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Modified Fatima method to solve non-linear differential equations with mix derivative

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Abstract

This research presents an effective approach for resolving nonlinear differential equations with mixed derivatives, focusing on the Ito problem. Unlike traditional methods that require decomposing or transforming the problem into a system of differential equations, our proposed approach directly tackles the problem's complexity while ensuring computational efficiency and precision. The Ito equation, recognized for its importance in simulating many physical and stochastic phenomena, poses considerable challenges owing to its nonlinearity and mixed-derivative terms. Our methodology uses this technique, which decomposes the mixed derivative to derive partial solutions for the other derivative of the equation, then leads to partial solutions for the equation itself, ultimately representing the entire or approximate solution to the equation. This approach guarantees reliable, accurate solutions while preserving the structural integrity of the original equation.

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1. Introduction

Differential equations are essential mathematical instruments for modeling many natural events and engineering processes. They are employed to represent dynamic changes across diverse systems, including physics, engineering, economics, and biology. Differential equations are employed to analyze the motion of objects, examine the spread of diseases, and forecast economic actions.

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Finding the right way to solve each model is crucial. Many researchers have developed effective methods to obtain analytical, approximate, and numerical solutions [1-5]. Despite advancements in analytical and computational methods for solving differential equations, significant challenges remain in resolving certain complex patterns. Therefore, this research aims to explore an improved approach for solving nonlinear differential equations with mixed derivatives. Among these equations is the Ito equation, which generalizes the bilinear KdV problem. Several studies have reformulated the Ito problem as systems of differential equations and solved them using various analytical and numerical methods. Methods such as the Darboux transformation [6], the Riccati–Bernoulli sub-ODE method [7], the extended F-expansion method [8], investigations of invariant tori in Ito-type stochastic dynamical systems [9], numerical methods [10], and stochastic integral transformations for jump-diffusion processes [11] have all been applied to study the Ito equation and related models. This paper presents a modified approach for solving nonlinear differential equations with mixed derivatives. The proposed method is explained clearly to ensure simplicity in application and implementation. Through this research, it is hoped that the developed approach will advance techniques for solving nonlinear differential equations and enhance scientific understanding in this area.

2. Fatima Method (FM)

In this section, we will present Fatima's method, Let's say we have a differential equation that looks like this: [12], [13]

$$L_t u(X,t) + L_X u(X,t) + L_{Xt} u(X,t) + \mathrm{R}(\mathrm{u}(X,\mathrm{t})) + N(u(X,t)) = \mathrm{g}(X,t), X = (x_1, x_2, \ldots)$$

Such that L_t, L_X, L_{Xt} represents the differential of t, x_i, x_i and t respectively, are at the top level, where I can take on any value between 1 and n, and so on. $g(X,t)$ symbolizes a forceful or inhomogeneous term, R represents the linear terms, N is a nonlinear analytic term. Remedies for $u(X,t)$ are partial solutions derived from the operators' equations $L_X u$, $L_t u$ and $L_{Xt} u$. These solutions are referred to as "partial." It has been determined that these partial solutions are substitutable and ultimately converge on the same accurate answer.

It is assumed that the operator L_t, of the minimal rank selected, should adhere to the most recognized conditions.

$$L_t(X,t) = g(X,t) - L_X u(X,t) - L_{Xt} u(\mathrm{X},\mathrm{t}) - R\big(u(\mathrm{X},\mathrm{t})\big)\big) - N(\mathrm{u}(\mathrm{X},\mathrm{t}))$$

Applying L_t^{-1}, then we obtain

$$u(x,y) = \emptyset_0 + L_t^{-1} g(\mathrm{X},\mathrm{t}) - L_t^{-1} L_X u(\mathrm{X},\mathrm{t}) - L_t^{-1} L_y u(\mathrm{X},\mathrm{t}) - L_t^{-1} R(u(X,t)) - L_t^{-1} N\big(u(\mathrm{X},t)\big)$$

Such that $\emptyset_0 = u\,(X,0)$, if $L = \frac{\partial}{\partial t}$, $\emptyset_0 = u(X,0) + t u_t(X,0)$, if $L = \frac{\partial^2}{\partial t^2}$, and so forth.

