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Oscillation of Nicholson's blowflies equation

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Abstract

In this paper, the oscillation property and the asymptotic behavior of all solutions to Nicholson's blowfly equation for variable-coefficient flies are studied. While most researchers have studied Nicholson's blowfly equation, with constant coefficients, and established some conditions to ensure the oscillation of all solutions of this equation about the equilibrium. In this research, the Nicholson's blowfly equation with variable coefficients was studied. By choosing the variable coefficient, we will have more than one equilibrium axis. This requires choosing points within the range of $\delta(\omega)$ and $P(\omega)$ functions that are consistent with the axis of optimal equilibrium. Sufficient conditions are established to ensure that these solutions oscillate about the equilibrium axis. To obtain these conditions, some auxiliary Lemmas are presented, and their effectiveness in obtaining the main results is demonstrated. Examples of the obtained results are also presented, and illustrate how to choose the optimal equilibrium as well as the optimal points of $\delta(\omega)$ and $P(\omega)$ which achieves oscillation of all solutions.

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1. Introduction

The data obtained by the Australian entomologist [1] from his experiments prompted scientists to conduct numerous entomological, mathematical, and statistical studies. In specific terms, [2] a delay differential equation was proposed to expound on the oscillatory behavior observed in the sheep blowfly *Lucilia cuprina*. 333 The population dynamics observed by [1]. According to the data obtained, it is assumed that any periodic change in the climate tends to

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influence the periodicity of oscillations with an internal source or in a harmonious relationship with the climate. [3] The dynamics of the Michelson equation have been found to be diffusive, with delays distributed over the entire spatial domain. Recent studies in fly dynamics have shown that accounting for low densities is more accurate, especially when density-dependent fly mortality is present. [2], [4] Results on stability are obtained by refining the assumptions for permanence established previously in [1]. This section provides a short summary of recent results concerning non-autonomous Nicholson's equation, an equation incorporating multiple pairs of time-varying delays and nonlinearities expressed through mixed monotone functions. The oscillation of solutions of Nicholson's blowflies' equation is discussed. The oscillation property and the parallel behavior property of all solutions of the Michelson fly equation with variable coefficients were studied. Appropriate conditions were also imposed for each hypothesis to obtain a decreasing function that approaches zero [5-10].

The following nonlinear delay differential equation is considered:

$$N'(\omega) = -\delta(\omega)N(\omega) + P(\omega)N(\omega-\tau)e^{-aN(\omega-\tau)}. \tag{1.1}$$

Were

$$\tau > 0 \ \delta, P \in C\left[[\ \omega_0, \infty); R^+\right], \tag{1.2}$$

The equilibrium points $N^*$of Eq. (1.1) must be satisfied

$$N^*\left(-\delta(\omega) + P(\omega)e^{-aN^*}\right) = 0. \tag{1.3}$$

Since $N^* > 0, 0 \neq a \in R$ then from (1.3) it follows

$$\frac{\delta^*}{P^*} = e^{-aN^*}, \text{or } N^* = \frac{1}{a}\ln\frac{P^*}{\delta^*}, \tag{1.4}$$

Hence $(P^* > \delta^*)$ sign a, $P^* \in P(D), \delta^* \in \delta(D), D = [\omega_0, \infty)$.

Where P^* and δ^* are optimally chosen values in the equation (1.3) such that N^* is the desired equilibrium. Using the change of variable $N(\omega) = N^* + \frac{1}{a}\xi(\omega)$ then $\xi(\omega) = 0$ if and only if $N(\omega) = N^*$, that is $\xi(\omega)$ fluctuates around zero exactly when $N(\omega)$ fluctuates around the equilibrium N^* $[3, 5, 7-10]$.

$$\xi(\omega) = aN(\omega) - aN^*, \xi'(\omega) = a\, N'(\omega)$$

$$\xi'(\omega) = a\left(-\delta(\omega)\left(\frac{1}{a}\,\xi(\omega) + N^*\right) + P(\omega)(\frac{1}{a}\,\xi(\omega-\tau) + N^*)e^{-a\left(\frac{1}{a}\xi(\omega-\tau) + N^*\right)}\right)$$

$$\xi'(\omega) = -\delta(\omega)\xi(\omega) - a\,N^*\delta(\omega) + P(\omega)\,\xi(\omega-\tau)e^{-\xi(\omega-\tau)-aN^*} + aN^*P(\omega)e^{-\xi(\omega-\tau)-aN^*}$$

$$\xi'(\omega) = -\delta(\omega)(\xi(\omega) + aN^*) + \frac{\delta^*}{P^*}P(\omega)e^{-\xi(\omega-\tau)}(\,\xi(\omega-\tau) + aN^*) \tag{1.5}$$

2. Some Basic Lemmas

In this section, there are some lemmas that mainly help in proving the main results.

