

Hybrid approach to solve delay differential equations

Hatim Kareem Khudhair [†]

Kasim Abbas Hussain ^{*}

Department of Mathematics

College of Science

Mustansiriyah University

Baghdad

Iraq

Abstract

In order to solve delay differential equations (DDEs), this work proposes a novel hybrid methodology that combines the Emad-Falih transform (EF) and homotopy perturbation methods (HPM), which is named (HPEFM). A perturbation theory is combined with an unusual transform known as the EF transform. The few numerical cases are solved to illustrate this new methodology. The goal for this study is to make computational jobs related to traditional methods easier. The results show the extent of the valuation decrease and are comparable to those of earlier studies.

Subject Classification: 65R10, 44A05, 47J05, 65Dxx.

Keywords: Emad-Falih transform, Homotopy perturbation method, Delay differential equations, Initial value problems, Hybrid method.

1. Introduction

Throughout history, in numerous domains of medicine, engineering, biology, physics, economics, and other disciplines, delay differential equations (DDEs) have always played significant roles [1-3]. Ordinary differential equations gain strength and provide a fairly accurate representation of actual events when delay is introduced. In contrast to ODEs, DDEs require a historical value or interval that serves as the model's beginning value or interval. Because of this, A delay equation are among the multidimensional equation make it difficult to study DDEs analytically, necessitating the use of a numerical technique. In recent years, numerical methods have been used to solve DDEs.

[†] E-mail: hatim.khudhair@uomustansiriyah.edu.iq

^{*} E-mail: kasimabbas@uomustansiriyah.edu.iq (Corresponding Author)

For example, in [4-5], the authors employ optimal homotopy ergodic methods to decompose linear and non-linear DDEs, whereas in [6-7], DDE formulas were derived using the homotopy analysis approach. A homotopy perturbations methods (HPM) be one for a methods that are used [8-9]. More often than not, applying HPM to differential equation involve introducing the homotopy parameter, which simplifies the system and typically yields a straightforward solution. In any event, HPM effectively reduces the number of calculations while maintaining the numerical solution's high level of precision. The HPM's drawback is that it can be a bit difficult to determine the small parameter. Furthermore, selecting an incorrect parameter produces an incorrect result. Combining the HPM with the EF transform could address this drawback [10-11]. The EF transform is use to decode problem without reverting to frequency domains due to its unit-preserving properties. The mentioned attribute enables the clarification to function smoothly and also improves accuracy. This study proposes a hybrid approach for solving linear and nonlinear DDEs that combines the EF transform with the HPM. The following DDEs is taken into account in this paper,

$$\frac{d^m u(t)}{dt^m} + Qu(t) + Ru(t - \tau) = v(t), \quad (1)$$

& initials condition $u^j(0) = u_0^j$.

Where a continuous function v , the nonlinear operator R , the linear operator Q are all represented.

2. EF Transform

We provide certain definitions of the EF transform and its differential properties in this section.

Definition 1: The EF transforms for functions $u(t)$ in [11] be:

$$EF\{u(t)\} = Z(\omega) = \frac{1}{\omega^2} \int_0^\infty e^{-\omega^2 t} u(t) dt, t \geq 0.$$

Definition 2: The function $Z(\omega)$ inverse EF transform is represented by $EF^{-1}\{Z(\omega)\}$ be the functions $u(t)$, s.t: $EF\{u(t)\} = Z(\omega) \Rightarrow EF^{-1}\{Z(\omega)\} = u(t)$.

Definition 3: The EF transforms for differentiable functions $u(t)$ be

- i. $EF\{u'(t)\} = \omega^2 Z(\omega) - \frac{u(0)}{\omega},$
- ii. $EF\{u''(t)\} = \omega^4 Z(\omega) - \omega u(0) - \frac{u'(0)}{\omega},$
- iii. $EF\{u^{(n)}(t)\} = \omega^{2n} Z(\omega) - \sum_{k=0}^{n-1} \frac{1}{\omega^{4-2n+k}} u^{(k)}(0).$

3. Formulation of HPEFM

The proposed hybrid approach HPEFM is used in this section to handle the general form of DDEs (1) as follows:

- (a) Applying EF transform on both sides of (1), we obtain

$$EF \left[\frac{d^m u}{dt^m} \right] + EF[Qu(t)] + EF[Ru(t - \tau)] = EF[v(t)]. \quad (2)$$

- (b) Using definition 3, we get

$$\begin{aligned} \omega^{2n} Z(\omega) - \sum_{k=0}^{n-1} \frac{1}{\omega^{4-2n+k}} u^{(k)}(0) + EF[Qu(t)] + EF[Ru(t - \tau)] \\ = EF[v(t)]. \end{aligned}$$

or

$$Z(\omega) = \frac{1}{\omega^{2n}} \left[\sum_{k=0}^{n-1} \frac{u^{(k)}(0)}{\omega^{4-2n+k}} - EF[Qu(t)] + EF[v(t)] - EF[Ru(t - \tau)] \right]. \quad (3)$$