The solving u is expressed in the shape of an infinite series in the format:
$u(\text{X}, t) = \sum_{n=0}^{\infty} \text{u}_n(X,t)$, such that $\text{u}_0 = \emptyset_0 + L_t{}^{-1} g(X,t)$

$$u_{n+1} = -L_t{}^{-1} L_X u_n(X,\text{t}) - L_t{}^{-1} L_y u_n(X,\text{t}) - L_t{}^{-1} R\big(u_n(X,t)\big) - L_t{}^{-1} N_n$$

Concerning the term "nonlinear," $N(u)$ is
$N(u) = \sum_{k=0}^{\infty} N_k$, Where N_k may be computed by

$$N_k = \frac{1}{k!}\left[\frac{\partial^k N(u)}{\partial t^k}\right]_{t=0}, \text{ where } \frac{\partial^k}{\partial t^k} u = k!\, u_k, k = 0,1,2, \ldots$$

3. Modification for Fatima Method(MFM)

This section outlines the steps involved in modifying the Fatima method [13], [14], [15], for solving DEs.

Consider the following nonlinear DEs as follow:

$$L_t u(X,t) + L_X u(X,t) + L_{Xt} u(X,t) + R(\text{u(X, t)}) + N(\text{u(X, t)}) = g(\text{X, t}), X = (x_1, x_2, \ldots) \quad (1),$$

where L_t is the differential for the largest order in t, L_X represents the highest order differential regarding x_i , $i = 1,2, \ldots$, L_{Xt} is the mix differential for the highest order amidst the x_i and t, R is the remaining linear term for lower derivatives, while .N denotes a mathematical term that is not linear, and the term $g(X,t)$ is a heterogeneous or pushing one. The partial solutions of $u(X,t)$ produced from the equations for operators $L_X u$, $L_t u$ and $L_{Xt} u$ yield partial solutions that are comparable and converge to the exact resolution. However, the choice of which operator $L_X u$ or $L_t u$ or L_{Xt} to utilize in solving the problem mostly hinges on two factors:

(i) To minimize the computing workload, it is advisable to select the operator of the lowest rank.

(ii) To speed up the evaluation of the solution's components, the lowest-order operator ought to be of utmost- established conditions.

Presuming that the operator $L_{xt} = \frac{\partial^n}{\partial t^{k_1} \partial x^{k_2}}$, $n = k_1 + k_2$; the given initial conditions $\left.\frac{\partial^{k_1} u_{k_2 x_i}(X,t)}{\partial t^{k_1}}\right|_{t=0} = f_0$, fulfills the two selection criteria, where $u_{k_2 x_i} = \frac{\partial^{k_2}}{\partial x_i{}^{k_2}} u$ and

$$f_0 = \begin{cases} u_{k_2 x_i}(X,0) \text{ for } L_t = \frac{\partial}{\partial t}, (k_1 = 1) \\ u_{k_2 x_i}(\text{X,0}) + t u_{k_2 x_i}(\text{X,0}) \text{ for } L_t = \frac{\partial^2}{\partial t^2}, (k_1 = 2) \\ u_{k_2 x_i}(\text{X,0}) + t u_{k_2 x_i}(\text{X,0}) + \frac{1}{2!} t^2 u_{k_2 x_i}(\text{X,0}) \text{ for } L_t = \frac{\partial^3}{\partial t^3}, (k_1 = 3) \end{cases},$$

and so on.

Note : The initial condition may be as follows $u(X,0) = f_0$.

To find the solution for equation (1) by MFM we follow these steps:

Step one rewrite the equation (1) as follows :

$$\mathbb{L}_t\big(u_{k_2 x_i}(X,t)\big) = g(\mathrm{X,t}) - L_{x_i}u(\mathrm{X,t}) - L_t u(\mathrm{X,t}) - R(u(\mathrm{X,t})) - N\big(u(X,t)\big)$$

$$(2),\ \mathbb{L}_t = \frac{\partial^{k_1}}{\partial t^{k_1}}$$

Step two applying the inverse operator of $\mathbb{L}_t$

$$\left(\mathbb{L}_t^{-1} = \underbrace{\int_0^t \int_0^t \dots \int_0^t (.)\,dt \dots dt}_{k_1 \text{ times}}\right)$$

for both parties of equation (2), thus, we obtain:

$$u_{k_2 x_i}(X,t) = f_0 + \mathbb{L}_t{}^{-1}\big(g(\mathrm{X,t})\big) - \mathbb{L}_t{}^{-1}\Big(L_{x_i}u(\mathrm{X,t}) + L_t u(\mathrm{X,t}) + R\big(u(\mathrm{X,t})\big) + N\big(u(X,t)\big)\Big)\ (3)$$

