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Practical uniform stability of two interconnected differential equations

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Abstract

This work investigates nonlinear systems, specifically perturbed triangle models with a customizable parameter, to determine their PU. IISS, or practically uniform integral input-to-state stability. Cascade interconnection techniques are used to conduct stability analyses under certain assumptions.

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1. Introduction

Nonlinear systems influenced by external factors are attracting growing attention. An input-to-stated stabilities, or Inp. SS were proposed by [1] and is significant to this area of study. As it applies the H_2 stability approach to nonlinear dynamics, this notion has proven useful in stability analysis and feedback system design [2].

As an alternative, a similar method, integral input-to-stated stabilities (In. ISS) is introduce in [3]. Then it was tweaked to handle continuous and

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discontinuous time models well [4-5]. The work in [6] focused on the stability of time-invariant cascade-connected systems, specifically on scenarios when an input. The ISS system is converted into a GAS system. According to [3], Int. ISS is nevertheless applicable in practice because of its intuitive interpretation, even if it is less strong in theory than Inp. SS. Systems with restricted responses to minor inputs and low overshoot are characterized by the International System of Stochastic Models (ISS), which is the nonlinear equivalent of the linear example of H_2 stability [7-9].

2. Preliminaries and Notation

$$\text{If } \frac{dx}{d\tau} = \mathbb{F}(t, x, v) \tag{1}$$

The variable τ represents time, $x \in \mathcal{R}^n$ be stated vector, & $v \in \Re^m$ a measurable input function.

The function classes used in the paper are defined as follows:

- $\mathcal{H}$: A set of strictly increasing functions and continuous $\alpha: \Re^+ \to \Re^+$ with $\alpha(0) = 0$.
- $\mathcal{H}_\infty$: Subclass of $\mathcal{H}$ containing unbounded functions.
- $\mathcal{J}$: Class of continuous functions $\mathcal{J}: \Re^+ \to \Re^+$ where $\mathcal{J}(r) \to 0$ as $r \to \infty$.

Lemma 2.1: *If* $\alpha_1, \alpha_2 \in \mathcal{H}_\infty$*, then:*

1. $\alpha_1^{-1} \in \mathcal{H}_\infty$.
2. $\alpha_1 \circ \alpha_2 \in \mathcal{H}_\infty$.
3. For all $r_1, r_2 \in \Re^+$:
$$\max\{\alpha_1(r_1), \alpha_2(r_2)\} \leq \alpha_1(r_1 + r_2) \leq \alpha_1(2r_1) + \alpha_2(2r_2)$$
4. It was a map $\alpha \in \mathcal{H}_\infty$ s.t:
$$\alpha(b_1 + b_2) \leq \alpha_1(b_1) + \alpha_2(b_2).$$

Definition 2.1: [10] The system is said to have the $\mathcal{L}^2$-to-$\mathcal{L}^\infty$ property if there exist $\alpha_1, \alpha_2 \in \mathcal{H}_\infty$ and $\beta \in \mathcal{HJ}$ where:

$$| x(t) | \leq \beta(| x_0 |, t) + \alpha_1\left(\int_0^t \alpha_2 (| v(s) |) ds\right).$$

Definition 2.2: [5] In the event that $\alpha_1, \alpha_2 \in \mathcal{H}_\infty$, a definite positive function α_3, and a gain $\gamma \in \mathcal{H}$ function that is differentiable on a continuous scale $U: \Re^n \to \Re^+$ is so-called an Int. ISS Lyapunov function in a way that:

$$\alpha_1(| x |) \leq U(x) \leq \alpha_2(| x |)$$
$$\frac{dU(x)}{d\tau} \leq -\alpha_3(| x |) + \gamma(| v |),$$

Definition 2.3: [5] If $\exists \gamma \in \mathcal{H}$ and a positive definite function β then $U: \Re^n \to \Re^+$ is considered Int. ISS Lyapunov, so satisfying:

$$\frac{dU(x,v)}{d\tau} \leq -\beta(| x |) + \gamma(| v |)$$

We now describe a cascade system where two subsystems interact under external inputs. Let $x = [x_1^T, x_2^T]^T \in \mathbb{R}^n$ be the state vector and $\lambda = [\lambda_1^T, \lambda_2^T]^T \in \mathbb{R}^k$ the input vector. The overall system, denoted Φ, is composed of two subsystems: Φ_1 and Φ_2, with Φ_1 depending on Φ_2 in a cascade structure. See Figure 1.

