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On approximation of functions by operators

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Abstract

The comparison of approximation functions between the two well-known operators, namely, Devalle'e Poussion integral operator and the modified Lupas operator in the L_p-spaces will be established by utilizing the modulus for the continuities for the order one for a function. A goal is to determine which one is the best approximation. spaces by utilizing the modulus for the continuity for a order one for functions. A crucial part is to specify which one is the best approximation.

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1. Introduction

The authors in [1] have presented an estimation of the best approximations for the unbounded function by appropriate trigonometric polynomial for the one variable in the weight spaces L_(p,α) (X. They analyzed the uniform convergence of modern operators of the function f by utilizing the asymptotic behavior of the operators. Moreover, Authors in [2] have studied the modification of Szasz-Mirakan-Schurer Kantorovich operators for functions preserving the exponential form e^(-bx),b>0. Other author in [3] analyzed the approximation of functions in C by employing the Weierstrass approximation theorem using Bernstein polynomials, the Jackson operator, the Fejer operator, and the Landau operator, respectively. They have found which approximation function is better. The best approximation of bounded functions with one variable will be discovered by considering two operators, namely, the De Vallee Poussin integral operator and the modified Lupos operators, and we have compared the approximation by utilizing the De Vallee operator and the approximation by utilizing the modified Lupos operator. It is obtained that the approximation by De Vallee is better than the approximation by the modified Lupos operator.

2. Main Definitions:

Definition 1: [4] Let $\Psi \in L_P[a,b]$, define the modulus of smoothness of Ψ on $[a,b]$ by the following of $\delta \in [0, b-a]$, $1 \leq P < \infty$:

$$\omega(\Psi,\delta)_P = \sup\{\|\Delta_h \Psi(\xi)\|_P : |h| \leq \delta, \xi, \xi + h \in [a,b]\}$$

with the property that:

i. $\omega(\Psi,\delta)_P$ is non-negative.

ii. $\omega(\Psi,\delta_1)_P \leq \omega(\Psi,\delta_2)_P$ for $\delta_1 \leq \delta_2$, $\delta_1, \delta_2 > 0$. (1)

iii. $\omega(\Psi,\lambda\delta)_P \leq (\lambda+1)\omega(\Psi,\delta)_P$, λ>0.

Definition 2: [1] Let $\Psi \in C_{2\pi}$, where $n \in$ N, then:

$$V_n(\Psi,\xi) = \frac{(2n)!!}{2\pi(2n-1)!!}\int_{-\pi}^{\pi} \Psi(t)\left[\cos\left(\frac{t-\xi}{2}\right)\right]^{2n} d \quad (2)$$

Then $V_n(\Psi,\xi)$ is called De-Valle's integral Poussin operator.

Definition 3: [5] Let Ψ be an integrable function on the $[0,\infty)$ & n is positive integers, the:

$$M_n(\Psi,\xi) = (n-1)\sum_{k=0}^{\infty} P_{n,k}(\xi)\int_0^{\infty} P_{n,k}(t)\Psi(t)dt \quad (3)$$

is called the Modified Lupas operators where:

$$P_{n,k}(\xi) = \binom{n+k-1}{k}\frac{\xi^k}{(1+\xi)^{n+k}}$$

Lemma 1: [6] *For m is positive integer, then $m!!$ can be defined as shown below:*

$$m!! = \begin{cases} \frac{2^{m+1/2}}{\sqrt{\pi}} \Gamma\left(\frac{m}{2}+1\right) \; if\ m\ is\ odd \\ 2^{m/2}\,\Gamma\left(\frac{m}{2}+1\right) \; if\ m\ is\ even \end{cases} \tag{4}$$

Lemma 2: *Let* $\Psi \in L_P(X)$, $X = [0, \infty)$, $0 \le P < \infty$, *then:*

$$\omega(\Psi, |\xi - t|)_P \le \omega\left(\Psi, \frac{|\xi-t|^2}{\delta}\right)_P + \omega(\Psi, \delta)_P \tag{5}$$

Proof: If $|\xi - t| \le \delta$, then by (1)(ii), we have:

$$\omega(\Psi, |\xi - t|)_P \le \omega(\Psi, \delta)_P \tag{6}$$

By utilizing (1)(i), we obtain the below inequality:

$\omega(\Psi, |\xi - t|)_P \le \omega\left(\Psi, \frac{|\xi-t|^2}{\delta}\right)_P + \omega(\Psi, \delta)_P$, Otherwise $|\xi - t| > \delta$, we similarly obtain

$$\omega(\Psi, |\xi - t|)_P \le \omega\left(\Psi, |\xi - t| \frac{|\xi-t|}{\delta}\right)_P \tag{7}$$

As a result, we obtain (6).

