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Positive and co-positive L_p approximation with interpolation constraint

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Abstract

It is well known, that if f be continuous functions define on $[a, b]$, $a, b \in \mathbb{R}$, there is interpolating polynomials P_n on $\{a_i\}_{i=1}^{m}$ satisfy: $\|f - P_n\|_p \le \frac{1}{n^r}\, \omega_t\left(f^{(r)}, \frac{1}{n}\right)_p$, where ω_t is the t – th usual modulus for the smoothness' , and $P_n^{(i)}(a_i) = f^{(i)}(a_i)$, $1 \le i \le m$, $m < r$, $m, r \in \mathbb{N}$. In this paper we shall answer the following questions: Is the above result true for intertwining, one-sided, positive, co-positive approximation.

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1. Introduction

In this paper we need the following notations and definition: from [1]. Where $I := [-1,1]$, and $k \ge 3$, we say that on interval I the map f be k-monotone, if $\Delta_h^k(f) \ge 0$, $\forall x \in I$ & h≥ 0, where $\Delta_h^k(f)$ is k-th symmetric difference of f, and write $f \epsilon \Delta^{(k)}(I)$. In particular and respectively $\Delta^{(2)}(I)$, $\Delta^{(1)}(I)$ and $\Delta^{(0)}(I)$, are the sets of all convex, monotone and nonnegative

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functions on I. From [2] let $s \in \mathbb{N}$, where $\mathbb{N}$ be sets for natural numbers. and $\mathbb{Y}_s$ a sets for family $Y_s := \{y_i\}_{i=1}^{s}$ for point y_i, such that $-1 < y_i < \cdots < y_1 < 1$, in $I := [-1,1]$, we defined $y_0 := 1$ and $y_{s+1} := -1$ and $\Delta^{(k)}(Y_s)$ is sets for the function that changed their k- monotonicity at point for Y_S. So $\mathbb{Y}_0 := \{\emptyset\}$ and $\Delta^{(k)}(Y_0) := \Delta^{(k)}(I)$ if $s = 0$. Let $q \in \mathbb{N}$, and $\mathbb{A}_{q,}$ be the set of all collection $A_q := \{\alpha_i\}_{i=1}^{q}$ of points α_i such that $-1 \leq \alpha_q < \cdots < \alpha_1 \leq 1$. Let $C^r(I)$ be spaces for each r −time continuous differential function on I. For given $r \in \mathbb{N}_{0,}$ where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, denoted as $\mathcal{J}_p^{(r)}(f, A_q)$ a set for the function satisfying interpolation constraint at points $A_{q,}$ such that $\mathcal{J}_p^{(r)}(f, A_q) := \{ g \in L_p^{(r)}(I) \mid f^{(i)}(\alpha) = g^{(i)}(\alpha)\}$, for all $\alpha \in A_q$ and $0 \leq i \leq r\}$. using notations $\mathcal{J}_p(f, A_q) := \mathcal{J}_p^{(0)}(f, A_q)$, if $r = 0$. In L_p spaces, $0 < p < 1$, given $f \in L_p^{(r)}(I), r \in \mathbb{N}_0$, and $L_p^{(r)}(I) := \{ f: I \to \mathbb{R}, f, f^{(r)} \in L_p(I)\}$, where $L_p := L_p(I) := \left\{ f: I \to \mathbb{R}, \|f\|_P = (\int_I \|f\|^p)^{1/p} < \infty \right\}$, [1]. And $\mathbb{R}$ is the set of real numbers. As usual, $\mathbb{P}_n$ be sets for every polynomial that has degrees $\leq n$. And $\sigma_n := n^{-2} + n^{-1}\sqrt{1 - x^2}$, [3 – 4]. And $\|\cdot\|_p$ is L_p – norm on I such that $\|f\|_P = (\int_I \|f\|^p)^{1/p}$. [1] And $\omega_t(f,\cdot; I)_p := \omega_t(f,\cdot)_p$ be t −th usual modulus for the smoothness for the f, such that $\omega_t(f, \delta; I)_p = \sup_{0<h\leq\delta} \|\Delta_h^t(f,.)\|_{L_p(I)}, \delta \geq 0$, [3], [4] And $\tau_t(f,\cdot; I)_p := \tau_t(f,\cdot)_p$ is τ−modulus for the smoothness for the f in $L_p(I)$, such that $\tau_t(f, \delta)_p := \tau_t(f, \delta; I)_{p(I)} = \left\|\omega_t(f, x; \delta)_p\right\|_{L_p(I)}$, [3], And $\mathcal{W}_p^r(I)$ is the set of all function from $L_p(I)$ s.t $f^{(r-1)}$ be absolutely continuous & $f^{(r)} \in L_p(I)$, [3] The degree of best approximation of f with interpolation constraints $E_n^{(k)}(f, Y_s;\, A_q)_p$ is defined as [2]:

$$E_n^{(k)}(f, Y_s;\, A_q)_p := \inf\{\|f - P_n\|_p \mid P_n \in \Delta^{(k)}(Y_s) \cap \mathcal{J}_p(f, A_q) \cap \mathbb{P}_n\}.$$

2. Main Results.

2.1 *k −Monotone L_p Space, $0 < p < 1$ Approximations with Generals Interpolatory Constraint are Impossible where $k \geq 1$*

First let us introduce the following auxiliary lemma:

Lemma 2.1: [5]. *Let $V_i = (x_i, y_i)$, $1 \leqq i \leqq n, n \geqq 2, a \leqq x_1 < \cdots < x_n \leq b$, is n distinct point on graphs for functions f. Then f be $n -$ convex iff every such set for the n distinct point, A graphs lies alternate above & below a curves $y = \pi_{n-1}(f; x; V_i)$, lies below where $x_{n-1} \leqq x \leqq x_n$. and $\pi_{n-1}(x; V_i) \leqq f(x), x_n \leqq x \leqq b$; & $\pi_{n-1}(x;\, V_i) \leqq f(x)(\geqq f(x))$ if $a \leqq x < x_1$, n being even (odd).*