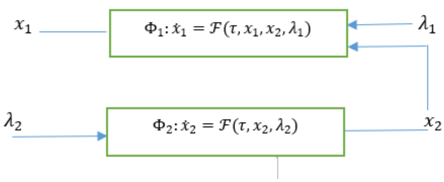

Figure 1
Cascade system *Φ*

Definition 2.4: [11] Let $\mathcal{Z}_1(n_1, m_1, \gamma_1, \gamma_{\lambda 1})$ and $\mathcal{Z}_2(n_2, m_2, \gamma_{\lambda 2})$ be defined by:

$$\dot{x}_1 = \mathbb{F}_1(\tau, x_1, x_2, \lambda_1)$$

$$\dot{x}_2 = \mathbb{F}_2(\tau, x_2, \lambda_2)$$

$$\mathbb{F}_i(\tau, 0, \dots, 0) = 0, \quad \forall \tau \in \Re^+, \quad i = 1, 2$$

Assume the following:

- A piece-wise continuous. function of τ & local Lipschitz functions in its argument are F_1 and F_2.
- There exist Lyapunov functions $U_1(\tau, x_1)$ and $U_2(\tau, x_2)$, both continuously differentiable.
- Their derivatives satisfy:

$$\frac{\partial U_1}{\partial \tau} + \frac{\partial U_1}{\partial x_1}\mathcal{F}_1 \leq -m_1(| x_1 |) + \gamma_1(| x_2 |) + \gamma_{\lambda 1}(| \lambda_1 |)$$

$$\frac{\partial U_2}{\partial \tau} + \frac{\partial U_2}{\partial x_2}\mathcal{F}_2 \leq -m_2(| x_2 |) + \gamma_{\lambda 2}(| \lambda_2 |)$$

Then both subsystems are Int. ISS (satisfying **Definition 2.2)**, and U_1, U_2 serve as valid Lyapunov functions.

As stated in Theorem 2.1 [5]: For the system (1), the following assertions are equivalent:

a. ISS is the system in question.

   The system is dissipative and satisfies weak detectability of the specified zero.

b. An integral smooth ISS Lyapunov function exists.

c. The system displays dissipative output behavior and is asymptotically stable at the origin on a global scale.

Remarks 2.1:

1- A system qualifies as Int. ISS if systems have Int. ISS Lyapunov functions.

2- An Int. The ISS system is characterized as 0-GAS, which denotes global asymptotic stability under zero input.

3- Presence for Int. ISS Lyapunov function guarantees the system's Int. ISS property**.**

Many other important concepts introduced in the paper [11] are reflected in this article. One of them a concept is trajectory-based definitions of input to stat stability, which differs from the Lyapunov-based definition. Then, it follows from **Definition 4.2** that an (Inp. SS) is implies to (Int. ISS).

Definition 2.5: A system is so-called an **integrals input-to-state practical stables (Int. ISPS)** where $\exists \alpha_1, \alpha_2 \in \mathcal{H}_\infty$, a function $\beta \in \mathcal{HJ}$, the constant $\varepsilon > 0$, the inequality holds:

$$|\, x(\tau)\, | \leq \beta(|\, x_0\, |, \tau) + \alpha_1\big(\int_0^\tau \alpha_2\,(|\, v(s)\, |)ds\big) + \varepsilon$$