Lemma 2.1: *Let's assume that* $0 < b_1 \leq P(\omega) \leq b, \delta(\omega) \geq \frac{\delta^*}{P^*}P(\omega)$. *Then every non-oscillatory solution to Eq. (1.5) goes to zero when* $\omega \to \infty$.

Proof: Consider that Eq. (1.5) possesses a non-oscillatory solution $\xi(\omega)$, let $\xi(\omega) > 0, \xi(\omega - \tau) > 0, \omega \geq \omega_0$. Define

$$W(\omega) = \xi(\omega) - \frac{\delta^*}{P^*}\int_0^{\omega-\tau} P(s+\tau)\xi(s)e^{-\xi(s)}\,ds \tag{2.1}$$

$$W'(\omega) = \xi'(\omega) - \frac{\delta^*}{P^*}P(\omega)\xi(\omega-\tau)e^{-\xi(\omega-\tau)}$$

Then $W'(\omega) \leq \xi'(\omega)$, and

$$W'(\omega) = -\delta(\omega)(\xi(\omega) + aN^*) + \frac{\delta^*}{P^*}P(\omega)e^{-\xi(\omega-\tau)}(\xi(\omega-\tau) + aN^*)$$
$$-\frac{\delta^*}{P^*}P(\omega)\xi(\omega-\tau)e^{-\xi(\omega-\tau)}$$

So,

$$W'(\omega) = -\delta(\omega)\xi(\omega) - \delta(\omega)aN^* + \frac{\delta^*}{P^*}P(\omega)e^{-\xi(\omega-\tau)}\xi(\omega-\tau)$$
$$+\frac{\delta^*}{P^*}P(\omega)e^{-\xi(\omega-\tau)}aN^* - \frac{\delta^*}{P^*}P(\omega)\xi(\omega-\tau)e^{-\xi(\omega-\tau)}$$
$$W'(\omega) = -\delta(\omega)\xi(\omega) + aN^*\left(\frac{\delta^*}{P^*}P(\omega)e^{-\xi(\omega-\tau)} - \delta(\omega)\right) \tag{2.2}$$
$$\leq \delta(\omega)(-\xi(\omega) + aN^*\big(e^{-\xi(\omega-\tau)} - 1\big) \leq 0$$

Hence $W(\omega)$ is nonincreasing. We claim that $W(\omega) > 0$ otherwise $W(\omega) < 0$. Thus, there is $\alpha > 0$ such that $W(\omega) \leq -\alpha$ from Eq. (2.1) we get

$$\xi(\omega) = W(\omega) + \frac{\delta^*}{P^*}\int_0^{\omega-\tau} P(s+\tau)\xi(s)e^{-a\xi(s)}ds \tag{2.3}$$
$$\leq -\alpha + \frac{\delta^*}{P^*}\int_0^{\omega-\tau} P(s+\tau)\xi(s)e^{-a\xi(s)}ds$$

Since $e^{-a\xi(s)} \leq 1$. So,

$$\xi(\omega) \leq -\alpha + \frac{\delta^*}{P^*}\int_0^{\omega-\tau} P(\omega+\tau)\xi(s)ds \leq -\alpha + b\frac{\delta^*}{P^*}\int_0^{\omega-\tau}\xi(s)ds$$
$$\xi(\omega) \leq -\alpha + b\frac{\delta^*}{P^*}\max\{\xi(s), 0 \leq s \leq \omega-\tau\}$$