Lemma 2.2: [3]. *Let f and g function in $L_p(I)$, such that $g \neq 0$, then*

$$\|f + g\|_p \leq \left(\|f\|_p^p + \|g\|_p^p\right)^{1/p} \leq 2^{\frac{1}{p}-1}\left(\|f\|_p + \|g\|_p\right).$$

Lemma 2.3: [6] *For any* $k \in \mathbb{N}, Y_s \in \mathbb{Y}_{s,}$ *and* $s \in \mathbb{N}_{0,}$ $\exists f \in \Delta^{(k)}(Y_s)$ *& sets* $A_q \in \mathbb{A}_{q,}$*where* $q \in k+2$, *such that,* $\Delta^{(k)}(Y_s) \cap \mathcal{J}_p(f, A_q) \cap \mathbb{P}_n = \emptyset, \forall n \in \mathbb{N}_{0.}$

Proof: Choosing A_q, where $q \in k+2$, such that

$$y_1 < \alpha_{k+2} < \alpha_{k+1} < \cdots < \alpha_2 < \gamma < \alpha_1. \tag{1}$$

Not that A_q is measurable set and $\mu(A_q) < \infty$, and $\chi_{A_q} \in L_p(\mu)$, such that $\chi_{A_q} = \begin{cases} 1, x \in A_q \\ 0, x \notin A_q \end{cases}$. Defining the function $f(x) = (-\gamma)_+^k = \begin{cases} (-\gamma)^k ; \, if \; x \le \gamma \\ 0 ; \, if \; x > \gamma \end{cases}$, the function $f \in \Delta^{(k)}(Y_s)$, because f is $k-$ monotone on $[y_1, 1]$, and $f \in L_p, 0 < p < 1$, because: $\|f\|_p = (\int_{y_1}^{\gamma} |(-\gamma)^k|^p \, \mathrm{dx})^{1/p} < \infty$. By using Lemma 2.2 that affects convergence analyses. Let $f \in \Delta^{(k)}(I), k \in \mathbb{N}$ and,

$$(f - \mathcal{L}_{k-1}) \in \Delta^{(0)}(\{m_i\}_{i=1}^k). \tag{2}$$

Suppose there exist a polynomial $P_n \in \Delta^{(k)}(Y_s) \cap \mathcal{J}_p(f, A_q) \cap \mathbb{P}_n$, so P_n be $k-$ monotones on $[y_1, 1]$, and by definition of $\mathcal{J}_p(f, A_q)$, P_n must satisfy: $P_n(\alpha_i) = 0$, and $P_n(\alpha_1) = f(\alpha_1) = (-\gamma)^k > 0$. By Lemma 2.1. If P_n is $k-$ monotone, it can be interpolated by polynomial ℓ_1 at points $\alpha_{k+2}, \cdots, \alpha_3$, such that $\ell_1 = \mathcal{L}_{k-1}(\alpha_{k+2}, \cdots, \alpha_3) \equiv 0$, and so from (2) that $P_n(x) = P_n - \ell_1(x) \ge 0$, for all $x \ge \alpha_3$. Similarly define another polynomial ℓ_2 at points $\alpha_{k+2}, \cdots, \alpha_4, \alpha_2$, such that $\ell_2 = \mathcal{L}_{k-1}(\alpha_{k+2}, \cdots, \alpha_{4,}\alpha_{2,}) \equiv 0$, here $-P_n(x) = (-1)^k(P_n - \ell_2(x)) \ge 0$, for all $\alpha_4 \le x \le \alpha_2$. From $\ell_1 \equiv 0$, we get $P_n(x) \ge 0$, for all $x \ge \alpha_3$. And from $\ell_2 \equiv 0$, we get $(-1)^k P_n(x) \ge 0$, $\forall \alpha_4 \le x \le \alpha_2$. but $k \ge 1$, this implies $P_n(x) \le 0$, on $[\alpha_4, \alpha_2]$. And on interval $[\alpha_3, \alpha_2]$, So a $k-$ monotone polynomial cannot be zero on one interval and positive on another these contradictions so the polynomials P_n not exists [10].

Now if we think about minimizing the number of interpolating points. Is it possible to have interpolating $k-$ monotone approximation? The answer it may be possible. But it may be not have a good $k-$ monotone interpolatory approximation.

Proposition 2.4: If $A_1 = \{\alpha\} \subset I$, then

$$E_n^{(k)}(f, Y_s;\, A_1)_p \le 2^{\frac{1}{p}} E_n^{(k)}(f, Y_s;\, \emptyset)_p, 0 < p < 1.$$

Proposition 2.5: If $Y_s \subset [-1,0)$, $f(x) = x_+^k \in \Delta^{(k)}(Y_s)$, and $A_{k+1} \subset (y_1, 0)$, then$E_n^{(k)}(f, Y_s;\, A_{k+1})_p = 1$.