- (c) Utilizing the inverse EF transform on (9), we acquire

$$u(t) = w(t) - EF^{-1} \left[\frac{1}{\omega^{2n}} (EF[Qu(t)] + EF[Ru(t - \tau)]) \right]. \quad (4)$$

$$\text{where } w(t) = EF^{-1} \left[\frac{1}{\omega^{2n}} \left(\sum_{k=0}^{n-1} \frac{u^{(k)}(0)}{\omega^{4-2n+k}} + EF[v(t)] \right) \right].$$

- (d) Then, HPM is performed

$$u(t) = \sum_{i=0}^{\infty} k^i u_i(t) = u_0(t) + ku_1(t) + k^2 u_2(t) + \dots, \quad (5)$$

and the nonlinear term $R(u(t - \tau))$ in (4) could be written as

$$R(u(t - \tau)) = \sum_{i=0}^{\infty} k^i h_i(u) = h_0(t) + kh_1(t) + k^2 h_2(t) + \dots, \quad (6)$$

where $h_i(u)$ is the Adomian polynomials that be

$$h_i(u_0 + u_1 + u_2 + \dots + u_n) = \frac{1}{n!} \frac{\partial^n}{\partial k^n} [R(\sum_{i=0}^{\infty} k^i u_i)]_{k=0}, n = 0, 1, 2, \dots \quad (7)$$

Putting (5), (6) and (7) in (4) and comparing the identical powers of k , we yield

$$\begin{aligned} k^0: u_0(t) &= w(t), \\ k^1: u_1(t) &= -EF^{-1} \left[\frac{1}{\omega^{2n}} (EF[Qu_0(t) + h_0(u)]) \right], \\ k^2: u_2(t) &= -EF^{-1} \left[\frac{1}{\omega^{2n}} (EF[Qu_1(t) + h_1(u)]) \right], \\ &\vdots \\ k^n: u_m(t) &= -EF^{-1} \left[\frac{1}{\omega^{2n}} (EF[Qu_{m-1}(t) + h_{m-1}(u)]) \right]. \end{aligned}$$

We may summarize this series by carrying out similar procedures to obtain an approximate resolution of DDEs (1), like

$$u(t) = u_0 + u_1 + u_2 + \dots$$

4. Numerical Test

In this section, two test problems of DDEs are solved to demonstrate the efficacy of the hybrid HPEFM technique, as shown in Tables 1 and 2. The efficiency of problems 1 and 2 is shown in Figures 1 and 2.

Problem 1: Consider the first-order nonlinear DDEs

$$u'(t) = 1 - 2u^2\left(\frac{t}{2}\right), \quad (8)$$

exact solutions are $\sin(t) = u(t)$ where $u(0) = 0$

Solution: After utilizing EF transform, we get

$$\omega^2 Z(\omega) - \frac{u(0)}{\omega} = \frac{1}{\omega^3} - EF \left[2u^2\left(\frac{t}{2}\right) \right],$$

Applying initial condition, it yields

$$Z(\omega) = \frac{1}{\omega^5} - \frac{1}{\omega^2} EF \left[2u^2\left(\frac{t}{2}\right) \right]. \quad (9)$$

On both sides (9) apply inverse EF transform, which gives

$$u(t) = t - EF^{-1} \left[\frac{1}{\omega^2} EF \left[2u^2\left(\frac{t}{2}\right) \right] \right]$$

After that, we use the Hybrid Approach HPEFM to extract Adomian polynomials

$$\sum_{i=0}^{\infty} k^i u_i(t) = t - EF^{-1} \left[\frac{2}{\omega^2} EF \left(\sum_{i=0}^{\infty} k^i u_i^2\left(\frac{t}{2}\right) \right) \right]$$

Considering the comparable power of k , we obtain

$$k^0: u_0(t) = t,$$

$$k^1: u_1(t) = -EF^{-1} \left[\frac{2}{\omega^2} EF \left(u_0^2\left(\frac{t}{2}\right) \right) \right] = -\frac{t^3}{6},$$

$$k^2: u_2(t) = -EF^{-1} \left[\frac{2}{\omega^2} EF \left(2u_0\left(\frac{t}{2}\right) u_1\left(\frac{t}{2}\right) \right) \right] = \frac{t^5}{120},$$

$\vdots$

Thus, an approximate solution will be

$$u(t) = t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040} + \frac{t^9}{362880} + \dots$$

Table 1
Problems 1 outcomes

t	Errors of HPEFM	Errors of LVIM[11]	Errors of STHPM[10]
0	0	0	0
0.1	0	0	0
0.2	0	0	0
0.3	0	0	0
0.4	0	0	0
0.5	$1.3e^{-11}$	$3.5e^{-10}$	$1.51e^{-6}$
0.6	$2.0e^{-10}$	$1.2e^{-9}$	$5.34e^{-6}$
0.7	$4.93e^{-10}$	$4.8e^{-9}$	$1.55e^{-5}$
0.8	$2.143e^{-9}$	$1.56e^{-8}$	$3.88e^{-5}$
0.9	$7.821e^{-9}$	$4.38e^{-8}$	$8.68e^{-5}$
1	$6.542e^{-8}$	$1.095e^{-7}$	$1.77e^{-4}$