Then by Lemma (1.5.4) [6], it follows that $\xi(\omega) < 0$, this is a contradiction, then $W(\omega) > 0$, this implies that $W(\omega)$ is bounded, hence $\xi(\omega)$ must be bounded otherwise $\xi(\omega)$is unbounded, so there is a sequence $\{\omega_n\}$, $n \to \infty$ such that $\lim_{n\to\infty}\omega_n = \infty$ and $\xi(\omega_n) = \max_{\omega_0 \leq s \leq \omega}\{\psi(s)\}$ from Eq. (2.1) we obtain

$$W(\omega_n) = \xi(\omega_n) - \frac{\delta^*}{P^*}\int_0^{\omega_n-\tau} P(s+\tau)\xi(s)e^{-a\xi(s)}\,ds, \tag{2.4}$$
$$\geq \xi(\omega_n) - b\frac{\delta^*}{P^*}\xi(\omega_n)\int_0^{\omega_n-\tau} e^{-a\xi(s)}\,ds,$$
$$W(\omega_n) \geq \xi(\gamma_n)\left(1 - b\frac{\delta^*}{P^*}\int_0^{\omega_n-\tau} e^{-a\xi(s)}\,ds\right).$$

From the last inequality, one can conclude that $\lim_{n\to\infty} W(\omega_n) = \lim\limits_{n\to\infty}\xi(\omega_n) = \infty$, a contradiction. Let $\liminf\limits_{\omega\to\infty}\xi(\omega) = l \geq 0$, so there is a sequence $\{\omega_n\}, n \to \infty$ such that $\lim_{n\to\infty}\omega_n = \infty$ and $\lim_{n\to\infty}\psi(\omega_n) = l$, and for large ω_{n_1}, $\psi(\omega_n) \geq l$ for $\omega_n \geq \omega_{n_1}$. Since $\xi(\omega)$ is bounded, then there is a constant $\lambda > 0, \xi(\omega) \leq \lambda$. We claim $l = 0$, otherwise if $l > 0$,

$$W(\omega_n) = \xi(\omega_n) - \frac{\delta^*}{P^*}\int_0^{\omega_n-\tau} P(s+\tau)\xi(s)e^{-a\xi(s)}\,ds$$
$$\leq \lambda - l\frac{\delta^* b_1}{P^*}\int_0^{\omega_n-\tau} e^{-al}ds = \lambda - l\frac{\delta^* b_1}{P^*}e^{-al}(\omega_n-\tau)$$

As $n \to \infty$, $\lim_{n\to\infty} W(\omega_n) = -\infty$, a contradiction. Hence $\liminf_{\omega\to\infty}\xi(\omega) = 0$. Since $W(\omega) \leq \xi(\gamma)$, we conclude that $\lim_{\omega\to\infty} W(\gamma) = 0$. Let $\limsup_{\omega\to\infty}\xi(\gamma) = L \geq 0$, so there exists $\omega_1 \geq \gamma_0$, such that for $\omega \geq \omega_1$, $\xi(\omega) \leq L$. We claim that $L = 0$ otherwise $L > 0$, by taking limit superior to both sides of (2.4)

$$\limsup_{n\to\infty}\xi(\omega_n) - \liminf_{n\to\infty}\left(\frac{\delta^*}{P^*}\int_0^{\omega_n-\tau} P(s+\tau)\,\xi(s)\,ds\right)$$
$$= \lim_{n\to\infty} W(\omega_n) = 0$$

Hence $\limsup_{n\to\infty}\xi(\omega_n) = 0$, this means $\lim_{\omega\to\infty}\xi(\omega) = 0$.

Lemma 2.2: *Let's assume that* $0 < P(\omega) \leq b_1$, $\delta(\omega) \geq \frac{\delta^*}{P^*}max\{P(\omega), P(\omega+\tau)\}$, *in addition to*

$$\liminf_{\omega\to\infty}\frac{\delta^*}{P^*}\int_{\omega-\tau}^{\omega} P(s+\tau)\,ds \leq 1. \tag{2.5}$$

Then every non oscillatory solution to Eq. (1.5) converge to zero when ω → ∞.