Proof: Let $f(x) = x_+^k = \begin{cases} x^k, \text{if } x \ge 0 \\ 0, \text{otherwise} \end{cases}$, $Y_s \subset [-1,0)$, and $A_{k+1} \subset (y_1, 0)$ is (a set of $k+1$ interpolation points on $(y_1, 0)$), where y_1 is left most point of Y_s, and $f \in \Delta^{(k)}(Y_s)$, s.t f be the $k-$ monotones on *the* Y_s, since $f(x) = 0$ on A_{k+1}, any polynomial P_n that interpolates f at A_{k+1} must satisfy $P_n(x) = 0$, $\forall x \in A_{k+1}, P_n$ is $k-$ monotone on Y_s if $k-$th derivative nonnegative on Y_s, sine $0 < p < 1$, the L_p space is not a normed but it is achieved Lemma 2.2. So by factor theorem if P_n has $k+1$ roots in $(y_1, 0)$, then $P_n(x) = (x-x_1)(x-x_2)\cdots(x-x_{k+1})\varphi(x)$, where $x_1, x_2, \cdots, x_{k+1} \in A_{k+1}$, and $\varphi(x)$ is a polynomial of degree $n-(k+1)$. However, sine $f(x)=0$ on Y_s and P_n must be $k-$ monotone on Y_s, the only way to achieve this is both $k-$ monotone and interpolation is for $P_n(x)=0$ on $(y_1, 0)$. If $P_n(x)=0$ on $(y_1, 0)$, then $P_n(x)=0$ on $Y_s \subset [-1,0)$, and sine $f(x)=0$ on Y_s. The error $\|f-P_n\|_{L_{p(y_1,0)}} = \|0-0\|_{L_{p(y_1,0)}} = 0$. If $P_n(x)=0$ on $(y_1, 0)$, then $P_n(x)=0$ on $Y_s \subset [-1,0)$, and sine $f(x)=x^k$ on $[0,\infty)$ and P_n be polynomials for the degrees n & it a best approximations for the $f(x)=x^k$ on $[0,\infty)$, such that minimal error is a chieved $\|f-P_n\|_{L_{p[0,\infty)}} = \|x^k - 0\|_{L_{p[0,\infty)}} = 1$. When $P_n(x)=0$, everywhere, the degree of best approximation of f $E_n^{(k)}(f, Y_s;\ A_{k+1})_p = 1$ is achieved.

3. Positive, Co-positive, One-sided and Intertwining Approximation with Interpolator Constraints

In this section we study different kinds of shape preserving approximation, In particular we focus on Positive, co-positive, one-sided, intertwine approximations with interpolators constraint. Below let us introduced some of details of key concepts with definitions. In [7] the one-sided approximation is a special case of intertwining approximation where $Q \le f \le \tilde{Q}$ on I, such that can be approximation of f by two polynomials Q and $\tilde{Q}$. With respect to $Y_s \in \mathbb{Y}_{s,}$ a polynomials $\{\tilde{Q}, Q\}$ is called an intertwining pair for f if $\tilde{Q}-f, f-Q \in \Delta^{(0)}(Y_s)$. Here $\tilde{Q} \ge f$ and $Q \le f$ on Y_s, and $\Delta^{(0)}(Y_s)$ is the set of function that change nonnegative on Y_s. If s = 0 it usual referr to as one-sided approximations [8]. A degree for intertwine one-sided approximations be define as infimum for $\|\tilde{Q}-Q\|$, where $\{\tilde{Q}, Q\}$ is an intertwining polynomials pair for f, it can be defined as:

$\breve{E}_n(f, Y_s) := \inf\{\|f-\tilde{Q}\| \ |\tilde{Q} \in \breve{\Delta}(f, Y_s) \cap \mathbb{P}_n\}$ For s=0, the degree of one-sided approximation is denoted by $\breve{E}_n(f) := \breve{E}_n(f, Y_0)$. Another kind for shapes preserve approximations called co-positive approximations if $f \in \Delta^{(0)}(Y_s)$, this means the polynomial $\tilde{Q} \in \Delta^{(0)}(Y_s)$. Here, f and $\tilde{Q}$ have the same sign on Y_s. such that polynomial $\tilde{Q}$ is co-positive approximation of f, if it is preserves the sign of f on a given interval. Note, the case of co-positive approximation for $k=0$, is not covered by Lemma 2.3. But on general sets A_q, co-positive polynomial approximation with interpolation conditions always possible. While, its rates for approximations may be worse compared to co-positive approximation without interpolation constrains.

4. Relationship Between Intertwining/One Sided and Co-positive/Positive Approximation

While intertwining/one-sided approximation implies co-positive/positive approximation, we want to prove that.

Proposition 4.1: On $L_{p,}$ spaces for $0<p<1$. If intertwining/one-sided approximation is possible then co-positive/positive approximation is also possible.

Proof: Suppose $\{\tilde{Q}, Q\}$ be intertwine polynomials pair of f, relative to $Y_s \in \mathbb{Y}_s$, we have $\tilde{Q} - f, f - Q \in \Delta^{(0)}(Y_s)$. Here $\tilde{Q} \geq f$ and $Q \leq f$ on Y_s, we consider the polynomials $\tilde{Q}$ and sine $\tilde{Q} - f \in \Delta^{(0)}(Y_s)$, we get $\tilde{Q} = (\tilde{Q} - f) + f$, since $f \in \Delta^{(0)}(Y_s)$ and $\tilde{Q} - f \in \Delta^{(0)}(Y_s)$ then $\tilde{Q} \in \Delta^{(0)}(Y_s)$, thus $\tilde{Q}$ is co-positive approximation of f and it is preserves the sign of f on Y_s. Similarly, the intertwining polynomial Q satisfy $f - Q \in \Delta^{(0)}(Y_s)$, and Q is co-positive approximation of f and it also preserves the sign of f on Y_s $[13, 14]$.

But the converse of above proposition is not always true such that co-positive/positive approximation does not imply intertwining approximation. In fact: If we suppose $f \in \Delta^{(0)}(Y_s)$, and let $\tilde{Q}$ be a co-positive approximation of f, this means that $\tilde{Q} \in \Delta^{(0)}(Y_s)$, and $\tilde{Q}$ preserves the sign of f on Y_s. We need a polynomials pair $\{\tilde{Q}, Q\}$ such that $\tilde{Q} \geq f$ and $Q \leq f$ on Y_s, if $\tilde{Q} - f \in \Delta^{(0)}(Y_s)$, while $\tilde{Q}$ satisfies $\tilde{Q} \geq f$, there is no polynomial Q may exist that satisfies $Q \leq f$ on Y_s. So the existence of single co-positive approximation $\tilde{Q}$ not guarantee the existence of a second polynomial Q that limits f from below.