Lemma 3: *The below equality holds:*

$$\int_0^\infty t^m P_{n,k}(t) dt = \frac{(m+k)!(n-m-2)!}{k!(n-1)!} \tag{8}$$

Note that $P_{n,k}(t)$ is described as shown below:

$$P_{n,k}(t) = \binom{n+k-1}{k} t^k (1+t)^{-(n+k)} \tag{9}$$

Proof:

$$\int_0^\infty t^m P_{n,k}(t) dt = \binom{n+k-1}{k} \int_0^\infty t^{m+k} (1+t)^{-(n+k)}\, dt \tag{10}$$

Let $t = \tan^2\theta$. If $= 0 \to \theta = 0, t = \infty \to \theta = \frac{\pi}{2}$. Then $t = 2\tan\theta$ $\sec^2\theta\, d\theta$. Thus $\int_0^\infty t^m P_{n,k}(t) dt$ can be rewritten in terms of the new transformation as shown below:

$$\int_0^\infty t^m P_{n,k}(t) dt = 2\binom{n+k-1}{k} \int_0^{\frac{\pi}{2}} (tan\,\theta)^{2m+2k+1} (\sec\theta)^{-2n-2k+2}\, d\theta$$

$$= 2\binom{n+k-1}{k} \int_0^{\frac{\pi}{2}} \frac{\sin^{2m+2k+1}\theta}{\cos^{2m+2k+1}\theta} (cos\,\theta)^{2n+2k-2}\, d\theta$$

$$= 2\binom{n+k-1}{k} \int_0^{\frac{\pi}{2}} sin^{2m+2k+1}\theta\, (cos\,\theta)^{2n-2m-3}\, d\theta \tag{11}$$

Since we know that:

$$\int_0^{\frac{\pi}{2}} \sin^{2a-1}\theta \cos^{2b-1}\theta\, d\theta = \frac{\pi a \pi b}{2\pi(a+b)} \tag{12}$$

Comparing (8) and (9), we would then result in the below values of a and b respectively.

$$a = m + k + 1, \;\; b = n - m - 1 \tag{13}$$

Then the integral $\int_0^\infty t^m P_{n,k}(t)dt$ can be calculated based on the above values of a and b respectively.

Lemma 4: [7] *The following integral equation holds:*

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos t)^{2n} dt = \frac{\Gamma(n+\frac{1}{2})\Gamma(\pi)}{\Gamma(n+1)} \tag{14}$$

where Γ is Gamma function.

Proof:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos t)^{2n} dt = 2\int_0^{\frac{\pi}{2}} (cos\, t)^{2n} (\sin t)^0\, dt \tag{15}$$

Comparing with (9), it would result in $2a-1=2n$ and $2b-1=0$ and then $a=\frac{2n+1}{2}$ $,=\frac{1}{2}$. Thus, the integral in (20) can be rewritten in terms of the values of a and b respectively and by (4), as shown below:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos t)^{2n} dt = 2\frac{\Gamma(\frac{2n+1}{2})\Gamma(\frac{1}{2})}{2\Gamma(\frac{2n+1}{2}+\frac{1}{2})} = \frac{\Gamma(n+\frac{1}{2})\sqrt{\pi}}{\Gamma(n+1)} = \begin{cases} \frac{(2n-1)!!}{(2n)!!}\sqrt{\pi} & if\ n\ is\ odd \\ \frac{(2n-1)!!}{(2n)!!}\sqrt{2\pi} & if\ n\ is\ even \end{cases} \tag{16}$$

4. Main Results

Theorem 1: [8] Let $\Psi \in C_{2\pi}(X)$, $X=[-\pi,\pi]$, $n \in \mathrm{N}$, then the following inequality holds:

$$|V_n(\Psi,\xi)-\Psi(\xi)| \le \frac{1}{\sqrt{2\pi}}\omega(\Psi,\frac{1}{\sqrt{n}})_P \tag{17}$$

Proof: From (2), one can obtain:

$$V_n(\Psi,\xi)-\Psi(\xi) = \frac{(2n)!!}{(2n-1)!!}\frac{1}{2\pi}\int_{-\pi}^{\pi}\Psi(t)[\cos\frac{t-\xi}{2}]^{2n}\, dt - \Psi(\xi) \tag{18}$$