Definition 2.6: Let $\alpha_1, \alpha_2 \in \mathcal{H}_\infty$, $\gamma_1, \gamma_2 \in \mathcal{H}$, and let $\alpha(\rho)$ be a positive scalar function such that $\alpha(\rho) \to 0$ as $\rho \to 0$. the $U\colon \mathfrak{R}^+ \times \mathfrak{R}^n \to \mathfrak{R}^+$ called PU. IISS Lyapunov if:

$$\alpha_1(|\, x\, |) \leq U(\tau, x) \leq \alpha_2(|\, x\, |)$$

and if the condition below is satisfied:

$$U(\tau, x) \ \geq \gamma_1(|\, v\, |) \Rightarrow \frac{d}{d\tau} U(\tau, x) \leq -\gamma_2(|\, x\, |) + \alpha(\rho)$$

$$\forall x \in \mathbb{R}^n,\ \tau \geq \tau_0 \ \&\ v \in \mathbb{R}^m.$$

Proposition 1.1: A system is practically uniformly integral input – to - stated stables (PU. IISS) iff there is a PU. IISS Lyapunov functions satisfying Definition 2.6.

Remark 1.1: The nonlinear system discussed in this work is affected by a parameter $\rho > 0$, which reduces the influence of perturbations as its value decreases.

3. Main results

3.1 *Practical uniform Int. ISS for the perturbed system*

We studied integral input-to-state stability (Int. ISS) of a nonlinear system influenced by a parameter and external perturbation.

Now, put the following nominal system:

$$\dot{x} = \mathbb{F}_\rho(\tau, x) \tag{2}$$

and

$$\dot{x} = \mathbb{F}_\rho(\tau, x) + \mathbb{Q}(\tau, x, v) \tag{3}$$

Is the perturbed system. So, assume the following:

- Although Fρ and Q are not known, they are both continuous in the τ & the local Lipschitzes in the x. An UGA is present in the system of nominal variables.
- Assuming τ and x are real numbers, a Lyapunov function U_ρ exists:

$$\alpha_{1\rho}(|\, x \,|) \le U_\rho(x) \le \alpha_{2\rho}(|\, x \,|)$$

$$\frac{\partial U_\rho}{\partial \tau} + \frac{\partial U_\rho}{\partial x}\mathbb{F}_\rho(\tau, x) \le -\alpha_{3\rho}(U_\rho) + \alpha(\rho)$$

- With $\alpha_{1\rho}, \alpha_{2\rho} \in \mathcal{H}_\infty$, $\alpha_{3\rho} \in \mathcal{H}$, and $\alpha(\rho)$ is positeve such that $\alpha(\rho) \to 0$ as $\rho \to 0$

  Let $0 < \psi < 1$, and define the gain function:

$$\omega(\varepsilon) = (\psi\alpha_{3\rho})^{-1} \circ \left(1 - \frac{\alpha_1}{\psi}(\alpha_{3\rho} \circ \alpha_{1\rho})^{-1}\right)^{-1} \circ \alpha_2 \,|\, \varepsilon \,|$$

Theorem 3.1.1: *Under the above conditions, the perturbed system is practically uniformly Int. ISS.*

Proof: Suppose that the derivative $U_\rho(x, v)$ trajectory the perturbed system satisfies:

$$\frac{d}{d\tau}U_\rho(x, v) \le -\alpha_{3\rho}(U_\rho) + \alpha(\rho) + \alpha_1(\rho) \,|\, x \,| + \alpha_2(\rho) \,|\, v \,|$$

Since $|\, x \,| \le \alpha_{1\rho}^{-1}(U_\rho)$, we substitute:

$$\frac{d}{d\tau}U_\rho(x, v) \le -\alpha_{3\rho}(U_\rho) + \alpha(\rho) + \alpha_1(\rho)\, \alpha_{1\rho}^{-1}(U_\rho) + \alpha_2(\rho) \,|\, v \,|$$

$$= -(1-\psi)\alpha_{3\rho}(U_\rho) - \psi\alpha_{3\rho}(U_\rho) + \alpha_1(\rho)\, \alpha_{1\rho}^{-1}(U_\rho) + \alpha_2(\rho) \,|\, v \,| + \alpha(\rho)$$

Now, for $U_\rho(x, v) \ge \omega(|\, v \,|)$, the condition:

$$\psi\alpha_{3\rho}(U_\rho) \ge \alpha_1(\rho)\alpha_{1\rho}^{-1}(U_\rho) + \alpha_2(\rho) \,|\, v \,|$$

is satisfied.

