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On coincidence point theorems via weakly compatible mappings

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Abstract

In this article, we present a novel space, said complex valued public metric spaces, and show some of its properties. Examine the convergence of sequences in this space. We will also demonstrate the standard $\mathbb{F}.\mathbb{P}$ theorems for mappings that are w-compatible.

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Keywords: *Fixed-points, Compatible weakly, Convergent sequence, Cauchy sequence.*

1. Introduction

Fixed-points ($\mathbb{F}.\mathbb{P}$) theory is the best potent and interesting tools in current mathematical analysis. It is a broad field of study in the mathematical sciences that combines geometry, algebra, and topology. Although Banach [1] made a contribution to this subject in 1922 by proving the $\mathbb{F}.\mathbb{P}$ theorem, sometimes known as the Banach $\mathbb{F}.\mathbb{P}$ theorem, which has many applications in other branches of science, and many researchers later developed the basic Banach $\mathbb{F}.\mathbb{P}$ theorem in different ways [2]-[4]. On the other hand, the $\mathbb{F}.\mathbb{P}$ Theorems for multivalued mappings were examined by Nadler [5] in 1969. Nadler's [5] contraction principle inspired numerous researchers to expand the concept in different ways refer to [6]-[8]. Also, Bakhtin [9] first proposed the idea of b-metric spaces. Subsequently, other researchers explored the theorems of $\mathbb{F}.\mathbb{P}$ and common $\mathbb{F}.\mathbb{P}$ in metric spaces, as shown in [10]-[11]. They then began to develop and extend it to multiple spaces, including b-metric space [12]-[13].

In 2006, some theorems for extensional mappings in G-metric spaces were introduced by authors [14]. They proved the existence of common $\mathbb{F}.\mathbb{P}$ and also

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proved its uniqueness for some mappings extensional in complete G-metric spaces. In 2015, other Author [15], They defined a new spaces called the complex-metric Gb spaces, and also proved some associated common $\mathbb{F}.\mathbb{P}$ theorems. The ($\mathbb{F}.\mathbb{P}$) and its applications in different settings have been the subject of numerous studies; see [16]-[24]. In 2014 the authors [25], presented the $\mathbb{F}.\mathbb{P}$ and common $\mathbb{F}.\mathbb{P}$ theorems in CVM spaces.

Definition 1.1: Let Φ and Ƕ a mappings. Then Φ and Ƕ are said to be weakly compatible if Φђ $=$ Ƕђ implies ΦǶђ $=$ ǶΦђ.

2. Main Results

We define a complex public metric space and several significant theorems in this section.

Definition 2.1: Let $U \neq \emptyset$, $M_1, M_2 : U \times U \to [1, \infty)$ are function and $\Lambda : U \times U \times U \to \mathcal{C}$ satisfy, $\forall$ ђ, ʅ, ʒ $\in U$

1) Λ(ђ, ʅ, ʒ) $\succcurlyeq 0$
2) If ђ $=$ ʅ $=$ ʒ, then Λ(ђ, ʅ, ʒ) $= 0$
3) Λ(ђ, ʅ, ʒ) $=$ Λ(ђ, ʒ, ʋ) $=$ Λ(ʅ, ђ, ʒ) $=$ Λ(ʅ, ʒ, ђ) $=$ Λ(ʒ, ђ, ʋ) $=$ Λ(ʒ, ʅ, ђ)
4) Λ(ђ, ʅ, ʒ) $\preccurlyeq$ M_1(ђ, ʋ)(Λ(ђ, σ, σ)) $+$ M_2(ʅ, ʒ)(Λ(σ, ʅ, ʒ)).

Then, we say that (U, Λ) the complex valued public metric (CVPM)-space.

Example 2.2: Let $U \neq \emptyset$ and $\Lambda : U \times U \times U \to \mathcal{C}$ defined as (ђ, ʅ, ʒ) $=$ $4i$(|ђ $-$ ʅ| $+$ |ʅ $-$ ʒ| $+$ |ђ $-$ ʒ|), ђ, ʅ, ʒ $\in U$. Then (U, Λ) is a (CVPM) - space.