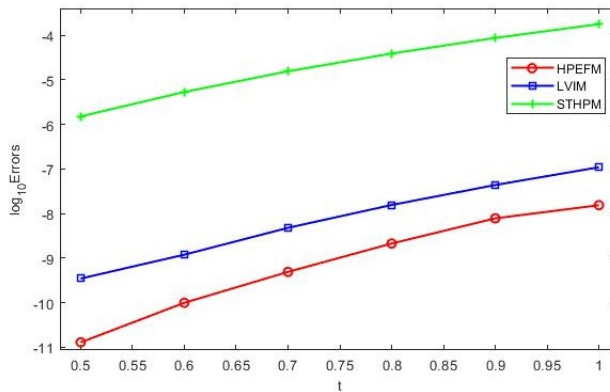


Figure 1
Efficiency curves of Problem 1

Problem 2: Consider the second order linear DDEs

$$u''(t) = \frac{3}{4}u(t) + u\left(\frac{t}{2}\right) - t^2 + 2. \quad (10)$$

exact solutions are $u(t) = t^2$, when $u(0) = 0, u'(0) = 0$

Solution: After utilizing EF transform, we get

$$\omega^4 Z(\omega) - \omega u(0) - \frac{u'(0)}{\omega} = EF \left[\frac{3}{4}u(t) + u\left(\frac{t}{2}\right) \right] - \frac{2}{\omega^7} + \frac{2}{\omega^3},$$

Applying the initial condition, it yields

$$Z(\omega) = \frac{2}{\omega^7} - \frac{2}{\omega^{11}} + \frac{1}{\omega^4} EF \left[\frac{3}{4}u(t) + u\left(\frac{t}{2}\right) \right], \quad (11)$$

On both sides (11) apply inverse EF transform, which gives

$$u(t) = t^2 - \frac{t^4}{12} + EF^{-1} \left[\frac{1}{\omega^4} EF \left[\frac{3}{4}u(t) + u\left(\frac{t}{2}\right) \right] \right],$$

After that, we use the Hybrid Approach HPEFM to extract Adomian polynomials

$$\sum_{i=0}^{\infty} k^i u_i(t) = t^2 - \frac{t^4}{12} + EF^{-1} \left[\frac{1}{\omega^4} EF \left[\frac{3}{4} \sum_{i=0}^{\infty} k^i u_i(t) + \sum_{i=0}^{\infty} k^i u_i\left(\frac{t}{2}\right) \right] \right],$$

Considering the comparable power of k , we obtain

$$k^0: u_0(t) = t^2 - \frac{t^4}{12},$$

$$k^1: u_1(t) = EF^{-1} \left[\frac{1}{\omega^4} EF \left[\frac{3}{4} u_0(t) + u_0\left(\frac{t}{2}\right) \right] \right] = \frac{t^4}{12} - \frac{13t^6}{5760},$$

$\vdots$

As a result, an approximate solutions be $u(t) = t^2 - \frac{17563t^{10}}{67947224800} + \dots$

Table 2
The Problem 2 outcomes

t	Errors of HPEFM	Errors of LVIM[11]	Errors of STHPM[10]
0	0	0	0
0.1	0	0	0
0.2	0	0	0
0.3	0	0	0
0.4	0	0	0
0.5	$1.0e^{-10}$	$3.0e^{-10}$	$3.0e^{-10}$
0.6	$3.0e^{-10}$	$2.1e^{-9}$	$1.6e^{-9}$
0.7	$7.1e^{-10}$	$9.7e^{-9}$	$7.3e^{-9}$
0.8	$1.8e^{-9}$	$3.68e^{-8}$	$2.78e^{-8}$
0.9	$4.15e^{-9}$	$1.196e^{-7}$	$9.01e^{-8}$
1	$1.12e^{-8}$	$3.249e^{-7}$	$5.585e^{-7}$

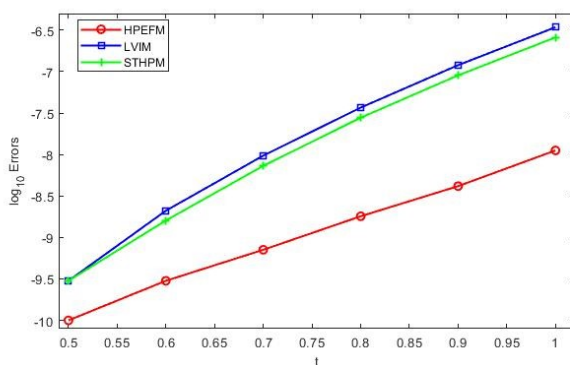


Figure 2
Efficiency curves of Problem 2

5. Conclusions

The Emad-Failh homotopy perturbation method (HPEFM) is a hybrid strategy we used in this paper to estimates solution to both linear and nonlinear DDEs. We provided the results in the form of series solutions after introducing the HPM to address the limitations of the EF transform in handling nonlinearities. Compared with the LVIM and STHPM approaches, the absolute error graphs showed the reliability and precision of HPEFM. Consequently, HPEFM was successfully used to solve both linear and nonlinear DDEs.

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Received May, 2025