It is well known that the Timan-Dzyadyk-Freud-Brudnyi direct theorem provides a rigorous mathematical framework for understanding the relationship between the smoothness of a function and its rate of polynomial approximation, highlighting the vital role of $\sigma_n(x)$ in achieving a balance between pointwise estimates and inverse results (see e.g. [9]).

4.3. *Summary for the Fundamental Result*

Let us summarize the results on co-positive, one-sided, intertwining polynomial approximation without interpolating constrains.

Table 1

Approximations without interpolation constraint

One-sided approximations		
$f \in C$	$\exists P_n \geq f$: $\lvert f(x) - P_n(x)\rvert \leq c\omega_t(f, \sigma_n(x))$	[6, Th.1]
Positive approximations		
$f \in C$	$\exists P_n \geq 0$: $\lvert f - P_n(x)\rvert \leq c\omega_t(f, \sigma_n(x))$	[5, Th. 3], or [6, cor. 2]
Intertwine approximations		
$f \in C$	$\breve{E}_n(f, Y_s) \nleq c\lVert f\rVert$	[6, Th. 13]
$f \in C^1$	$\exists P_n \geq \breve{\Delta}(f, Y_s)$: $\lvert f(x)(x) - P_n(x)\rvert \leq c\,\sigma_n(x)\,\omega_t(f', \sigma_n(x))$	[6, Th. 5]
	Co-positive approximations	
$f \in C$	$\exists P_n$, copositives & f: $\lvert f(x) - P_n(x)\rvert \leq c\omega_3(f, \sigma_n(x))$	[6, Th. 7]
	$\omega_3(f, \sigma_n(x))$ can not replace by $\omega_4(f, n^{-1})$	[11, 12]
$f \in C^1$	$\exists P_n$, copositive & f: $\lvert f(x)\, p_n(x)\rvert \leq c\sigma_n(x)\omega_t(f', \sigma_n(x))$	[5, Th.1] Or [6, cor. 6]

Now let us study Table 1, with interpolation constrain. And put the conditions on approximation polynomials that interpolated functions f with its derivative on L_p spaces, $0 < p < 1$, this work provides a rigorous mathematical analysis between (Hermit interpolation [8-10] and function approximation) such that at a certain point $A_q \in \mathbb{A}_{q,}$ approximation polynomials must interpolate the function and its derivatives (if possibly), where at such points the function value matches not only the polynomial but also its derivatives. This kind of interpolation is known as Hermit interpolation. In space $\breve{\Delta}(f, Y_s)$ any continuous function f at the points in $Y_s \in \mathbb{Y}_s$, satisfy inherently interpolation conditions this mean the approximation set sub set of interpolation set, it mean $\breve{\Delta}(f, Y_s) \subseteq \mathcal{J}_p(f, Y_s)$. Also the paper investigates whether the interpolation conditions can be extended to derivatives of the function f. Accuracy of approximation do not just rely on smoothness of f, but also location points of interpolation, such that the points in A_q lie in the interior of interval I or on the boundary of interval I, and so depend on odd or even of highest derivative of f. At the points from $Y_s \in \mathbb{Y}_s$, the interpolation conditions of derivatives is more complex than usual approximation. For $A_q \in \mathbb{A}_{q,}$ and $Y_s \in \mathbb{Y}_s$, always taking into consideration assumed that $A_q \cap Y_s = \emptyset$, such that $A_q \cap Y_s := \{\lambda_i\}_{i=1}^{q+s}$, with $-1 \le \lambda_{q+s} < \lambda_{q+s-1} < \cdots < \lambda_2 < \lambda_1 \le 1$, and $d(Y_s, A_q) := \min_{\mathrm{P}} = \{\lambda_{i-1} - \lambda_i | 1 \le i \le q+s+1\}$, where $\lambda_0 := 1$, $\lambda_{q+s+1} := -1$, so $d(\emptyset, A_q) := d(A_q)$, $d(Y_s, \emptyset) := d(Y_s)$, and $\min_{\mathrm{P}}(S)$ is from the set S of non-negative real numbers that satisfy smallest positive number.

5. The Fundamental Results

In this section we shall introduce our main results.

5.1. *Intertwining and one-sided approximation with interpolator conditions*

We put the next notation $\dagger_{\text{intertwing}} := \bigcup_{r\ge1} \dagger_r$, where

$$\dagger_r := \left\{(r, a_1, a_2, a_3) \middle| \begin{matrix} a_1 = r-2, a_2 = r-1, a_3 = r-1, \text{if r is odd;} \\ a_1 = r-1, a_2 = r-2, a_3 = r-1, \text{if r is even} \end{matrix}\right\}.$$

Hence

$$\dagger_{\text{intertwing}} := \{(2j+1, 2j-1, 2j, 2j) | j \in \mathbb{N}_0\} \cup \{(2j, 2j-1, 2j-2, 2j-1) | j \in \mathbb{N}\}. \tag{4}$$

So, the only condition in which $a_i's$ can be negative is $\dagger_1 := \{(1, -1, 0, 0)\}$, when r =1. The next is our fundamental result for intertwining L_p space, $0 < p < 1$, approximation with interpolatory conditions.