This confirms the system is PU. IISS.

3.2 *PU. IISS to cascade system of perturbed triangular system.*

Theorem (PU. IISS+ PU. IISS) 3.2.1

Consider two interconnected subsystems satisfying the following:

$$\frac{\partial U_{1\rho}}{\partial \tau} + \frac{\partial U_{1\rho}}{\partial x_1}\mathcal{F}_1(\tau, x_1, x_2, \lambda_1) \le -\gamma_{2\rho}(U_1) + \gamma_{1\rho}(|\, x_2 \,|) + \gamma_{\lambda 1\rho}(|\, \lambda_1 \,|) + \alpha_{1\rho}(\rho)$$

$$\frac{\partial U_{2\rho}}{\partial \tau} + \frac{\partial U_{2\rho}}{\partial x_2}\mathcal{F}_2(\tau, x_2, \lambda_2) \le -\gamma_{3\rho}(U_2) + \gamma_{\lambda 2\rho}(|\, \lambda_2 \,|) + \alpha_{2\rho}(\rho)$$

Assume:

- $U_{1\rho}$ and $U_{2\rho}$ are positive definite, radially unbounded, and continuously differentiable.
- $\gamma_1, \gamma_2, \gamma_3$ are in $\mathcal{H}$.

- $\alpha_{1\rho}, \alpha_{2\rho} > 0$, and $\lim_{\rho\to 0}\alpha_{i\rho}(\rho) = 0$

  Define that

$$\omega_\rho(|\varepsilon|) = (\psi\gamma_{3\rho})^{-1} \circ \left(1 - \frac{1}{\psi}\gamma_{1\rho} \circ \left(\bar{\alpha}_{2\rho} \circ \gamma_{3\rho}\right)^{-1}\right)^{-1} \circ \gamma_{\lambda\rho}(|\varepsilon|)$$

  Then the entire cascade system is PU. IISS.

Proof: Let $U_\rho = U_{1\rho} + U_{2\rho}$, We define $\alpha_\rho = \alpha_{1\rho} + \alpha_{2\rho}, and\ \gamma_{\lambda\rho}(|\lambda|) = \gamma_{\lambda 1\rho}(|\lambda_1|) + \gamma_{\lambda 2\rho}(|\lambda_2|)$.

Now, express $|x_2|$ using the inequality $|x_2| \le \bar{\alpha}_{2\rho}^{-1}(U_{2\rho})$, then $\gamma_{1\rho}(|x_2|) \le \gamma_{1\rho} \circ \bar{\alpha}_{2\rho}^{-1}(U_{2\rho})$, given by

$$\frac{d}{d\tau}U_\rho \le -\gamma_{2\rho}(U_{1\rho}) - \gamma_{3\rho}(U_{2\rho}) + \gamma_{1\rho} \circ \bar{\alpha}_{2\rho}^{-1}(U_{2\rho}) + \gamma_{\lambda\rho}(|\lambda|) + \alpha_\rho$$

Split the term $-\gamma_{3\rho}(U_{2\rho})$:

$$= -\gamma_{2\rho}(U_{1\rho}) - (1-\psi)\gamma_{3\rho}(U_{2\rho}) - \psi\gamma_{3\rho}(U_{2\rho}) + \gamma_{1\rho} \circ \bar{\alpha}_{2\rho}^{-1}(U_{2\rho}) + \gamma_{\lambda\rho}(|\lambda|) + \alpha_\rho$$

Define a new function $\gamma_4 \in \mathcal{H}$ such that $\gamma_4(U_\rho) = \gamma_{2\rho}(U_{1\rho}) + (1-\psi)\gamma_{3\rho}(U_{2\rho})$, then $\frac{d}{d\tau}U_\rho \le -\gamma_4(U_\rho) + \left[\gamma_{1\rho} \circ \bar{\alpha}_{2\rho}^{-1} - \psi\gamma_{3\rho}\right](U_\rho) + \gamma_{\lambda\rho}(|\lambda|) + \alpha_\rho$

If for $U_\rho \ge \omega_\rho(|\lambda|)$, the expression:

$\left[\gamma_{1\rho} \circ \bar{\alpha}_{2\rho}^{-1} - \psi\gamma_{3\rho}\right](U_\rho) + \gamma_{\lambda\rho}(|\lambda|)$ is nonpositive, then $\frac{d}{d\tau}U_\rho \le -\gamma_4(U_\rho) + \alpha_\rho$

This proves that the system be the PU. IISS.