Then

$$|V_n(\Psi,\xi)-\Psi(\xi)| \le \frac{(2n)!!}{(2n-1)!!}\frac{1}{2\pi}\int_{\xi-\pi}^{\xi+\pi}|\Psi(t)-\Psi(\xi)|[cos\frac{t-\xi}{2}]^{2n}\, dt \tag{19}$$

$$|V_n(\Psi,\xi)-\Psi(\xi)| \le \frac{(2n)!!}{(2n-1)!!}\frac{1}{2\pi}\int_{\xi-\pi}^{\xi+\pi}\omega(\Psi,|t-\xi|)_P[cos\frac{t-\xi}{2}]^{2n}\, dt \tag{20}$$

By utilizing (1)(iii), we obtain the below inequality:
Hence:

$$|V_n(\Psi,\xi)-\Psi(\xi)| \le \frac{(2n)!!}{(2n-1)!!}\frac{1}{2\pi}\omega(\Psi,\frac{1}{\sqrt{n}})_P\int_{\xi-\pi}^{\xi+\pi}[|t-\xi|\sqrt{n}+1][cos\frac{t-\xi}{2}]^{2n}\, dt \tag{21}$$

$$\le \frac{(2n)!!}{(2n-1)!!}\frac{1}{2\pi}\omega\left(\Psi,\frac{1}{\sqrt{n}}\right)_P\{\int_{\xi-\pi}^{\xi+\pi}|t-\xi|\sqrt{n}[cos\frac{t-\xi}{2}]^{2n}\, dt + \int_{\xi-\pi}^{\xi+\pi}[cos\frac{t-\xi}{2}]^{2n}\, dt \tag{22}$$

Assume that $=\frac{t-\xi}{2}$, then $du=\frac{1}{2}dt$. If $t=\xi-\pi$, then $u=-\frac{\pi}{2}$. If $t=\xi+\pi$, then $u=\frac{\pi}{2}$.

Then the right side of the inequality (19) can be rewritten as shown below:

$$\text{R.H.S} = \frac{(2n)!!}{(2n-1)!!}\frac{1}{2\pi}\omega\left(\Psi,\frac{1}{\sqrt{n}}\right)_P\left\{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}2|u|\sqrt{n}[\cos u]^{2n}\,du + 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}[\cos u]^{2n}\,du\right\} \tag{23}$$

By (16), we obtain the below equality:

$$\text{R.H.S}= \frac{(2n)!!}{(2n-1)!!}\frac{1}{2\pi}\omega\left(\Psi,\frac{1}{\sqrt{n}}\right)_P\{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}2|u|\sqrt{n}[\cos u]^{2n}\,du + \frac{(2n-1)!!}{(2n)!!}\sqrt{2\pi}\}$$

Hence:

$$\text{R.H.S}= \frac{(2n)!!}{(2n-1)!!}\frac{1}{\pi}\omega\left(\Psi,\frac{1}{\sqrt{n}}\right)_P\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}|u|\sqrt{n}[\cos u]^{2n}\,du + \frac{1}{\sqrt{2\pi}}\omega\left(\Psi,\frac{1}{\sqrt{n}}\right)_P \tag{24}$$

$$\leq [\frac{1}{\sqrt{2\pi}} + \frac{(2n)!!}{(2n-1)!!}\frac{\sqrt{n}}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}|u|(\cos u)^{2n}\,du]\,\omega\left(\Psi,\frac{1}{\sqrt{n}}\right)_P \tag{25}$$

Since $\frac{(2n)!!}{(2n-1)!!} < 2\sqrt{n}$. Then

$$\frac{\pi}{2}\int_0^{\frac{\pi}{2}}\sin u\,(\cos u)^{2n}\,du = \frac{\pi}{2}\frac{1}{2n} < \frac{\pi}{4n} \tag{26}$$

Then
$$\text{R.H.S} \leq [\frac{1}{\sqrt{2\pi}} + \frac{4n}{\pi}\frac{\pi}{4n}]\omega\left(\Psi,\frac{1}{\sqrt{n}}\right)_P \tag{27}$$

$$|V_n(\Psi,\xi) - \Psi(\xi)| \leq \frac{1}{\sqrt{2\pi}}\omega\left(\Psi,\frac{1}{\sqrt{n}}\right)_P, \tag{28}$$

$$|V_n(\Psi,\xi) - \Psi(\xi)| \leq \frac{1}{\sqrt{\pi}}\omega\left(\Psi,\frac{1}{\sqrt{n}}\right)_P, \tag{29}$$

$V_n(\Psi,\xi)$ is the best approximation from n is odd.