Proof: Consider that Eq. (1.5) possesses a non oscillatory solution $\xi(\omega)$, let $\xi(\omega) > 0$, $\xi(\omega-\tau) > 0$. Set

$$W_1(\omega) = \xi(\omega) + \frac{\delta^*}{P^*}\int_{\omega-\tau}^{\omega} P(s+\tau)\xi(s)e^{-\xi(s)}\,ds. \tag{2.6}$$

Hence $W_1(\omega) \geq \xi(\omega)$

$$W_1'(\omega) = \xi'(\omega) + \frac{\delta^*}{P^*}P(\omega+\tau)\xi(\omega)e^{-\xi(\omega)} - \frac{\delta^*}{P^*}P(\omega)\xi(\omega-\tau)e^{-\xi(\omega-\tau)}.$$

Then

$$W_1'(\omega) = -\delta(\omega)\xi(\omega) - \delta(\omega)aN^* + \frac{\delta^*}{P^*}P(\omega)\xi(\omega-\tau)\,e^{-\xi(\omega-\tau)}$$
$$+\frac{\delta^*}{P^*}\,P(\omega)e^{-\xi(\omega-\tau)}aN^* + \frac{\delta^*}{P^*}P(\omega+\tau)\xi(\omega)e^{-\xi(\omega)}$$
$$-\frac{\delta^*}{P^*}P(\omega)\xi(\omega-\tau)e^{-\xi(\omega-\tau)},$$

$$W_1'(\omega) = -\delta(\omega)\xi(\omega) - aN^*\left(\delta(\omega) - \frac{\delta^*}{P^*}P(\omega)e^{-\xi(\omega-\tau)}\right)$$
$$+\frac{\delta^*}{P^*}P(\omega+\tau)\xi(\omega)e^{-\xi(\omega)}. \tag{2.7}$$

$$W_1'(\omega) \leq -aN^*\left(\delta(\omega) - \frac{\delta^*}{P^*}P(\omega)e^{-\psi(\omega-\tau)}\right) \leq 0$$

$W_1(\omega)$ is positive, nonincreasing, and so $W_1(\omega)$ is bounded, hence $\xi(\omega)$ must be bounded. Let $\liminf_{\omega\to\infty} \xi(\omega) = l \geq 0$. Since $\xi(\omega)$ is bounded, then there is a constant $\lambda > 0, \xi(\omega) \leq \lambda$, so there is a sequence $\{\omega_n\}$, $n \to \infty$ such that $\lim_{n\to\infty} \omega_n = \infty$ and $\lim_{n\to\infty} \xi(\omega_n) = l$, and for large ω_{n_2}, $\xi(\omega) \geq l, \omega \geq \omega_{n_2}$. We claim that $l = 0$, for if $l > 0$, then from (2.6) it follows $\xi(\omega_n) = W_1(\omega_n) - \frac{\delta^*}{P^*}\int_{\omega_n-\tau}^{\omega_n} P(s+\tau)\xi(s)e^{-a\xi(s)}\,ds$,

$$\leq W_1(\omega_n) - \frac{l\delta^* b_1}{P^*}\int_{\omega_n-\tau}^{\omega_n} e^{-al} ds \leq W_1(\omega_n) - \frac{l\delta^* b_1}{P^*}\int_0^{\omega_n} e^{-al} ds$$

$$= W_1(\omega_n) - \frac{l\delta^* b_1}{P^*} e^{-a\lambda}\omega_n$$

As $n \to \infty$, $\lim_{n\to\infty} \xi(\omega_n) = -\infty$, a contradiction. Hence $\liminf_{\omega\to\infty} \xi(\omega) = 0$. Since $W(\omega) \leq \xi(\omega)$, assume that $\lim_{\omega\to\infty} W_1(\omega) = L \geq 0$. To show that $L = 0$, for if not then $L > 0$,

$$\xi(\omega) = W_1(\omega) - \frac{\delta^*}{P^*}\int_{\omega-\tau}^{\omega} P(s+\tau)\xi(s)e^{-\xi(s)}\,ds,$$

$$> W_1(\omega) - \frac{\delta^*}{P^*}\int_{\omega-\tau}^{\omega} P(s+\tau)W_1(s)\,ds,$$

$$\frac{\delta^*}{P^*}\int_{\omega-\tau}^{\omega} P(s+\tau)\,ds W_1(\omega-\tau) \geq W_1(\omega) - \xi(\omega),$$

$$\frac{\delta^*}{P^*}\int_{\omega-\tau}^{\omega} P(s+\tau)\,ds \geq \frac{W_1(\omega)-\xi(\omega)}{W_1(\omega-\tau)}. \tag{2.8}$$

By taking limit inferior of both sides to (2.8) leads to

$$\liminf_{\omega\to\infty} \frac{\delta^*}{P^*}\int_{\omega-\tau}^{\omega} P(s+\tau)\,ds > 1$$

Taking into account condition (2.5), the last inequality reaches a contradiction. Hence $\lim_{\omega\to\infty} W_1(\omega) = 0$, this implies $\lim_{\omega\to\infty} \xi(\omega) = 0$.