3.3. *Perturbations term that depended on the parameters*

Consider a cascade systems described as:

$$\dot{x}_1 = \mathbb{F}_1(\tau, x_1, x_2, \lambda_1) + \mathbb{Q}_{1\rho}(\tau, x_1, x_2) \tag{4}$$

$$\dot{x}_2 = \mathbb{F}_2(\tau, x_2, \lambda_2) + \mathbb{Q}_{2\rho}(\tau, x_2) \tag{5}$$

Assume:

- $\mathbb{F}_1$ and $\mathbb{F}_2$ exhibit continuity in τ and local Lipschitz conditions piecewise in their arguments.
- The systems satisfy $\mathbb{F}_1(\tau, 0,0,0) = 0$ and $\mathbb{F}_2(\tau, 0,0) = 0$.

Assumption Ω1: The nominal sub system $\dot{x}_1 = \mathcal{F}_1(\tau, x_1, 0, \lambda_1)$ is PUiISS. There are Lyapunov functions $U_1(\tau, x_1)$ s.t: [12]

$$C_1 |x_1|^2 \le U_1(\tau, x_1) \le C_2 |x_1|^2$$

$$\frac{\partial U_1}{\partial \tau} + \frac{\partial U_1}{\partial x_1}\mathbb{F}_1(\tau, x_1, 0, \lambda_1) \le -\gamma_1(U_1) + \gamma_{\lambda 1}(|\lambda_1|) + \alpha_1$$

$$\left| \frac{\partial U_1}{\partial x_1} \right| \leq C_3 \mid x_1 \mid, \quad \left| \frac{\partial U_1}{\partial x_1} \mathbb{Q}_{1\rho}(\tau, x_1, x_2) \right| < \kappa_1(\rho)$$

with constants $C_1, C_2, C_3, \alpha_1 > 0$, and $\kappa_1(\rho) \to 0$ as $\rho \to 0$. Assume also that $\left| \frac{\partial U_1}{\partial x_1} \right| < L$ for some $L < \infty$.

Assumption Ω2: [13] The nominal subsystem $\dot{x}_2 = \mathbb{F}_2(\tau, x_2, \lambda_2)$ is PU. IISS with a Lyapunov function $U_2(\tau, x_2)$ s.t:

$$D_1 \mid x_2 \mid^2 \leq U_2(\tau, x_2) \leq D_2 \mid x_2 \mid^2$$

$$\frac{\partial U_2}{\partial \tau} + \frac{\partial U_2}{\partial x_2} \mathbb{F}_2(\tau, x_2, \lambda_2) \leq -\gamma_2(U_2) + \gamma_{\lambda 2}(\mid \lambda_2 \mid) + \alpha_2$$

$$\left| \frac{\partial U_2}{\partial x_2} \mathbb{Q}_{2\rho}(\tau, x_2) \right| < \kappa_2(\rho)$$

with constants $D_1, D_2, \alpha_2 > 0$, and $\kappa_2(\rho) \to 0$ as $\rho \to 0$

We define that**:** $\omega(\mid \varepsilon \mid) = (\psi\gamma_4)^{-1} \circ \left(1 - \frac{C_3 L}{2\psi \min(C_1, D_1)} \gamma_4^{-1}\right)^{-1} \circ \gamma_\lambda(\mid \varepsilon \mid)$

Theorem 3.3.1: *When assumptions Ω1 & Ω2 hold, so cascade systems be the PU. IISS.*