Step three A Fatima technique allows for the deconstruction of the variable u into an endless number of constituents

$$u(x,y,t) = \sum_{n=0}^{\infty} \mathrm{u_n(X,t)} = \mathrm{u_0+u_1+u_2} + \cdots,\ (4)$$

and an infinite series of N_k can be used to express the nonlinear portion $N(u)$, that is

$$N(u) = \sum_{k=0}^{\infty} N_k,\ (5)$$

The value of N_k can be computed by $N_k = \frac{1}{k!}\left[\frac{\partial^k N(u)}{\partial t^k}\right]$ (6) ,

where $\frac{\partial^k}{\partial t^k}u = k!\,u_k$, $k = 0,1,2,\dots$

Step four equation (3) becomes :

$$\sum_{n=0}^{\infty}(\mathrm{u_n})_{k_2 x_i}(\mathrm{X,t}) = f_0 + \mathbb{L}_t{}^{-1}\big(g(X,t)\big) - \mathbb{L}_t{}^{-1}L_{x_i}\big(\textstyle\sum_{n=0}^{\infty} u_n(X,t)\big) - \mathbb{L}_t{}^{-1}L_t\big(\sum_{n=0}^{\infty} u_n(\mathrm{X,t})\big) - \mathbb{L}_t{}^{-1}\Big(R\big(\sum_{n=0}^{\infty} u_n(X,t)\big)\Big) - \mathbb{L}_t{}^{-1}\big(\sum_{k=0}^{\infty} N_k\big)\ .$$

The constituents $(u_n)_{k_2 x_i}(X,t)$, where n greater than or equal to 0 can be calculated iteratively utilization the given relation:

$$\left.\begin{aligned}(u_0)_{\mathrm{k_2x_i}}(\mathrm{X,t}) &= \mathrm{f_0} + \mathbb{L}_t^{-1}\big(\mathrm{g(X,t)}\big)\\ (\mathrm{u_{k+1}})_{\mathrm{k_2x_i}}(\mathrm{X,t}) &= -\mathbb{L}_t^{-1}\mathrm{L}_{\mathrm{x}_i}\mathrm{u_k} - \mathbb{L}_t^{-1}\mathrm{L_t u_k} - \mathbb{L}_t^{-1}\big(\mathrm{R(u_k)}\big) - \mathbb{L}_t^{-1}(\mathrm{N_k}),\ \mathrm{k} \geq 0\end{aligned}\right\},$$

(7) where each The determination of a component can be achieved by utilizing the component that comes before it.

Step five finally to find the elements $u_n(X,t)$, n greater than or equal to 0 of solution $u(X,t)$ taking $\mathbb{L}_{x_i}^{-1} = \underbrace{\int\int \dots \int (.)\,dx_i \dots dx_i}_{k_2 \text{ times}}$ to equation (7), a solution that is presented in the form of a series has been derived.

4. Application

This method may effectively solve any problem, regardless of whether it is linear or nonlinear, and involves a combination of derivatives. Here are two examples that demonstrate this, the Ito equation is one of the two differential equations with mixed derivatives that MFM will be used to solve in this section.

4.1 *Example*

The authors in [14] used Heun's approach with the traditional 4th order Runge-Kutta method to solve the Ito equation and obtained numerical solution. In [16] other author used a Darboux transformation and also got the general solution, authors in [17-22] used a three-soliton limit method to obtain solutions regarding rogue waves (general solutions that obtained). Here, The MFM is utilized to resolve the Ito problem, which is solved precisely, whereas various techniques are applied to tackle this system. and got a numerical solution; that is, the presented approach has been better than others.