3. Main Results

In this section, we introduce three results related to the lemmas raised in the previous section.

Theorem 3.1: *Let's assume that* $0 < b_1 \leq P(\omega) \leq b$, $\delta(\omega) \geq \frac{\delta^*}{P^*} P(\omega)$.

$$\liminf_{\omega\to\infty} \int_{\omega-\mu}^{\omega} \delta(s)ds > \frac{1}{e}, \mu \in (0,\tau]. \tag{3.1}$$

Then every solution of equation (1.1) oscillates around the equilibrium N^* *or tends to* N^* *as* $\omega \to \infty$.

Proof: Assume that Eq. (1.1) has no oscillatory solution $N(\omega)$, let $N(\omega) > N^*, N(\omega-\tau) > N^*$, that is $\xi(\omega) > 0, \xi(\omega-\tau) > 0$. From Lemma 2.1, $\xi(\omega)$ is bounded and tends to zero as $\omega \to \infty, W(\omega)$ is nonincreasing, $\xi'(\omega) \geq W'(\omega)$, and $W(\omega) > 0$. From Eq. (2.2) we get

$$W'(\omega) + \delta(\omega)\,\xi(\omega) + \delta(\omega)\,aN^* - aN^*\frac{\delta^*}{P^*} P(\omega)e^{-\xi(\omega-\tau)} = 0,$$

$$W'(\omega) + \delta(\omega)\big(\xi(\omega) + aN^*(1 - e^{-\xi(\omega-\tau)})\big) \le 0, \tag{3.2}$$

Since $\xi(\omega) \ge W(\omega), \omega \in [\omega_0, \infty)$, so there is $\mu \in (0, \tau], \omega_1 \ge \omega_0$, such that $\xi(\omega) \ge W(\omega - \mu), \omega \in [\omega_1, \infty)$

$$\frac{\xi(\omega)+aN^*(1-e^{-\xi(\omega-\tau)})}{W(\omega-\mu)} \ge 1 + \frac{aN^*\left(1-e^{-\xi(\omega-\tau)}\right)}{W(\omega-\mu)} \ge 1, \omega \in [\omega_1, \infty).$$

Hence

$$\xi(\omega) + aN^*\left(1 - e^{-\xi(\omega-\tau)}\right) \ge W(\omega - \mu). \tag{3.3}$$

Substituting (3.3) in (3.2) leads to

$$W'(\omega) + \delta(\omega)\, W(\omega - \mu) \le 0. \tag{3.4}$$

by Theorem 2.3.3 [6] it follows that (3.4) has no eventually positive solution. This leads to a contradiction.

Theorem 3.2: [7] *Let's assume that* $0 < b_1 \le P(\omega) \le b\,, \delta(\omega) \ge \frac{\delta^*}{P^*}$ $max\{P(\omega), P(\omega + \tau)\}$, *and*

$$\limsup_{\omega\to\infty} \int_{\omega_1}^{\omega} (\delta(s) - aN^* \tfrac{\delta^*}{P^*} P(s))ds = \infty. \tag{3.5}$$

Then every solution of equation (1.1) oscillates around equilibrium N^**or tends to* N^* *as* $\omega \to \infty$.