Lemma 5: [9] *Let $X = [0,\infty), \Psi \in C[0,\infty), \xi \in X$. Then $M_n(\Psi,\xi)$ is given in terms of the infinite series shown below:*

$$M_n(\Psi,\xi) = (n-1)\sum_{k=0}^{\infty}P_{n,k}^{(\xi)}\int_0^{\infty}P_{n,k}^{(t)}\Psi(t)dt\ldots \tag{30}$$

with the below properties:

(1) $M_n(1,\xi) = 1$ (2) $M_n(t,\xi) = \xi + \alpha(\xi)$ (3) $M_n(t^2,\xi) = \xi^2 + \beta(\xi)$

Proof: Starting with (1):

$$M_n(1,\xi) = \frac{1}{(1+\xi)^n}\sum_{k=0}^{\infty}\binom{n+k-1}{k}\left(\frac{\xi}{1+\xi}\right)^k. \tag{31}$$

Since $\xi \in [0,\infty)$, then $\frac{\xi}{1+\xi} \in [0,1]$, we use the below identity:

$$\sum_{k=0}^{\infty}\binom{n+k-1}{k}y^k = (1-y)^{-n} = \frac{1}{(1+\xi)^n}\left(1-\frac{\xi}{1+\xi}\right)^{-n} = \frac{1}{(1+\xi)^n}\left(\frac{1+\xi-\xi}{1+\xi}\right)^{-n}$$
$$= \frac{1}{(1+\xi)^n}\left(\frac{1}{1+\xi}\right)^{-n} = \frac{1}{(1+\xi)^n}(1+\xi)^n = 1 \tag{32}$$

We prove (2). Starting with the below infinite summation:

$$M_n(t,\xi) = (n-1)\sum_{k=0}^{\infty}P_{n,k}^{(\xi)}\int_0^{\infty}P_{n,k}^{(t)}\,t\,dt \tag{33}$$

By (8), one can obtain the below equality.

$$M_n(t,\xi) = (n-1)\sum_{k=0}^{\infty} P_{n,k}^{(\xi)} \frac{(k+1)!(n-1-2)!}{k!(n-1)!} = (n-1)\sum_{k=0}^{\infty} P_{n,k}^{(\xi)} \frac{(k+1)k!(n-3)!}{k!(n-1)(n-2)(n-3)!}$$

$$= \frac{1}{n-2}\sum_{k=0}^{\infty} P_{n,k}^{(\xi)}(k+1) = \frac{1}{n-2}\left(\sum_{k=0}^{\infty} kP_{n,k}^{(\xi)} + \sum_{k=0}^{\infty} P_{n,k}^{(\xi)}\right) = \frac{1}{n-2}(n\xi+1) = \frac{(n\xi+1)}{n-2}$$

$$= \xi + \frac{(2\xi+1)}{n-2} = \xi + \alpha(\xi) \tag{34}$$

Provided that the function $\alpha(\xi)$ is given as shown below:

$$\alpha(\xi) = \frac{(2\xi+1)}{n-2} \tag{35}$$

To prove (3), we start from the below equality:

$$M_n(t,\xi) = (n-1)\sum_{k=0}^{\infty} P_{n,k}^{(\xi)} \int_0^{\infty} P_{n,k}^{(t)}\, t^2 dt \tag{36}$$

By utilizing (8), we get the below equality:

$$M_n(t,\xi) = (n-1)\sum_{k=0}^{\infty} P_{n,k}^{(\xi)} \frac{(k+2)!(n-4)!}{k!(n-1)(n-2)(n-3)(n-4)!}$$

$$= \frac{1}{(n-1)(n-2)}\sum_{k=0}^{\infty} P_{n,k}^{(\xi)}(k^2+3k+2)$$

$$= \frac{1}{(n-1)(n-2)}\left(\sum_{k=0}^{\infty} P_{n,k}^{(\xi)} k^2 + \sum_{k=0}^{\infty} P_{n,k}^{(\xi)}(3k) + 2\sum_{k=0}^{\infty} P_{n,k}^{(\xi)}\right)$$

$$= \frac{1}{(n-1)(n-2)}(\xi^2 n^2 + n\xi^2 + n\xi + 3n + 2) + \xi^2$$

$$= \xi^2 + \frac{4n\xi^2 + n\xi - 2\xi^2 + 3n + 2}{n^2 - 3n + 2} = \xi^2 + \beta(\xi) \tag{37}$$