Assume that the form of the Ito equation with (2+1) dimension is

$$u_{tt} + u_{xxxt} + 6u_x u_t + 3uu_{xt} + 3u_{xx}\int_{-\infty}^{x} u_t dx' + u_{yt} - 5u_{xt} = 0 \; (8),$$

with initial condition

$u(x,0,t) = \frac{-2}{3} + 2sech^2(x+2t)$, we can write the eq.(8) as follow

$u_{yt} = -u_{tt} - u_{xxxt} - 6u_x u_t + 3uu_{xt} - 3u_{xx}\int_{-\infty}^{x} u_t dx' + 5u_{xt}$, or as

$$u_{yt} = -(u_{tt} + u_{xxxt} - 5u_{xt}) - \left(6u_x u_t + 3uu_{xt} + 3u_{xx}\int_{-\infty}^{x} u_t dx'\right) \; (9)$$

$$u_{yt} = -\big(R(u) + N(u)\big), \; (10)$$

where $R(u) = u_{tt} + u_{xxxt} - 5u_{xt}$ and $N(u) = 6u_x u_t + 3uu_{xt} + 3u_{xx}\int_{-\infty}^{x} u_t dx'$, It is possible to write the nonlinear part as a sum of nonlinear terms as follows

$N(u) = n^1 + n^2 + n^3$ such that $n^1 = 6u_x u_t$, $n^2 = 3uu_{xt}$ and $n^3 = 3u_{xx}\int_{-\infty}^{x} u_t dx'$

$L_y u_t = -R(u) - N(u)$, such that $L_y = \frac{\partial}{\partial y}$,

When we integrate equation (9) with respect to (y), we obtaint

$$L_y^{-1} L_y u_t = -L_y^{-1} R(u) - L_y^{-1} N(u) \text{ , where } L_y^{-1} = \int_0^y (.)\, dy$$

$$u_t(x,y,t) - u_t(x,0,t) = -L_y^{-1} R(u) - L_y^{-1} N(u) \Longrightarrow$$

$$u_t(x,y,t) = u_t(x,0,t) + -L_y^{-1} R(u) - L_y^{-1} N(u)$$

Since $u(x,0,t) = \frac{-2}{3} + 2sech^2(x+2t)$

Let $u_0 = \frac{-2}{3} + 2sech^2(x+2t)$, note that we'll put the symbol $\emptyset$ instead of $(x+2t)$ for ease of writing only.

From $u_0 = \frac{-2}{3} + 2sech^2\emptyset$, we get

$\Longrightarrow u_{0t} = -8sech^2\emptyset \tanh\emptyset$,

$u_{0tt} = -16sech^4\emptyset - 4u_{0t}\tanh\emptyset$

$u_{0x} = \frac{1}{2}\, u_{0t}$

$u_{0xx} = -4sech^4\emptyset + 8sech^2\emptyset\, tanh^2\emptyset$

$u_{0xt} = 2\, u_{0xx}$

$u_{0xxx} = 32sech^4\emptyset \tanh\emptyset - 16sech^2\emptyset\, tanh^3\emptyset$

$u_{0xxxt} = 64sech^6\emptyset - 352sech^4\emptyset \tanh^2\emptyset + 64sech^2\emptyset tanh^4\emptyset$