Proof: Assume that Eq. (1.1) has non oscillatory solution $N(\omega)$, let $N(\omega) > N^*, N(\omega - \tau) > N^*$. That is $\xi(\omega) > 0, \xi(\omega - \tau) > 0$. From Lemma 2.2, we conclude that $\xi(\omega)$ and $W_1(\omega)$ are bounded and their limits are zero as $\omega \to \infty$. $W_1(\omega)$ is positive nonincreasing. From Eq. (2.7) one can obtain

$$W_1'(\omega) = -\delta(\omega)\xi(\omega) - aN^*\left(\delta(\omega) - \tfrac{\delta^*}{P^*}P(\omega)e^{-\xi(\omega-\tau)}\right)$$
$$+\tfrac{\delta^*}{P^*}P(\omega+\tau)\xi(\omega)e^{-\xi(\omega)}.$$

$$W_1'(\omega) \le -aN^*\delta(\omega) + aN^*\tfrac{\delta^*}{P^*}P(\omega)e^{-\xi(\omega-\tau)} + \tfrac{\delta^*}{P^*}P(\omega+\tau)\xi(\omega)e^{-\xi(\omega)},$$

$W_1'(\omega) \le aN^*\left(-\delta(\omega) + aN^*\tfrac{\delta^*}{P^*}P(\omega)\right) + \tfrac{\delta^*}{P^*}P(\omega+\tau)\xi(\omega)e^{-\xi(\omega)}$. Since $\lim\limits_{\omega\to\infty} \xi(\omega) = 0$ for each $\varepsilon > 0$, and ω_1 greater enough such that $|\xi(\omega)| < \varepsilon$,

$$W_1'(\omega) < aN^*(-\delta(\omega) + \tfrac{\delta^*}{P^*}P(\omega)) + \varepsilon,$$

$$W_1'(\omega) \le aN^*(-\delta(\omega) + \tfrac{\delta^*}{P^*}P(\omega)). \tag{3.6}$$

Integrating (3.6) from ω_1 to ω we obtain

$$W_1(\omega) - W_1(\omega_1) \le aN^* \int_{\omega_1}^{\omega} (-\delta(s) + \tfrac{\delta^*}{P^*}P(s))ds. \tag{3.7}$$

Taking into account condition (3.5) inequality (3.7) implies $\lim_{\omega\to\infty} W_1 = (\omega) - \infty$, a contradiction.

4. Examples

In the current section, some exemplars are presented to explain and verify the results presented in the preceding section.

Example 1: Consider the Nicholson's Blowflies equation.

$$N'(\omega) = -\frac{b+e^{-\omega}(e^{\tau}+1)}{b+e^{-\omega}} N(\omega) + e^{a(b+e^{-\omega+\tau})} N(\omega-\tau) e^{-aN(\omega-\tau)} \qquad (4.1)$$

Where $b,\ a,\ \tau > 0,\ \delta(\omega) = \frac{b+e^{-\omega}(e^{\tau}+1)}{b+e^{-\omega}}, P(\omega) = e^{\frac{1}{2}(b+e^{-\omega+\tau})}$. Let $a = \frac{1}{2},\ b = 2,\ \tau = 2,$ then $\delta(\omega) = \frac{2+e^{-\omega}(e^{2}+1)}{2+e^{-\omega}}, P(\omega) = e^{\frac{1}{2}(2+e^{-\omega+2})}$ If we choose $\delta^* = 1,\ P^* = 2.75$, then $N^* = 2\ln\frac{2.75}{1} \cong 2$. It is obvious that $2.75 \le P(\omega) \le 109, \delta(\omega) > \frac{P^*}{\delta^*} P(\omega) = 0.36P(\omega), \gamma \ge \frac{5}{2}$ and

$$\liminf_{\omega\to\infty} \int_{\omega-2}^{\omega} \delta(s)ds \ge \liminf_{\omega\to\infty} \int_{\omega-2}^{\omega} ds = 2 > \frac{1}{e}.$$

Hence all conditions of Theorem 3.1 satisfy, so according to Theorem 3.1 every solution of (4.1) either oscillates or converge to equilibrium N^* as $\omega \to \infty$. The solution $N(\omega) = e^{-\omega} + b$ is such a solution converge to equilibrium 2. We can notice that the optimal oscillation patterns of the solution $N(\omega)$ depend on the value of k and n [9].