Example 2.3: Let $U \neq \emptyset$ and define $\Lambda : U \times U \times U \to \mathcal{C}$ by Λ(ђ, ʅ, ʒ) $=$ (|ђ $-$ ʅ| $+$ |ʅ $-$ ʒ| $+$ |ђ $-$ ʒ|) $+ 2i$, $\forall$ ђ, ʅ, ʒ $\in U$, then (U, Λ) is not (CVPM)-space.

Definition 2.4: Let (U, Λ) is a (CVPM)- space, we say (U, Λ) has M-property if satisfy: Λ(ђ, ʅ, ʒ) $\succcurlyeq$ Λ(ђ, ђ, ʋ) $\succ 0$, $\forall$ ђ, ʅ, ʒ $\in U$ and ђ $\neq$ ʅ.

Proposition 2.5: Let (U, Λ) be a (CVPM) $-$space has M-property, *for all* ђ, ʅ, ʒ, σ $\in U$, we have:

1) If Λ(ђ, ʅ, ʒ) $= 0$ then ђ $=$ ʅ $=$ ʒ
2) Λ(ђ, ʅ, ʒ) $\preccurlyeq$ M_1(ʅ, ђ)(Λ(ђ, ђ, ʋ)) $+$ M_2(ђ, ʒ)(Λ(ђ, ђ, ʒ))
3) Λ(ђ, ʅ, ʋ) $\preccurlyeq$ [M_1(ʅ, ђ) $+$ M_2(ʅ, ʋ)](Λ(ʅ, ђ, ђ))

Proposition 2.7: Let(U, Λ) is (CVPM)_space. The follows are equivalent:

1) {$ђ_{m}$} CVP-convergent
2) Λ($ђ_{m}$, $ђ_{m}$, ђ) $\to 0$ as $m \to \infty$
3) Λ($ђ_{m}$, ђ, ђ) $\to 0$ as $m \to \infty$

Proposition 2.8: Let (U, Λ) a (CVPM)- space. The follows are equivalent:

1) $\{\hbar_{\mathfrak{m}}\}$ CVP _ Cauchy
2) For all $\sigma \succ 0$, $\exists\ \mathfrak{m}_0 \in \mathcal{N}$; $\Lambda(\hbar_{\mathfrak{m}}, \hbar_{\eta}, \hbar_{\eta}) \prec \sigma$, $\forall\ \eta, \mathfrak{m} \geq \mathfrak{m}_0$

Lemma 2.9: $\{\hbar_{\mathfrak{m}}\}$ *is* (CVP)- *converges to* $\hbar \Leftrightarrow |\Lambda(\hbar, \hbar_{\mathfrak{m}}, \hbar_{\mathfrak{m}})| \to 0, as\ \mathfrak{m} \to \infty$

Lemma 2.10: $\{\hbar_{\mathfrak{m}}\}$ *is* (CVP)-*Cauchy sequence* $\Leftrightarrow |\Lambda(\hbar_{\mathfrak{m}}, \hbar_{\eta}, \hbar_{\eta})| \to 0$ *as* $\mathfrak{m}, \eta \to \infty$

Theorem 2.11: *Let* Φ *and* Ƕ *are maps of* $(CVPM)$- *space* (U, Λ) *satisfying the following*:

i) $\Phi U \subseteq ǶU,$
ii) $\Lambda(\Phi\hbar, \Phi\mathfrak{l}, \Phi\mathfrak{l}) \preccurlyeq \mathfrak{z}_1 \Lambda(Ƕ\hbar, Ƕ\mathfrak{l}, Ƕ\mathfrak{l}) + \mathfrak{z}_2 \frac{\Lambda(\Phi\hbar, Ƕ\hbar, Ƕ\hbar)\ \Lambda(\Phi\mathfrak{l}, Ƕ\mathfrak{l}, Ƕ\mathfrak{l})}{1 + \Lambda(Ƕ\hbar, Ƕ\mathfrak{l}, Ƕ\mathfrak{l})} + \mathfrak{z}_3 \frac{\Lambda(Ƕ\hbar, \Phi\mathfrak{l}, \Phi\mathfrak{l})\ \Lambda(Ƕ\hbar, Ƕ\mathfrak{l}, Ƕ\mathfrak{l})}{1 + \Lambda(Ƕ\hbar, Ƕ\mathfrak{l}, Ƕ\mathfrak{l})} + \mathfrak{z}_4 \frac{\Lambda(\Phi\hbar, Ƕ\hbar, Ƕ\hbar)\ \Lambda(Ƕ\hbar, Ƕ\mathfrak{l}, Ƕ\mathfrak{l})}{1 + \Lambda(Ƕ\hbar, Ƕ\mathfrak{l}, Ƕ\mathfrak{l})} + \mathfrak{z}_5 \frac{\Lambda(Ƕ\hbar, \Phi\mathfrak{l}, \Phi\mathfrak{l})\ \Lambda(\Phi\mathfrak{l}, Ƕ\mathfrak{l}, Ƕ\mathfrak{l})}{1 + \Lambda(Ƕ\hbar, Ƕ\mathfrak{l}, Ƕ\mathfrak{l})}$, $\forall\ \hbar, \mathfrak{l} \in U$, where $0 < \mathfrak{z}_1 + \mathfrak{z}_2 + \mathfrak{z}_3 + \mathfrak{z}_4 + \mathfrak{z}_5 < 1.$
iii) $ǶU$ is a complete subspace of U.
iv) $\sum_{i=\mathfrak{m}}^{\eta-1} M_i(\mathfrak{l}_i, \mathfrak{l}_{\eta})\omega^i \to 0$ as $i \to \infty$ and $M_2(\mathfrak{l}_i, \mathfrak{l}_i) = 1$. Then Φ , Ƕ has a coincidence element.Also, if Φ, Ƕ w-compatible. Then, Ƕ has a unique the common $\mathbb{F}.\mathbb{P}$.

Proof: Let $\hbar_0 \in U$. Then $\{\hbar_{\mathfrak{m}}\}$ and $\{\mathfrak{l}_{\mathfrak{m}}\}$ belong in$U$; $\mathfrak{l}_{\mathfrak{m}} = Ƕ\hbar_{\mathfrak{m}+1} = \Phi\hbar_{\mathfrak{m}}$, $\mathfrak{m} = 0, 1, 2$