Since $u_{1t} = -L_y^{-1}[R(u_0) + N_0(u)]$, where
$R(u_0) = u_{0tt} + u_{0xxxt} - 5u_{0xt} =$ and
$N_0 = n^1{}_0 + n^2{}_0 + n^3{}_0$, where
$n^1{}_0 = 6u_{0x}u_{0t} = 192sech^4\emptyset\ \tanh^2\emptyset$,
$n^2{}_0 = 3u_0u_{0xt} = 16sech^4\emptyset - 32sech^2\emptyset\ tanh^2\emptyset - 48sech^6\emptyset + 96sech^4\emptyset\tanh^2\emptyset$,
$n^3{}_0 = 3u_{0xx}\int_{-\infty}^{x} u_{0t}dx' = -48sech^6\emptyset + 96sech^4\ \emptyset\ tanh^2\emptyset$
then we get
$u_{1t} = -L_y^{-1}[R(u_0) + N_0(u)]$
$u_{1t} = y[24sech^4\emptyset - 48sech^2\emptyset tanh^2\emptyset + 64sech^6\emptyset - 352sech^4\emptyset\ \tanh^2\emptyset + 64sech^2\emptyset tanh^4\emptyset + 192sech^4\emptyset\ \tanh^2\emptyset + 16sech^4\emptyset - 32sech^2\emptyset\ tanh^2\emptyset - 48sech^6\emptyset + 96sech^4\emptyset\ \tanh^2\emptyset - 48sech^6\emptyset + 96sech^4\emptyset\ tanh^2\emptyset]$,
so we getting the partial solution $u_1$of eq. (10) after integrating with t and assuming the integral constant as zero, thus
$u_1 = -4ysech^2\emptyset\tanh\emptyset$, since $\emptyset = (x + 2t)$, i.e
$u_1 = -4ysech^2(x + 2t)\tanh(x + 2t)$,
also since
$u_{2t} = -L_y^{-1}[R(u_1) + N_1(u)]$, and from u_1 we get
$u_{1t} = y[-8sech^4\emptyset + 16sech^2\emptyset\ tanh^2\emptyset]$
$u_{1x} = \left[\frac{1}{2}u_{1t}\right]$
$u_{1xx} = y[32sech^4\emptyset\ \tanh\emptyset - 16sech^2\emptyset\ tanh^3\emptyset]$
$u_{1tt} = [4u_{1xx}]$
$u_{1xt} = [2u_{1xx}]$
$u_{1xxx} = y[32sech^6\emptyset - 176sech^4\emptyset\ tanh^2\emptyset + 32sech^2\emptyset\ tanh^4\emptyset]$ $u_{1xxxt} = y[-1088sech^6\emptyset\ \tan\emptyset + 16646sech^4\emptyset\ tanh^3\emptyset - 128sech^2\emptyset\ tanh^5\emptyset]$,
so
$R(u_1) = u_{1tt} + u_{1xxxt} - 5u_{1xt}$, and $N_1 = \frac{\partial}{\partial y}(n^1{}_0 + n^2{}_0 + n^3{}_0)$ such that
$\frac{\partial}{\partial y}n^1{}_0 = \frac{\partial}{\partial y}6u_xu_t = 6[u_{0x}u_{1t} + u_{0t}u_{1x}] = y[144sech^6\emptyset - 768sech^4\emptyset\ tanh^3\emptyset]$
$\frac{\partial}{\partial y}n^2{}_0 = \frac{\partial}{\partial y}3uu_{xt} = 3[u_0u_{1xt} + u_{0xt}u_1] = y[-128sech^4\emptyset\tanh\emptyset + 64sech^2\emptyset tanh^3\emptyset + 384sech^6\tan\emptyset - 192sech^4\emptyset\ tanh^3\emptyset]$
$\frac{\partial}{\partial y}n^3{}_0 = \frac{\partial}{\partial y}3u_{xx}\int_{-\infty}^{x} u_tdx' = 3\left[u_{0xx}\int_{-\infty}^{x} u_{1t}\,dx + u_{1xx}\int_{-\infty}^{x} u_{0t}\,dx\right] = y[480sech^6\emptyset\tan\emptyset - 384sech^4\emptyset tanh^3\emptyset]$,
thus we get
$u_{2t} = -L_y^{-1}[y(-192sech^4\emptyset\ \tanh\emptyset + 96sech^2\emptyset\ tanh^3\emptyset - 1088sech^6\emptyset\ \tan\emptyset + 16646sech^4\emptyset\ tanh^3\emptyset - 128sech^2\emptyset\ tanh^5\emptyset + 144sech^6\emptyset - 768sech^4\emptyset\ tanh^3\emptyset - 128sech^4\emptyset\ \tanh\emptyset + 64sech^2\emptyset\ tanh^3\emptyset + 384sech^6\emptyset\ \tan\emptyset - 192sech^4\emptyset\ tanh^3\emptyset + 480sech^6\emptyset\tan\emptyset - 384sech^4\emptyset\ tanh^3\emptyset)]$
After we integrate with respect to (y) and then to (t) we get