Theorem 5.2: *Intertwining L_p space,* $0 < p < 1$, *approximation with interpolatory conditions. Let* $A_q \in \mathbb{A}_{q,}$ $Y_s \in \mathbb{Y}_s$, $t, q, s \in \mathbb{N}$, *and* $(r, a_1, a_2, a_3) \in \dagger_{intertwing}$ *defined in (4). Then*

$\|f(x) - P_n(x)\|_p \le c(t, r, s, q)\sigma_n^r(x) \quad \tau_t(f^{(r)}, \sigma_n(x))_p, x \in I,$ for some $P_n \in \mathcal{J}_p^{(a_1)}(f, A_q) \cap \mathcal{J}_p^{(a_2)}(f, Y_s) \cap \mathcal{J}_p^{(a_3)}(f, \{\pm1\}) \cap \breve{\Delta}(f, Y_s) \cap \mathbb{P}_n,$. for any

$f \in L_p^{(r)}$, & $n \geq c\left(d\left(Y_s, A_q\right)\right)$, if $a_1 = 1$, so no condition P_n interpolated f at point $A_q \in \mathbb{A}_q$, and the constraint $P_n \in \mathcal{J}_p^{(-1)}\left(f, A_q\right)$, is empty, where $\mathcal{J}_p^{(-1)}\left(f, A_q\right) := \{\, g: I \mapsto \mathbb{R}\}$, the class of all real valued function on I. When $s = 0$, and making the constraint $P_n \in \mathcal{J}_p^{(a_2)}(f, Y_0)$, is empty. The one-sided approximation results are like and may be used in the notification of Theorem 5.2, such that

$$\dagger_{\text{onesided}} := \left\{(r, a_1, a_3) \middle| (r, a_1, a_2, a_3) \in \dagger_{\text{intertwing}}, \text{for some } a_2 \geq 0\right\}$$
$$=\{(2j+1, 2j-1, 2j) | j \in \mathbb{N}_0\} \cup \{(2j, 2j-1, 2j-1) | j \in \mathbb{N}\}. \quad (5)$$

Theorem 5.3: *One-sided L_p space, $0 < p < 1$, approximation with interpolatory conditions. Let $A_q \in \mathbb{A}_q$, $t, q \in \mathbb{N}$, and $(r, a_1, a_3) \in \dagger_{onesided}$ defined in (5). Then $\|f(x) - P_n(x)\|_p \leq c(t, r, q)\sigma_n^r(x)\ \tau_t(f^{(r)}, \sigma_n(x))_p, x \in I$, for some $P_n \in \mathcal{J}_p^{(a_1)}\left(f, A_q\right) \cap \mathcal{J}_p^{(a_3)}(f, \{\pm 1\}) \cap \mathbb{P}_n$, s.t $P_n \geq f$, on I, for any $f \in L_p^{(r)}$, and $n \geq c\left(d\left(A_q\right)\right)$, if $a_1 = 1$, thus no condition P_n interpolates f at point $A_q \in \mathbb{A}_q$.*

5.4. *Co-positive and positive L_p space, $0 < p < 1$, approximation with interpolatory conditions.*

In the section we need the following notation for the indicator set $\dagger_{\text{co-positive}}$, this notation should be more explicit including the order of modulus t. Such that $\dagger_{co-positive}^{(0)} := \cup_{r \geq 0} \dagger_r^{(0)}$, where $\dagger_0^{(0)} := \{(0,2,0,0,0)\}$, $\dagger_1^{(0)} := \{(1, t, -1, 0, 0,) | t \in \mathbb{N}\} \cup \{(1,3,0,0,0), (1,2,0,1,0)\}$,

$$\dagger_2^{(0)} := \{(2, t, 1, 0, 1,) | t \in \mathbb{N}\} \cup \{(2,3,1,1,1)\},$$

and $\dagger_r^{(0)} := \{(r, t, a_1, a_2, a_3) | t \in \mathbb{N} \text{ and } (r, a_1, a_2, a_3) \in \dagger_r\}$

$$= \left\{(r, t, a_1, a_2, a_3) \middle| \begin{array}{l} t \in \mathbb{N}, \text{and } a_1 = r-2, a_2 = r-1, a_3 = r-1, \text{if r is odd;} \\ \quad a_1 = r-1, a_2 = r-2, a_3 = r-1, \text{if r is even} \end{array}\right\},$$

for $r \geq 3$. So that

$$\dagger_{co-positive}^{(0)} := \{(2j+1, t, 2j-1, 2j, 2j) |\, j \in \mathbb{N}_0, t \in \mathbb{N}\} \cup$$
$$\{(2j, t, 2j-1, 2j-2, 2j-1) |\, j, t \in \mathbb{N}\} \cup$$
$$\{(0, 2, 0, 0, 0), (1, 3, 0, 0, 0), (1, 2, 0, 1, 0), (2, 3, 1, 1, 1, 1)\}. \quad (6)$$

Theorem 5.5: *Co-positive L_p space, $0 < p < 1$, approximation with interpolatory conditions. Let $Y_s \in \mathbb{Y}_s$, $A_q \in \mathbb{A}_q$, $q, s \in \mathbb{N}$, and $(r, t, a_1, a_2, a_3) \in \dagger_{co-positive}^{(0)}$ defined in (6). Then $\|f(x) - P_n(x)\|_p \leq c(t, r, s, q)\sigma_n^r(x)$ $\tau_t(f^{(r)}, \sigma_n(x))_p, x \in I$, for some $P_n \in \mathcal{J}_p^{(a_1)}\left(f, A_q\right) \cap \mathcal{J}_p^{(a_2)}(f, Y_s) \cap \mathcal{J}_p^{(a_3)}$ $(f, \{\pm 1\}) \cap \mathbb{P}_n \cap \Delta^{(0)}(Y_s), \forall f \in L_p^{(r)} \cap \Delta^{(0)}(Y_s)$, if $a_1 = 1$, so no condition P_n interpolates f at point $A_q \in \mathbb{A}_q$ for the intertwining/ one-sided approximation the positive approximation results are the same the co-positive approximation* *results in the notification of Theorem 5.5., when* $s = 0$, *and making the constraint* $P_n \in \mathcal{J}_p^{(a_2)}(f, Y_0)$, *is empty. (7)*