$$\Lambda(\mathfrak{l}_{\mathfrak{m}+1}, \mathfrak{l}_{\mathfrak{m}}, \mathfrak{l}_{\mathfrak{m}}) = \Lambda(\Phi\hbar_{\mathfrak{m}+1}, \Phi\hbar_{\mathfrak{m}}, \Phi\hbar_{\mathfrak{m}}) \preccurlyeq \mathfrak{z}_1 \Lambda(Ƕ\hbar_{\mathfrak{m}+1}, Ƕ\hbar_{\mathfrak{m}}, Ƕ\hbar_{\mathfrak{m}}) + \mathfrak{z}_2 \frac{\Lambda(\Phi\hbar_{\mathfrak{m}+1}, Ƕ\hbar_{\mathfrak{m}+1}, Ƕ\hbar_{\mathfrak{m}+1})\Lambda(\Phi\hbar_{\mathfrak{m}}, Ƕ\hbar_{\mathfrak{m}}, Ƕ\hbar_{\mathfrak{m}})}{1+\Lambda(Ƕ\hbar_{\mathfrak{m}+1}, Ƕ\hbar_{\mathfrak{m}}, Ƕ\hbar_{\mathfrak{m}})} + \mathfrak{z}_3 \frac{\Lambda(Ƕ\hbar_{\mathfrak{m}+1}, \Phi\hbar_{\mathfrak{m}}, \Phi\hbar_{\mathfrak{m}})\Lambda(Ƕ\hbar_{\mathfrak{m}+1}, Ƕ\hbar_{\mathfrak{m}}, Ƕ\hbar_{\mathfrak{m}})}{1+\Lambda(Ƕ\hbar_{\mathfrak{m}+1}, Ƕ\hbar_{\mathfrak{m}}, Ƕ\hbar_{\mathfrak{m}})} + \mathfrak{z}_4 \frac{\Lambda(\Phi\hbar_{\mathfrak{m}+1}, Ƕ\hbar_{\mathfrak{m}+1}, Ƕ\hbar_{\mathfrak{m}+1})\ \Lambda(Ƕ\hbar_{\mathfrak{m}+1}, Ƕ\hbar_{\mathfrak{m}}, Ƕ\hbar_{\mathfrak{m}})}{1+\Lambda(Ƕ\hbar_{\mathfrak{m}+1}, Ƕ\hbar_{\mathfrak{m}}, Ƕ\hbar_{\mathfrak{m}})} + \mathfrak{z}_5 \frac{\Lambda(Ƕ\hbar_{\mathfrak{m}+1}, \Phi\hbar_{\mathfrak{m}}, \Phi\hbar_{\mathfrak{m}})\Lambda(\Phi\hbar_{\mathfrak{m}}, Ƕ\hbar_{\mathfrak{m}}, Ƕ\hbar_{\mathfrak{m}})}{1+\Lambda(Ƕ\hbar_{\mathfrak{m}+1}, Ƕ\hbar_{\mathfrak{m}}, Ƕ\hbar_{\mathfrak{m}})}$$

So, $|\Lambda(\mathfrak{l}_{\mathfrak{m}+1}, \mathfrak{l}_{\mathfrak{m}}, \mathfrak{l}_{\mathfrak{m}})| \leq$

$$\mathfrak{z}_1 |\Lambda(\mathfrak{l}_{\mathfrak{m}}, \mathfrak{l}_{\mathfrak{m}-1}, \mathfrak{l}_{\mathfrak{m}-1})| + \mathfrak{z}_2 \frac{|\Lambda(\mathfrak{l}_{\mathfrak{m}+1}, \mathfrak{l}_{\mathfrak{m}}, \mathfrak{l}_{\mathfrak{m}})||\Lambda(\mathfrak{l}_{\mathfrak{m}}, \mathfrak{l}_{\mathfrak{m}-1}, \mathfrak{l}_{\mathfrak{m}-1})|}{|1+\Lambda(\mathfrak{l}_{\mathfrak{m}}, \mathfrak{l}_{\mathfrak{m}-1}, \mathfrak{l}_{\mathfrak{m}-1})|} + \mathfrak{z}_4 \frac{|\Lambda(\mathfrak{l}_{\mathfrak{m}+1}, \mathfrak{l}_{\mathfrak{m}}, \mathfrak{l}_{\mathfrak{m}})||\Lambda(\mathfrak{l}_{\mathfrak{m}}, \mathfrak{l}_{\mathfrak{m}-1}, \mathfrak{l}_{\mathfrak{m}-1})|}{|1+\Lambda(\mathfrak{l}_{\mathfrak{m}}, \mathfrak{l}_{\mathfrak{m}-1}, \mathfrak{l}_{\mathfrak{m}-1})|}.$$