$u_2 = \frac{y^2}{2!}[-4sech^4\emptyset + 8sech^2\emptyset tanh^2\emptyset]$, since $\emptyset = (x + 2t)$, i.e

$u_2 = \frac{y^2}{2!}[-4sech^4(x + 2t) + 8sech^2(x + 2t)tanh^2(x + 2t)]$,and so on

Since, $u = \sum_{k=0}^{\infty} u_k \implies$

$u = \frac{-2}{3} + 2sech^2(x + 2t) - 4ysech^2(x + 2t)\tanh(x + 2t) + \frac{y^2}{2!}[-4sech^4(x + 2t) + 8sech^2(x + 2t)tanh^2(x + 2t)] + \cdots$

$u = \left[\frac{-2}{3} + 2sech^2(x + 2t) + y\frac{\partial}{\partial y}\left(\frac{-2}{3} + 2sech^2(x + y + 2t)\right) + \frac{y^2}{2!}\frac{\partial^2}{\partial y^2}\left(\frac{-2}{3} + 2sech^2(x + y + 2t)\right) + \cdots\right]_{t=0}$

Thus, $u = \frac{-2}{3} + 2sech^2(x + y + 2t)$, this is the Ito equation's precise solution.

4.2 *Example*

The Kadomtsev-Petviashivili equation talks about nonlinear waves that change slowly in a medium that spreads them out. It also talks about how weakly nonlinear long water waves that move almost in one direction are changed [9], [18], here we'll solve this equation.

Let's examine the 4th order nonlinear Kadomtsev-Petviashvili equation.

$$u_{xt} - 6uu_{xx} - 6(u_x)^2 + u_{xxxx} + 3u_{yy} = 0 \ . \ (12)$$

With IC: $u_x(x,y,0) = -\frac{1}{2}csc^2\left(\frac{1}{2}(x+y)\right)\coth\left(\frac{1}{2}(x+y)\right)$.

This is one way to write Equation (12): $u_{xt} = N(u) - \mathrm{R(u)}$, (13)

where $N = 6(n^1 + n^2) = 6(uu_{xx} + (u_x)^2)$ and $R(u) = u_{xxxx} + 3u_{yy}$

Here $(u) = \frac{\partial}{\partial t}$, i.e.,

$L(u_x) = N(u) - \mathrm{R(u)}$, take $\mathrm{L}^{-1}$for both parties of the equation (13) to get

$u_x = u_x(x, y, 0) + L^{-1}(N(u) - \mathrm{R(u)})$

Let $u_{0x}(x,y,0) = -\frac{1}{2}csc^2\left(\frac{1}{2}(x+y)\right)\coth\left(\frac{1}{2}(x+y)\right)$

$\implies u_0 = \frac{1}{2}csch^2\left(\frac{1}{2}(x + y)\right)$, note that we'll put the symbol ζ instead of $(x + y)$ for ease of writing only.

So the $u_0$becomes $u_0 = \frac{1}{2}csch^2\left(\frac{1}{2}\zeta\right)$

assuming that the integral constant is equal to zero.

since, $u_{1x} = L^{-1}(N_0 - \mathrm{R(u_0)})$, where $N_0 = 6n^1{}_0 + 6n^2{}_0$, and $\mathrm{R}(u_0) = u_{0xxxx} + 3u_{0xx}$,

from u_0, we get

$u_{0xx} = u_{0yy} = \frac{1}{4}csch^4\left(\frac{1}{2}\zeta\right) + \frac{1}{2}csch^2\left(\frac{1}{2}\zeta\right)coth^2\left(\frac{1}{2}\zeta\right)$

$u_{0xxx} = -csch^4\left(\frac{1}{2}\zeta\right)coth\left(\frac{1}{2}\zeta\right) - \frac{1}{2}csch^2\left(\frac{1}{2}\zeta\right)coth^3\left(\frac{1}{2}\zeta\right)$

$u_{0xxxx} = \frac{1}{2}csch^6\left(\frac{1}{2}\zeta\right) + \frac{11}{4}\ csch^4\left(\frac{1}{2}\zeta\right)\ coth^2\left(\frac{1}{2}\zeta\right) + \frac{1}{2}csch^2\left(\frac{1}{2}\zeta\right)coth^4\left(\frac{1}{2}\zeta\right)$