Theorem 5.6: *Positive* L_p *space,* $0 < p < 1$, *approximation with interpolatory conditions. Let* $A_q \in \mathbb{A}_q$, $q \in \mathbb{N}$, *and* $(r, t, a_1, a_3) \in \dagger^{(0)}_{positive}$ *defined in (7). Then* $\|f(x) - P_n(x)\|_p \le c(t, r, q)\sigma_n^r(x) \quad \tau_t(f^{(r)}, \sigma_n(x))_p, x \in I$, *for some* $P_n \in \mathcal{J}_p^{(a_1)}(f, A_q) \cap \mathcal{J}_p^{(a_3)}(f, \{\pm 1\}) \cap \mathbb{P}_n$, *such that* $P_n \ge 0$, *and* $\forall f \in L_p^{(r)}$, *s.t* $f \ge 0$, & $n \ge c\left(d\left(A_q\right)\right)$, *if* $a_1 = 1$, *so no condition* P_n *interpolated* f *at point* $A_q \in \mathbb{A}_q$.

6. Negative Results

Remember where function $h: I \to \mathbb{R}$ which absolutely cont. $(r-1)$st derivative . Taylor formulas hold.

$$h(x) = \sum_{j=0}^{r-1} \frac{1}{j!} h^{(j)}(a)(x-a)^j + \frac{1}{(r-1)!} \int_a^x (x-u)^{r-1} h^{(r)}(u)\, du, x \in I.$$

6.1. *Negative results for one-sided intertwine approximations with interpolator constraint.*

Aims for next lemma to prove one-sided/intertwining approximation negative results in L_p space, $0 < p < 1$, with interpolatory conditions.

Lemma 6.1.1: *negative results of one-sided approximation in* L_p *space,* $0 < p < 1$, *with interpolatory conditions for* $f \in L_p^{(r)}, r \ge 0$. *Let* $r \in \mathbb{N}_0$, *the following are satisfied:*

(i) If $\beta \in (-1,1)$, (β be interior points for the *I*), so
$\mathcal{J}_p^{(\check{r})}(f, \{\beta\}) \cap \left\{h \in L_p^{(r+1)} \,|\, h \ge f \text{ on } (\beta - \varepsilon, \beta + \varepsilon), \text{for some } \varepsilon > 0\right\} = \emptyset$, where
$$\check{r} := 2[r/2] = \begin{cases} r, \text{when } r \text{ be the even,} \\ r-1, \text{where } r \text{ be an odd,} \end{cases}$$
for some $f \in L_p^{(r)}$ s.t $f \equiv 0$, on [-1,1].

(ii) If $\beta = \pm 1$, (β is an endpoint of *I*), then
$\mathcal{J}_p^{(\check{r})}(f, \{\beta\}) \cap \left\{h \in L_p^{(r+1)} \,|\, h \ge f \text{ on } (\beta - \varepsilon, \beta + \varepsilon) \cap I, \text{for some } \varepsilon > 0\right\} = \emptyset$, for some function $f \in L_p^{(r)}$.

Lemma 6.1.2: *Negative results of intertwining approximation in* L_p *space,* $0 < p < 1$, *with interpolatory conditions at* Y_s, *for* $f \in L_p^{(r)}, r \ge 1$. *Let* $\beta \in (-1,1)$, *and* $r \in \mathbb{N}$, *then*

$\mathcal{J}_p^{(\check{r})}(f,\{\beta\}) \cap \left\{h \in L_p^{(r+1)} \mid h \le f \text{ on } (\beta-\varepsilon,\beta), \text{and } h \ge f \text{ on } (\beta,\beta+\varepsilon) \text{ for some } \varepsilon > 0\right\} = \emptyset$, *where* $\check{r} := 2[r/2]-1 = \begin{cases} r-1, \text{for } r \text{ be even}, \\ r, \text{for } r \text{ be odd}, \end{cases}$ *for some* $f \in L_p^{(r)}$, *s.t* $f \equiv 0$, *on [-1,*β*]*.

Proof: $\beta \in (-1,1)$, $r \in \mathbb{N}$, and $0 < p < 1$, Define f as

$$f(x) := \begin{cases} \frac{1}{(r-1)!}\int_\beta^x (x-u)^{r-1}\varphi(u)\,du, x \ge \beta, \\ 0, x < \beta, \end{cases}$$

where $\varphi(u) = (-1)^{[r/2]}|u-\beta|^r \ln|u-\beta|$, for $u \ne \beta$, and $\varphi(\beta) = 0$.

Clearly, $f \in \left(\mathcal{W}_p^r(I) \cap C^r(I)\right) \subset L_p^{(r)}$, since $\varphi \in L_p^{(r)}$, as $\int_\beta^{\beta+1}|u-\beta|_p^r \ln|u-\beta|_p\, du < \infty$, for $p > 0$. Suppose there exists $h \in \left(\mathcal{W}_p^r(I) \cap C^r(I)\right) \subset L_p^{(r)}$, such that satisfying $h^{(i)}(\beta) = f^{(i)}(\beta) = 0$, for $0 \le i \le \check{r}$, and $h \le f$ on $(\beta-\varepsilon,\beta)$, and $h \ge f$ on $(\beta,\beta+\varepsilon)$ for some $\varepsilon > 0$. For $\beta \ne x$,

$$h(x) - f(x) = \frac{1}{(r-1)!}\int_\beta^x (x-u)^{r-1}\left[h^{(r)}(u) - \varphi(u)\right] du,$$

then $r = \check{r}$ where r, be the odd such that $h^{(r)}(0) = 0$, and $\check{r} = r-1$, when r be even, in L_p use continuity of $h^{(r)}$ to show $h^{(r)}(\beta) = 0$. Indeed, for $x > \beta$, and $h^{(r)}(u) = \int_\beta^u h^{(r+1)}(v)\,dv$. Then

$$h(x) - f(x) = \frac{1}{(r-1)!}\int_\beta^x (x-u)^{r-1}\left[\int_\beta^u h^{(r+1)}(v)\,dv - \varphi(u)\right]du.$$