We can conclude that, $\Lambda(\mathfrak{l}_{\mathfrak{m}}, \mathfrak{l}_{\mathfrak{m}+1}, \mathfrak{l}_{\mathfrak{m}+1}) \preccurlyeq \omega\Lambda(\mathfrak{l}_{\mathfrak{m}-1}, \mathfrak{l}_{\mathfrak{m}}, \mathfrak{l}_{\mathfrak{m}})$

$$\preccurlyeq \omega^2 \Lambda(\mathfrak{l}_{\mathfrak{m}-2}, \mathfrak{l}_{\mathfrak{m}-1}, \mathfrak{l}_{\mathfrak{m}-1}) \dots\dots.. \preccurlyeq \omega^{\mathfrak{m}} \Lambda(\mathfrak{l}_0, \mathfrak{l}_1, \mathfrak{l}_1)$$

Now, $r\ all\ \eta > \mathfrak{m}$,

$$\Lambda(ʅ_ɱ, ʅ_ɳ, ʅ_ɳ) \preccurlyeq M_1(ʅ_ɱ, ʅ_ɳ)\big(\Lambda(ʅ_ɱ, ʅ_{ɱ+1}, ʅ_{ɱ+1})\big) + M_2(ʅ_ɳ, ʅ_ɳ)\big(\Lambda(ʅ_{ɱ+1}, ʅ_ɳ, ʅ_ɳ)\big)$$

$$\begin{aligned}\Lambda(ʅ_ɱ, ʅ_ɳ, ʅ_ɳ) \preccurlyeq{}& M_1(ʅ_ɱ, ʅ_ɳ)\big(\Lambda(ʅ_ɱ, ʅ_{ɱ+1}, ʅ_{ɱ+1})\big)\\ &+ \Big[M_1(ʅ_{ɱ+1}, ʅ_ɳ)\Lambda(ʅ_{ɱ+1}, ʅ_{ɱ+2}, ʅ_{ɱ+2})\\ &+ [M_1(ʅ_{ɱ+2}, ʅ_ɳ)\Lambda(ʅ_{ɱ+2}, ʅ_{ɱ+3}, ʅ_{ɱ+3})] + \cdots\\ &+ [M_1(ʅ_{ɳ-1}, ʅ_ɳ)\Lambda(ʅ_{ɳ-1}, ʅ_ɳ, ʅ_ɳ)]\Big]\end{aligned}$$

$$\Lambda = \sum_{i=ɱ}^{ɳ-1} M_1(ʅ_i, ʅ_ɳ)\omega^i\, \Lambda(ʅ_0, ʅ_1, ʅ_1) \Rightarrow \Lambda(ʅ_ɱ, ʅ_ɳ, ʅ_ɳ) \to 0,$$

Hence, $\{ʅ_ɱ\}$ is a Cauchy sequence in $ꜦU$. But $ꜦU$ is the complete a subspace of U, thus $\exists\, e \in ꜦU$; $ʅ_ɱ \to e$ as $ɱ \to \infty$. If $\ell \in Ꜧ^{-1} e$. Then $Ꜧ\ell = e$. Now, to prove $\Phi\ell = e$ Putting $ђ = \ell$ and $ʅ = ђ_{ɱ-1}$ in (ii), we get, $\Lambda(\Phi\ell, e, e) \preccurlyeq 0$, by (1) of proposition (2.5) , implies that $\Phi\ell = e$. and hence ℓ is the point of coincidence between Φ and $Ꜧ$. Now, since Φ and $Ꜧ$ are weakly compatible, so, $e = \Phi\ell = Ꜧ\ell$, implies that, $\Phi e = \PhiꜦ\ell = Ꜧ\Phi\ell = Ꜧe$.Now, to prove $Ꜧe = e$. If $Ꜧe \neq e$. We have, $|\Lambda(e, Ꜧe, Ꜧe)| \leq (ʒ_1 + ʒ_3)|\Lambda(e, Ꜧe, Ꜧe)|$, which is a contradiction.

3. Discussion and Conclusion

Some properties of metric spaces were proven, along with the study of sequence convergence and Cauchy sequences. Convergence equivalences and Cauchy's equivalences were proven, as were common $\mathbb{F}.\mathbb{P}$ for weakly compatible mappings. A unique common $\mathbb{F}.\mathbb{P}$ was obtained.

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