Example 2: Consider the Nicholson's blowflies equation

$$N'(\omega) = -(b + \frac{5}{4n}\sin\omega)N(\omega) + e^{ae^{\frac{1}{n}\cos\omega+k}}(b + \frac{1}{4n}\sin\omega)$$
$$N(\omega - 2\pi)e^{-aN(\omega-2\pi)}. \qquad (4.2)$$

Let $a = \frac{2}{9}, b = 2, k = 1, n = 1, \delta(\omega) = 2 + \frac{5}{4}\sin\omega, P(\omega) = e^{ae^{\frac{1}{n}\cos\omega+k}}$ $(2 + \frac{1}{4}\sin\omega)$. If we choose $\delta_1^* = 1.25, P_1^* = 5.25$, then $N_1^* = \frac{9}{2}\ln\frac{5.25}{1.25} \cong 7.24$. It is obvious that $\frac{6}{5} \le \delta(\omega) \le \frac{13}{4}, \frac{12}{5} \le P(\omega) \le \frac{31}{3}, \delta(\omega) > \frac{\delta^*}{P^*}P(\omega)$ and

$$\liminf_{\omega\to\infty} \int_{\omega-2\pi}^{\omega} \delta(s)ds = \liminf_{\omega\to\infty} \int_{\omega-2\pi}^{\omega} \left(k + \frac{1}{n}\sin s\right) ds = 2\pi k > \frac{1}{e}.$$

Hence, all conditions of Theorem 3.1 are satisfied, so according to Theorem 3.1 every solution of (4.1) oscillates, $N(\omega) = e^{\frac{1}{n}\cos\omega+k}$ is such an oscillatory solution about N^*. The optimal oscillation patterns of the solution N(ω) around 3.5 depend on the values of k and n.

Example 3: Consider the Nicholson's blowflies equation

$$N'(\omega) = -\left(1 + \frac{2e^{-\frac{3\pi}{4}} - b}{b+e^{-\gamma}\cos 2\omega}\right) N(\omega) + 2e^{-\frac{3\pi}{4}} e^{a(b-e^{-\omega+\frac{3\pi}{4}}\sin 2\omega}$$
$$N\left(\omega - \frac{3\pi}{4}\right) e^{-aN\left(\omega-\frac{3\pi}{4}\right)}. \qquad (4.3)$$

$a > 0, b \geq \frac{1}{4}, \delta(\omega) = 1 + \frac{2e^{-\frac{3\pi}{8}} - b}{b + e^{-\omega}\cos 2\gamma}, P(\omega) = 2e^{-\frac{3\pi}{8}} e^{a\left(b - e^{-\omega + \frac{3\pi}{4}}\sin 2\omega\right)}$.

Let $\mathrm{b} = 1, a = \ln\frac{P^*}{\delta^*}, \delta^* = \sup \delta(\omega) = 0.758, P^* = \sup P(\omega) \cong 3.96$, then $N^* = \frac{1}{a}\ln\frac{P^*}{\delta^*} \cong 1.304$. It is obvious that $0.15 \leq P(\omega) \leq 3.92, \delta(\omega) > \frac{\delta^*}{P^*}P(\omega)$ and

$$\liminf_{\omega\to\infty} \int_{\omega-\frac{3\pi}{4}}^{\omega} \delta(s)ds = \lim_{\omega\to\infty} \int_{\omega-\frac{3\pi}{4}}^{\omega} \left(1 + \frac{2e^{-\frac{3\pi}{8}} - 1}{1 + e^{-s}\cos 2s}\right) ds$$
$$\geq \lim_{\omega\to\infty} \int_{\omega-\frac{3\pi}{4}}^{\omega} \frac{1}{2} ds = \frac{3\pi}{8} > \frac{1}{e}.$$

Hence, all the conditions of Theorem 3.2 are satisfied, and thus, according to Theorem 3.2, every solution of (4.1) is oscillatory. For instance, $N(\omega) = e^{-\omega}\cos 2\omega + 1.3$ is an oscillatory solution about N^*. We can notice that $N(\omega)$ oscillates about 1.3.

5. Discussion and Conclusion

In this research, the effect of the Oscillation of Nicholson's Blowflies Equation was shown, in addition to its effect on the behavior of the solutions to these equations. We have found some sufficient conditions to ensure the oscillation of each solution of Nicholson's Blowflies equation, as well as other conditions to ensure the divergence of non-oscillating to solutions. We believe that the extracted conditions are weak and easily applicable conditions that can be generalized to be applicable in higher order equations of the Oscillation of Nicholson's Blowflies Equation. Examples were presented to illustrate the results.

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