Now, if $x \in (\beta-\varepsilon,\beta)$, (left of β), we get $h(x) \le f(x) = 0$, this implies $h(x) \le 0$, and if $x \in (\beta,\beta+\varepsilon)$, (right of β), we get $h(x) \ge f(x) = 0$, this implies $h(x) \ge 0$, and near $u = \beta$: For $u \ge \beta$, and $\varphi(u) = (-1)^{[r/2]}(u-\beta)^r \ln(u-\beta)$, if r is even, $\varphi(u) \to 0^+$ as $u \to \beta^+$; if r is odd, $\varphi(u) \to 0^-$, which contradiction with intertwining condition $h \ge f$ on one-side and $h \le f$ one other side. (Since $h(x) - f(x)$ changes its sign near β, when $\int_\beta^u h^{(r+1)}(v)\,dv$, dominates to $\varphi(u)$).

6.2. *Positive and co-positive* L_p *space,* $0 < p < 1$, *approximation with interpolatory conditions.*

In this work, here the first lemma although less general than the next Lemma 6.2.2.

Lemma 6.2.1: *Negative results of positive approximation in* L_p *space,* $0 < p < 1$, *with interpolatory conditions, for* $f \in L_p$. *For any* $\beta \in I, D > 0$, *and* $n \in \mathbb{N}$, *there exists* $f \in L_p(I), 0 < p < 1$, *which is non-negative on* I, *for any polynomial* $P_n \in \mathbb{P}_n$, *which is non-negative on* I, *satisfying* $f(\beta) = P_n(\beta)$, *we have* $\|f - P_n\|_p > D\,\tau_3(f,1)$.

Proof: Let $\beta \in [0,1], D > 0, n \in \mathbb{N}$, and $\varepsilon \in \left(0,\frac{\beta}{2}\right)$. Define the function:

$f(x) := \max\{(x-\beta)(x-\beta+1/\varepsilon), 0\}$, On $I = [-1,1]$, this function is non-negative and $f(\beta) = 0$. Let us define the polynomial. $S(x) = (x-\beta)(x-\beta+1/\varepsilon)$, Note that $S(x)$ is quadratic polynomial, and $S(x) = f(x)$ on $[\beta-\varepsilon, \beta]$, outside this interval $f(x) = 0$. Now estimate $\|f-S\|_p$. Since $S(x) = f(x)$ on $[\beta-\varepsilon, \beta]$, and $f(x) = 0, S(x)$ reach its maximum value at $x = \beta - \frac{1/\varepsilon}{2}$, because $S(x) = (x^2 - \beta x - (1/\varepsilon)x - \beta x + \beta^2 - (1/\varepsilon)\beta)$, and $S'(x) = (2x - \beta + (1/\varepsilon) - \beta)$, and $S'(x)$ =$(2x - 2\beta + (1/\varepsilon)) = 0$, then $2(x-\beta) + (1/\varepsilon) = 0$, this implies $x = \beta - \frac{1/\varepsilon}{2}$.

We have on $[\beta-\varepsilon, \beta]$, $\|f-S\|_p = \|S\|_{L_{p[\beta-\varepsilon,\beta]}} = \left(\int_{\beta-\varepsilon}^{\beta} |S(x)|^p\, dx\right)^{\frac{1}{p}}, = \left(\int_{\beta-\varepsilon}^{\beta} |(x-\beta)(x-\beta+1/\varepsilon)|^p\, dx\right)^{\frac{1}{p}}$=$(\int_{\beta-\varepsilon}^{\beta} |x^2 - \beta x + (1/\varepsilon)x - \beta x + \beta^2 - (1/\varepsilon)\beta|^p\, dx)^{\frac{1}{p}}$. Since $\beta, \varepsilon > 0, \beta \in [0,1], \varepsilon \in \left(0, \frac{\beta}{2}\right)$. Assume $\beta\varepsilon = C$, we get

$$\|f-S\|_p \le \left(\int_{\beta-\varepsilon}^{\beta} (x^{2p} + \beta^p x^p + (1/\varepsilon)^p x^p + \beta^p x^p + \beta^{2p} + (1/\varepsilon)^p \beta^p) dx\right)^{\frac{1}{p}} = \left(\frac{x^{2p+1}}{2p+1} + \beta^p \frac{x^{p+1}}{p+1} + (1/\varepsilon)^p \frac{x^{p+1}}{p+1} + \beta^p \frac{x^{p+1}}{p+1} + \beta^{2p} x + \beta^p (1/\varepsilon)^p x \Big|_{\beta-\varepsilon}^{\beta}\right)^{\frac{1}{p}} = \left(\frac{\beta^{2p+1}}{2p+1} + \frac{\beta^{2p+1}}{p+1} + (1/\varepsilon)^p \frac{\beta^{p+1}}{p+1} + \frac{\beta^{2p+1}}{p+1} + \beta^{2p+1} + \beta^{p+1}(1/\varepsilon)^p - \frac{(\beta-\varepsilon)^{2p+1}}{2p+1} - \beta^p \frac{(\beta-\varepsilon)^{p+1}}{p+1} - (1/\varepsilon)^p \frac{(\beta-\varepsilon)^{p+1}}{p+1} - \beta^p \frac{(\beta-\varepsilon)^{p+1}}{p+1} - \beta^{2p}(\beta-\varepsilon) - \beta^p (1/\varepsilon)^p (\beta-\varepsilon)\right)^{\frac{1}{p}} \le \left(\frac{\beta^{2p+1}}{2p+1} + \frac{\beta^{2p+1}}{p+1} + \frac{\beta^{p+1}}{\varepsilon^p (p+1)} + \frac{\beta^{2p+1}}{p+1} + \beta^{2p+1} + \frac{\beta^{p+1}}{\varepsilon^p} - \frac{\beta^{2p+1}}{2^{2p+1}(2p+1)} - \frac{\beta^{2p+1}}{2^{p+1}(p+1)} - \frac{\beta^{p+1}}{\varepsilon^p 2^{p+1}(p+1)} - \frac{\beta^{2p+1}}{2^{p+1}(p+1)} - \frac{\beta^{2p+1}}{2} - \frac{\beta^{p+1}}{2\varepsilon^p}\right)^{\frac{1}{p}} \le \left(\frac{\beta^{2p+1}}{2p+1} + \frac{\beta^{2p+1}}{p+1} + \frac{\beta^{p+1}}{\varepsilon^p (p+1)} + \frac{\beta^{2p+1}}{p+1} + \beta^{2p+1} + \frac{\beta^{p+1}}{\varepsilon^p}\right)^{\frac{1}{p}} \le \left(\frac{1}{2p+1} + \frac{1}{p+1} + \frac{C}{(p+1)} + \frac{1}{p+1} + 1 + C\right)^{\frac{1}{p}} \le (1+1+C+1+1+C)^{\frac{1}{p}} = C(p).$$