Proof: Define total Lyapunov function $U = U_1 + U_2$, and take the time derivative:

$$\frac{dU}{d\tau} = \frac{\partial U_1}{\partial \tau} + \frac{\partial U_1}{\partial x_1}\left(\mathcal{F}_1 + \mathcal{Q}_{1\rho}\right) + \frac{\partial U_2}{\partial \tau} + \frac{\partial U_2}{\partial x_2}\left(\mathcal{F}_2 + \mathcal{Q}_{2\rho}\right)$$

Apply the assumptions:

$$\leq -\gamma_1(U_1) - \gamma_2(U_2) + \gamma_{\lambda 1}(\mid \lambda_1 \mid) + \gamma_{\lambda 2}(\mid \lambda_2 \mid) + \alpha_1 + \alpha_2 + C_3 L \mid x_1 \mid\mid x_2 \mid + \kappa_1(\rho) + \kappa_2(\rho)$$

Using: $\mid x_1 \mid\mid x_2 \mid \leq \frac{1}{2C_1} U_1 + \frac{1}{2D_1} U_2$

we obtain:$\frac{dU}{d\tau} \leq -\gamma_4(U) + \gamma_\lambda(\mid \lambda \mid) + \alpha + \kappa(\rho) + \frac{C_3 L}{2\min(C_1, D_1)} U$

$$= -(1-\psi)\gamma_4(U) - \psi\gamma_4(U) + \frac{C_3 L}{2\min(C_1, D_1)} U + \gamma_\lambda(\mid \lambda \mid) + \kappa(\rho) + \alpha$$

If $\left[\frac{C_3 L}{2\min(C_1, D_1)} - \psi\gamma_4\right](U) + \gamma_\lambda(\mid \lambda \mid) \leq 0$, for $U \geq \omega(\mid \lambda \mid)$, then

$$\frac{dU}{d\tau} \leq -(1-\psi)\gamma_4(U) + \alpha + \kappa(\rho)$$

So the system is PU. IISS.

3.4. *Nominal System Based on a Parameter*

We now examine a triangular system composed of two perturbed subsystems affected by a positive parameter ρ:

$$\dot{x}_1 = \mathbb{F}_{1\rho}(\tau, x_1) + \mathbb{Q}(\tau, x, v) \quad (6)$$

$$\dot{x}_2 = \mathbb{F}_{2\rho}(\tau, x_2) \quad (7)$$

where:$x = [x_1^T, x_2^T]^T \in \Re^{2n}$

- v the local essential bound input such that $v: \Re^+ \to \Re^m$.
- $\mathbb{F}_{1\rho}, \mathbb{F}_{2\rho}, \mathbb{Q}$ are differentiable with respect to x and τ.
- Q is piecewise continuous and locally Lipschitz.

Assumption Ʊ1: For $i = 1,2$, assume the subsystem $\dot{x_i} = \mathbb{F}_{i\rho}(\tau, x_i)$ the latst estimate the global practical uniform asymptotic stables. Then, for each i, there is Lyapunov functions $U_{i\rho}(\tau, x_i)$ s.t:

$$\alpha_{i\rho}(\mid x_i \mid) \leq U_{i\rho}(\tau, x_i) \leq \alpha_{(2i+1)\rho}(\mid x_i \mid)$$

$$\frac{\partial U_{i\rho}}{\partial \tau} + \frac{\partial U_{i\rho}}{\partial x_i}\mathcal{F}_{i\rho}(\tau, x_i) \leq -\gamma_{(2i+2)\rho}(U_{i\rho}) + \alpha_i(\rho)$$

with $\alpha_{i\rho}, \alpha_{(2i+1)\rho} \in \mathcal{H}_\infty, \gamma_{(2i+2)\rho} \in \mathcal{H}$ and $\alpha_i(\rho) > 0$, and $\lim_{\rho\to 0}\alpha_i(\rho) = 0$