$n^1{}_0 = 6u_0u_{0xx} \implies n^1{}_0 = \frac{3}{4}csch^6\left(\frac{1}{2}\zeta\right) + \frac{3}{2}csch^4\left(\frac{1}{2}\zeta\right)coth^2\left(\frac{1}{2}\zeta\right)$

$n^2{}_0 = 6(u_{0x})^2 \implies n^2{}_0 = \frac{3}{2}csch^4\left(\frac{1}{2}\zeta\right)coth^2\left(\frac{1}{2}\zeta\right)$

$u_{1x} = t\left[\frac{1}{4}csch^6\left(\frac{1}{2}\zeta\right) + \frac{1}{4}csch^4\left(\frac{1}{2}\zeta\right)coth^2\left(\frac{1}{2}\zeta\right) - \frac{1}{2}csch^2\left(\frac{1}{2}\zeta\right)coth^4\left(\frac{1}{2}\zeta\right) - \frac{3}{4}csch^4\left(\frac{1}{2}\zeta\right) - \frac{3}{2}csch^2\left(\frac{1}{2}\zeta\right)coth^2\left(\frac{1}{2}\zeta\right)\right]$

$u_{1x} = t\left[-\mathrm{csch}^4\left(\frac{1}{2}\zeta\right) - 2csch^2\left(\frac{1}{2}\zeta\right)coth^2\left(\frac{1}{2}\zeta\right)\right] \implies$

$u_{1x} = t\, d\left[2csch^2\left(\frac{1}{2}\zeta\right)coth\left(\frac{1}{2}\zeta\right)\right]$, after we integrate u_{1x} for x, by assuming that when the integration constant is set to 0, what we get:

$u_1 = 2tcsch^2\left(\frac{1}{2}\zeta\right)coth\left(\frac{1}{2}\zeta\right)$, i.e

$u_1 = 2tcsch^2\left(\frac{1}{2}(\mathrm{x+y})\right)coth\left(\frac{1}{2}(\mathrm{x+y})\right)$

Since $u_{2x} = L^{-1}\left(N_1 - \mathrm{R}(u_1)\right)$, where $N_1 = 6n^1{}_1 + 6n^2{}_1$, and $\mathrm{R}(u_1) = u_{1xxxx} + 3u_{1xx}$

$u_{1x} = t\left[-csch^4\left(\frac{1}{2}\zeta\right) - 2csch^2\left(\frac{1}{2}\zeta\right)coth^2\left(\frac{1}{2}\zeta\right)\right]$

$u_{1xx} = t\left[4csch^4\left(\frac{1}{2}\zeta\right)coth\left(\frac{1}{2}\zeta\right) + 2csch^2\left(\frac{1}{2}\zeta\right)coth^3\left(\frac{1}{2}\zeta\right)\right]$

$u_{1xxx} = t\left[-2csch^6\left(\frac{1}{2}\zeta\right) - 11csch^4\left(\frac{1}{2}\zeta\right)coth^2\left(\frac{1}{2}\zeta\right) - 2csch^2\left(\frac{1}{2}\zeta\right)coth^4\left(\frac{1}{2}\zeta\right)\right]$

$u_{1xxxx} = t\left[17csch^6\left(\frac{1}{2}\zeta\right)coth\left(\frac{1}{2}\zeta\right) + 26csch^4\left(\frac{1}{2}\zeta\right)coth^3\left(\frac{1}{2}\zeta\right) + 2csch^2\left(\frac{1}{2}\zeta\right)coth^5\left(\frac{1}{2}\zeta\right)\right]$

$n^1{}_1 = \frac{\partial}{\partial t}[6uu_{xx}] = 6[uu_{xxt} + u_tu_{xx}] = 6u_0u_{1xx} + 6u_1u_{0xx}$

$n^1{}_1 = t\left[15csch^6\left(\frac{1}{2}\zeta\right)coth\left(\frac{1}{2}\zeta\right) + 12csch^4\left(\frac{1}{2}\zeta\right)coth^3\left(\frac{1}{2}\zeta\right)\right]$

$$n^2{}_1 = \frac{\partial}{\partial t}6(u_x)^2 = 12u_{0x}u_{1x}$$

$n^2{}_1 = t\left[6csch^6\left(\frac{1}{2}\zeta\right)coth\left(\frac{1}{2}\zeta\right) + 12csch^4\left(\frac{1}{2}\zeta\right)coth^3\left(\frac{1}{2}\zeta\right)\right]$