So that $\|f-S\|_p = C(p)$. Now let us estimate the (third- order) $\tau-$ modulus of smoothness $\tau_3(f,1)_p$:

$\tau_3(f,1)_p = \tau_3(f-S,1)_p \le C(p)\|f-S\|_p$. This implies

$$\tau_3(f,1)_p \le C(p). \tag{8}$$

Now, suppose there exists a polynomial $P_n \in \mathbb{P}_n$, which is non-negative, and satisfies $f(\beta) = P_n(\beta) = 0$, with

$$\|f - P_n\|_p \le D\tau_3(f,1)_p. \tag{9}$$

Then by the triangle inequality:

$\| P_n - S\|_p \le 2^{\frac{1}{P}-1}(\| P_n - f\|_p + \|f-S\|_p) \le 2^{\frac{1}{P}-1}(D\tau_3(f,1)_p + \|f-S\|_p)$. Then using (8) and (9) to obtain: $\| P_n - S\|_p \le 2^{\frac{1}{P}-1}\big(C(p) + C(p)\big) = C(p)$. The derivative of $S(x)$ $at\ x = \beta\ is\ S'(x) = \frac{1}{\varepsilon}$. Since $P_n \in \mathbb{P}_n$ is a non-negative on I and $P_n(\beta) = f(\beta) = 0$, we have $P_n'(\beta) \leq 0$, thus $S'(\beta) - P_n'(\beta) \geq \frac{1}{\varepsilon}$. By Dzyubenko inequality [11-12] for polynomials the derivative of $S(x) - P_n(x)$ satisfies: $\|S' - P_n'\|_p \leq C(p)\, n\, \| S - P_n\|_p$. Since each of S and P_n are polynomials So we get:

$$\frac{C(p)}{\varepsilon} = C(p)\, S'(\beta) \leq S'(\beta) - P_n'(\beta) \leq \|S' - P_n'\|_p. \quad \text{Then} \quad \frac{1}{\varepsilon} \leq n\, C(p).$$

This implies $\varepsilon \geq \frac{1}{n}\, C(p)$. As $\varepsilon \to 0^+$, the left hand side approches 0, while the right hand side approches to a constant, this contradictions. So the prove completed

7. Discussion and Conclusions

We can obtain positive and co-positive $L_{p,}$ approximation with interpolation condition in the following cases:

A_1. If $a = 0$ and $r = 0$, then such P_n exists but only if $t = 2$. If $t = 3$, so P_n no longer exist. where $a = 0$ and $r = 1$, so P_n does not exist if $t = 4$ and exists if $t = 3$. If $a = 1$ and $r = 1$, then such (for any t) P_n does not exist. If $r \geq 2$ is even, then P_n no exist where $a = r$ and exist when $a = r - 1$.

A_2. When $r = 0$ or $r = 1$ then such P_n do not exist . where $r \geq 2$ be even, so P_n do no exist where $a = r$ & exist if $a = r - 1$. Such P_n exist if $a = r - 2$ & no exists $where\ a = r - 1$, when $r \geq 3$ be odd.

A_3. If $r = 0$ or $r = 1$, then such P_n does not exist . Such P_n no exists if $a_1 = r$ or $a_2 = r - 1$ and exists if $a_1 = r - 1$ and $a_2 = r - 2$, if $r \geq 2$ is even. Such P_n exist if $a_1 = r - 2$ and $a_2 = r - 1$, and no exists where $a_1 = r - 1$ or $a_2 = r$, if $r \geq 3$ is odd.

A_4. These a most involve answers. $r = 0$: Such P_n does not exist if $t = 3$ and exists if $t = 2$, If $a_1 = a_2 = 0$. $r = 1$:Such P_n does not exist if $t = 4$ and exists if $t = 3$, if $a_1 = a_2 = 0$. Such $P_n$does not exist if $a_1 = 1$. Such P_n does not exist if $t = 3$ and exists if $t = 2$, if $a_1 = 0$ and $a_2 = 1$. $r = 2$: Such (for any t) P_n exists if $a_1 = 1$ and $a_2 = 0$. However, $a_1 = 1$ cannot be increased to 2. Such P_n does not exist if $t = 4$ and exists if $t = 3$, if $a_1 = 1$ and $a_2 = 1$. Such P_n does not exist if $a_2 = 2$. $r \geq 3$: an answer is same as in A_3.

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