The function $\beta(\xi)$ is given by the fraction:

$$\beta(\xi) = \frac{4n\xi^2 + n\xi - 2\xi^2 + 3n + 2}{n^2 - 3n + 2} \tag{38}$$

Theorem 2: [10] Let $\Psi \in L_P(X)$, $X = [0,\infty)$, then

$$|\Psi(\xi) - M_n(\Psi,,\xi)| \le 3\omega\left(\Psi, \sqrt{\beta(\xi) - \xi\alpha(\xi)}\right)_P \tag{39}$$

Proof: By (5),

$$\Psi(\xi) - \Psi(t) \le \omega(\Psi, |\xi - t|)_P \le \omega(\Psi, |\xi - t|\frac{|\xi-t|}{\delta})_P \le \omega(\Psi, \frac{|\xi-t|^2}{\delta})_P$$

Provided that $\delta > 0$, and $|\xi - t| > \delta$. By (1),

$$\Psi(\xi) - \Psi(t) \le \omega\left(\Psi, \frac{(\xi-t)^2}{\delta}\right)_P + \omega(\Psi,\delta)_P \le \omega\left(\Psi, \frac{(\xi-t)^2}{\delta^2}\delta\right)_P + \omega(\Psi,\delta)_P \tag{40}$$

By utilizing (1)(iii), we get the below inequality:

$$\le (\frac{(\xi-t)^2}{\delta^2} + 1)\omega(\Psi,\delta)_P \tag{41}$$

By (1)(i),

$$\le \left(\frac{(\xi-t)^2}{\delta^2} + 1\right)\omega(\Psi,\delta)_P + \omega(\Psi,\delta)_P \le \left(\frac{(\xi-t)^2}{\delta^2} + 2\right)\omega(\Psi,\delta)_P$$

$$\le (2 + (\xi - t)^2\delta^{-2})\omega(\Psi,\delta)_P \tag{42}$$

Then we obtain the below inequality:

$$\Psi(\xi) - M_n(\Psi,,\xi) \leq (2 + M_n(\xi - t)^2\delta^{-2})\omega(\Psi,\delta)_P$$
$$= (2 + \delta^{-2}M_n(\xi^2 - 2\xi t + t^2))\omega(\Psi,\delta)_P$$
$$= [2 + \delta^{-2}\xi^2 - \delta^{-2}\left(2\xi\left(\xi + \alpha(\xi)\right)\right) + \delta^{-2}(\xi^2 + \beta(\xi))]\omega(\Psi,\delta)_P$$
$$= [2 + \delta^{-2}\xi^2 - 2\delta^{-2}\xi^2 - 2\delta^{-2}\xi\alpha(\xi) + \delta^{-2}\xi^2 + \delta^{-2}\beta(\xi)]\omega(\Psi,\delta)_P$$
$$= [2 + \frac{\beta(\xi)-2\xi\alpha(\xi)}{\delta^2}]\omega(\Psi,\delta)_P \text{ , Put } \delta = \sqrt{\beta(\xi) - 2\xi\alpha(\xi)} \qquad (43)$$

Then δ can be calculated based on the values of $\alpha(\xi)$ and $\beta(\xi)$ respectively.

$$\delta = \sqrt{\frac{4n\xi^2+n\xi-2\xi^2+3n+2}{n^2-3n+2} - 2\xi(\frac{2\xi+1}{n-2})} = \sqrt{\frac{4n\xi^2+n\xi-2\xi^2+3n+2}{n^2-3n+2} - (\frac{4\xi^2+2\xi}{n-2})} \qquad (44)$$

If $\delta \to 0$ as $n \to \infty$, then:

$$|\Psi(\xi) - M_n(\Psi,\xi)| \leq (2+1)\omega\left(\Psi,\sqrt{\beta(\xi) - 2\xi\alpha(\xi)}\right)_P \leq 3\omega(\Psi,\delta)_P.$$

Thus, $M_n(\Psi,\xi) \to \Psi(\xi)$ as $n \to \infty$. This completes the proof.

5. Discussion and Conclusion

In this research, we compared two operators (17, 39) to determine which yields better approximations of functions. It was found that (17) gives a better approximation than (39) after comparing (17) and (39).

6. Future Work

The same concepts in this research can be applied to other fields and other function operators; for example, see [7-9].

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