$u_{2x} = \frac{t^2}{2!}\left[4csch^6\left(\frac{1}{2}\zeta\right)coth\left(\frac{1}{2}\zeta\right) - 4csch^4\left(\frac{1}{2}\zeta\right)coth^3\left(\frac{1}{2}\zeta\right) - 8csch^2\left(\frac{1}{2}\zeta\right)coth^3\left(\frac{1}{2}\zeta\right) - 12csch^4\left(\frac{1}{2}\zeta\right)coth\left(\frac{1}{2}\zeta\right)\right]$

$u_{2x} = \frac{t^2}{2!}\left[-16csch^4\left(\frac{1}{2}\zeta\right)coth\left(\frac{1}{2}\zeta\right) - 8csch^2\left(\frac{1}{2}\zeta\right)coth^3\left(\frac{1}{2}\zeta\right)\right]$

$u_{2x} = \frac{t^2}{2!}\left[-8csch^4\left(\frac{1}{2}\zeta\right)coth\left(\frac{1}{2}\zeta\right) - 8\left(csch^4\left(\frac{1}{2}\zeta\right)coth\left(\frac{1}{2}\zeta\right) + csch^2\left(\frac{1}{2}\zeta\right)coth^3\left(\frac{1}{2}\zeta\right)\right)\right]$, after we integrate u_{2x} concerning x, and assuming the constant of integration equals zero, we get:

$u_2 = \frac{t^2}{2!}\left[4csch^4\left(\frac{1}{2}\zeta\right) + 8csch^2\left(\frac{1}{2}\zeta\right)coth^2\left(\frac{1}{2}\zeta\right)\right]$, i.e

$u_2 = \frac{t^2}{2!}\left[4csch^4\left(\frac{1}{2}(\mathrm{x+y})\right) + 8csch^2\left(\frac{1}{2}(\mathrm{x+y})\right)coth^2\left(\frac{1}{2}(\mathrm{x+y})\right)\right]$, and so on, from equation (4):

$u = \sum_{n=0}^{\infty} u_n(x, y)$

$u = \frac{1}{2} csch^2\left(\frac{1}{2}(\text{x+y})\right) + 2t csch^2\left(\frac{1}{2}(\text{x+y})\right) coth\left(\frac{1}{2}(\text{x}+\text{y})\right) +$

$\frac{t^2}{2!}\left(4 csch^4\left(\frac{1}{2}(\text{x}+\text{y})\right) + 8 csch^2\left(\frac{1}{2}(\text{x}+\text{y})\right) coth^2\left(\frac{1}{2}(\text{x+y})\right)\right) + \cdots$

$u = \left[\frac{1}{2} csch^2\left(\frac{1}{2}(\text{x}+\text{y}-4\text{t})\right)\right]_{t=0} + t\left[\frac{\partial}{\partial t}\left(\frac{1}{2} csch^2\left(\frac{1}{2}(\text{x}+\text{y}-4\text{t})\right)\right)\right]_{t=0} +$

$\frac{t^2}{2!}\left[\frac{\partial^2}{\partial t^2}\left(\frac{1}{2} csch^2\left(\frac{1}{2}(\text{x+y-4t})\right)\right)\right]_{t=0} + \ldots$

The given response is close to the precise answer: $u = \frac{1}{2} csch^2\left(\frac{1}{2}(\text{x}+\text{y}-4\text{t})\right)$.

5. Conclusion

This study introduces an enhanced approach to the Fatima technique, specifically designed to solve nonlinear differential equations that involve mixed derivatives. This approach demonstrated efficacy in solving equations of this nature, particularly in comparison to alternative methods such as the way for decomposing A domain or iterative iteration method. This method was more straightforward and concise in the solution steps and precise in reaching the exact solution. Furthermore, it has the capability to effectively solve highly non-linear equations that posed a challenge to solve without transforming them into a set of differential equations, such as the Ito problem.

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