Assumption Ʊ2: The perturbation term satisfies:

$$\mid \frac{\partial U_{1\rho}}{\partial x_1}\mathbb{Q}(\tau, x, v) \mid \leq \alpha_3(\rho) \mid x_1 \mid + \alpha_4(\rho) \mid x_2 \mid + \gamma_{7\rho}(\mid v \mid)$$

where $\alpha_3, \alpha_4 > 0$ $as\ well\ as$ $\gamma_{7\rho} \in \mathcal{H}$

Define auxiliary functions, From Lemma 2.1, define:

$$\gamma_\rho(\varepsilon) = \min\left\{\gamma_{4\rho}\left(\frac{\varepsilon}{2}\right), \gamma_{6\rho}\left(\frac{\varepsilon}{2}\right)\right\}, \delta_\rho(\varepsilon) = \max\{\alpha_3(\rho)\alpha_{1\rho}^{-1}(\varepsilon), \alpha_4(\rho)\alpha_{2\rho}^{-1}(\varepsilon)\}$$

and $\omega(\varepsilon) = (\psi\gamma_\rho)^{-1} \circ \left(1 - \frac{1}{\psi}\delta_\rho \circ \gamma_\rho^{-1}\right)^{-1} \circ \gamma_{7\rho}(\mid \varepsilon \mid)$.

Theorem 3.4.1: *The triangular perturbed system is PU. IISS, If both Ʊ1 and Ʊ2 are satisfied.*

Proof: Let $U_\rho = U_{1\rho} + U_{2\rho}$, then $\frac{dU_\rho}{d\tau} \leq -\gamma_{4\rho}(U_{1\rho}) + \alpha_1(\rho) - \gamma_{6\rho}(U_{2\rho}) + \alpha_2(\rho) + \alpha_3(\rho) \mid x \mid + \alpha_4(\rho) \mid x \mid + \gamma_{7\rho}(\mid v \mid)$

Since $\mid x \mid \leq \alpha_{1\rho}^{-1}(U_{1\rho}) + \alpha_{2\rho}^{-1}(U_{2\rho})$, we combine terms:

$$\frac{dU_\rho}{d\tau} \leq -\gamma_\rho(U_\rho) + \delta_\rho(U_\rho) + \gamma_{7\rho}(\mid v \mid) + \alpha_1(\rho) + \alpha_2(\rho)$$

Let $\alpha(\rho) = \alpha_1(\rho) + \alpha_2(\rho)$, then:

$$\frac{dU_\rho}{d\tau} \leq -(1-\psi)\gamma_\rho(U_\rho) - \psi\gamma_\rho(U_\rho) + \delta_\rho(U_\rho) + \gamma_{7\rho}(\mid v \mid) + \alpha(\rho)$$

So, $\frac{dU_\rho}{d\tau} \leq -(1-\psi)\gamma_\rho(U_\rho) + \alpha(\rho)$

Thus, the system is PUiISS.

Example 1(Stability of a Triangular Cascade System): Consider the nonlinear system:

$$\dot{x_1} = -\rho\ln(x_1) + \left(\frac{x_1 x_2}{v} + \tau^2\right)v \tag{8}$$

$$\dot{x}_2 = -\rho \ln(x_2) + 1 + \frac{\tau}{1+x_2^2} \tag{9}$$

In comparison

$$\mathcal{F}_{1\rho}(\tau, x_1) = -\rho \ln(x_1), \mathcal{F}_{2\rho}(\tau, x_2) = -\rho \ln(x_2) + 1 + \frac{\tau}{1+x_2^2},$$

$$Q(\tau, x, v) = \left(\frac{x_1 x_2}{v} + \tau^2\right) v$$

Use the Lyapunov functions $U_{1\rho} = \rho \ln(x_1)$ and $U_{2\rho} = \rho \ln(x_2)$, these functions meet the conditions of assumptions Ʊ1 and Ʊ2.

4. Conclusion

This work established, under appropriate conditions, a perturbed triangular system for the nonlinear differential equation exhibiting PU. IISS. We demonstrate that a system remains stable despite external perturbations. The result made significant contributions to understanding the Inp. SS and its integral variant Int. ISS, particularly in the nonlinear system influenced by disturbance.

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