^4(4a^2\mathrm{C1}^4+6ab\mathrm{C1}^2(C2\\
&+C1u)\alpha^2+b^2(\mathrm{C2}+\mathrm{C1}u)^2\alpha^4+2a^2\mathrm{C1}^4Log[C2+C1(u\\
&+\frac{a\mathrm{C1}}{b\alpha^2})])))))\\
&/((a\mathrm{C1}^2+b(\mathrm{C2}+\mathrm{C1}u)\alpha^2)^4(2a\mathrm{C1}^2+b(\mathrm{C2}+\mathrm{C1}u)\alpha^2)^3))\\
&+\frac{Log\left[C2+C1u+\frac{a\mathrm{C1}^2}{b\alpha^2}\right]}{\alpha}\end{aligned}$$

and so on, we obtained successive approximations of $p_i(u,t)$ but for the brevity is not list.

4. Numerical Analysis

For a study for accuracy for a achieve approximates solutions with theoretical solution, the conveniences function for a absolute error will be:

$$R_i = |p(u,t) - p_i(u,t)|\, , i = 0,1,2, \dots \tag{15}$$

For Examples 1 and 2 will be taken the values of the parameters will be taken equal to: $a = \ \alpha \ = C1 = C2 = 0.1$ to calculate the approximate solutions and the absolute error. The values of R_i in Eq. (15) which are obtained by BCM are compared with those obtained by ADM that obtained in [10], [13-15] it observed in the Tables 1, and 2 below, where the values $u = t =${0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1}.

Table 1

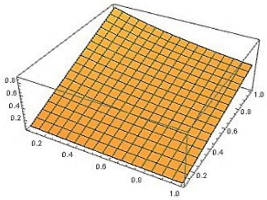
Comparison between the absolute error obtained by BCM and ADM for Example 1.

Absolute error by BCM			Absolute error by ADM		
R_0	R_1	R_2	Y_0	Y_1	Y_2
0.0904984	0.000410736	9.28524×10^6	0.0904984	0.000410736	0.000823959
0.165293	0.00137365	0.0000558615	0.165293	0.00137365	0.00276254
0.228147	0.00262245	0.000144946	0.228147	0.00262245	0.00528518
0.281709	0.00400552	0.000269169	0.281709	0.00400552	0.00808715
0.327898	0.00543511	0.000418624	0.327898	0.00543511	0.0109907
0.36814	00686027	0.000584253	0.36814	0.00686027	0.0138915
0.403513	0.00825175	0.00075875	0.403513	0.00825175	0.0167293
0.434851	0.00959333	0.000936574	0.434851	0.00959333	0.0194699
0.462808	0.0108767	0.00111369	0.462808	0.0108767	0.0220955
0.487902	0.0120984	0.00128723	0.487902	0.0120984	0.0245984

Table 2

Comparison between the absolute error obtained by BCM and ADM for Example 2.

Absolute error by BCM			Absolute error by ADM		
R_0	R_1	R_2	Y_0	Y_1	Y_2
0.0181098	0.000899234	0.000899678	0.0181098	0.000899234	0.00134563
0.0360781	0.00177899	0.00178089	0.0360781	0.00177899	0.00369226
0.053909	0.00263963	0.00264416	0.053909	0.00263963	0.00722104
0.0716062	0.00348149	0.00348999	0.0716062	0.00348149	0.0120995
0.0891733	0.00430493	0.00431887	0.0891733	0.00430493	0.0184826
0.106614	0.00511026	0.00513128	0.106614	0.00511026	0.0265133
0.123931	0.00589781	0.00592765	0.123931	0.00589781	0.0363237
0.141129	0.00666792	0.00670845	0.141129	0.00666792	0.0480353
0.0181098	0.000899234	0.000899678	0.0181098	0.000899234	0.00134563
0.0360781	0.00177899	0.00178089	0.0360781	0.00177899	0.00369226



Figures 1
3D diagram for R_0

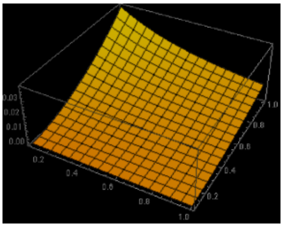


Figure 2
3D diagram for R_1

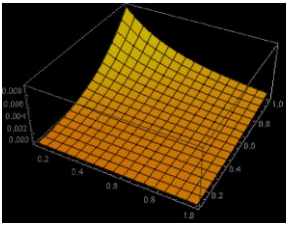


Figure 3
3D diagram for R_2

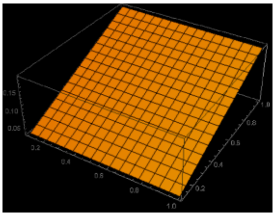


Figure 4
3D diagram for R_0

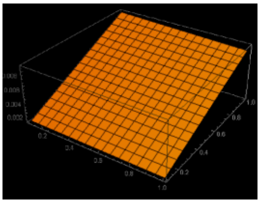


Figure 5
3D diagram for R_1

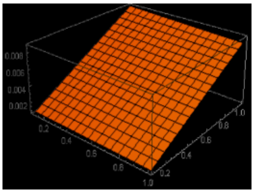


Figure 6
3D diagram for R_2

From Tables 1 and 2, we observe that BCM yields better results than ADM; accordingly, BCM is more accurate than ADM. The data of (R_0, R_1, R_2) can also be illustrated with Figure 1-6 above.

5. Conclusion

Based on the obtained results, we can point out the following conclusions:

1. In this research, the solution was successfully achieved by using BCM on the Heat Equation with Exponential Nonlinearity.
2. The results that were acquired from BCM are reliable results and are a close series of solutions with easy-to-calculate components that do not require any difficult assumptions, such as the ADM.
3. The implementation of the BCM is easy and fast to find exact and fast solutions compared to other methods. The BCM won't take long for calculations or characterization, and the results align more rapidly than with the other methods.
4. When comparing the approximate solutions acquired from the BCM method with the previous solutions by ADM, we find that the method used in this research is faster and more accurate.

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